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Probability Theory and Related Fields

On the Asymptotic Behaviour of the Free Gas and Its Fluctuations in the Hydrodynamical Limit

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Abstract. We consider the time evolved states $\bar{P}_t = \bar{P} \circ \theta_t^{-1}$ of the free motion $\theta_t(q, v) = (q + tv, v)$, $q, v \in \mathbb{R}^d$, starting in some non-equilibrium state $\bar{P}$ and look at the associated process X_t^ε of fluctuations of the actual number $\theta_{t/\varepsilon}(\mu)\left(\frac{1}{\varepsilon} A \times B\right)$ of particles of the realization μ in $\frac{1}{\varepsilon} \cdot A$ with velocities in B at time t/ε around its mean as $\varepsilon \to 0$ (i.e., in the hydrodynamic limit). It is shown that under natural conditions on the initial state $\bar{P}$, especially a mixing condition in the space variables, for each t the laws of the fluctuations become Gaussian in the hydrodynamic limit in the following sense: $\bar{P} \circ (X_t^\varepsilon)^{-1} \Rightarrow \bar{Q}_t$ as $\varepsilon \to 0$, where $\Rightarrow$ denotes weak convergence and $\bar{Q}_t$ is a centered Gaussian state, which is translation invariant in the space variables. Furthermore the time evolution $(\bar{Q}_t)_t$ is also given by the free motion in the sense that $\bar{Q}_t = \bar{Q}_0 \circ \theta_t^{-1}$. On the other hand we shall see that $\bar{P}_t \Rightarrow P_{z \cdot \lambda \times \sigma}$ as $t \to \infty$, where $P_{z \cdot \lambda \times \tau}$ is the Poisson process with intensity measure $z \cdot \lambda \times \tau$, i.e., the equilibrium state for the free motion with particle density z and velocity distribution τ. In the hydrodynamic limit this behaviour corresponds to the ergodic theorem for the fluctuation process: $\bar{Q}_t \Rightarrow \bar{Q}$ as $t \to \infty$. Here $\bar{Q}$ is a centered Gaussian state describing the equilibrium fluctuations, i.e., the fluctuations of $P_{z \cdot \lambda \times \tau}$. Thus we prove the central limit theorem for the ideal gas: fluctuations are Gaussian even in non-equilibrium. The proofs rest on an adaption of the method of moments for sequences of generalized fields.

Introduction

The natural object to be considered in the kinetic theory of gases would be a gas of infinite many particles in $Y = \mathbb{R}^d$ $(d > 1)$, starting in some non-equilibrium state and evolving under the action of the Newtonian dynamic. Unfortunately in this case nothing can be done with respect to the problems of convergence to equilibrium resp. time evolution of fluctuations. Therefore it is of interest to replace the Newtonian dynamics by simpler models of deterministic time dynamics. Here we consider the simplest case where there is no interaction among the particles, and now the above problems become accessible to a complete mathematical treatment.

Thus we consider the free motion in Y of infinitely many particles. The phase of this gas is given by a position $q\in Y$ and a velocity $v\in Y$ for each molecule, and will change with time according to $\theta_t(q,v)=(q+tv,v)$. At time 0 the phases are distributed according to some law $\bar{P}$, describing a non-equilibrium state of the gas. The main assumptions on $\bar{P}$ are translation invariance and good mixing properties in the space variables.

We consider the time evolved states $\bar{P}_t=\bar{P}\circ\theta_t^{-1}$ and look at

$$\nu^1_{\bar{P}_{t/\varepsilon}}\left(\frac{1}{\varepsilon}\cdot A\times B\right)=\int\theta_{t/\varepsilon}(\mu)\left(\frac{1}{\varepsilon}\cdot A\times B\right)\bar{P}(d\mu),$$

where A, B are bounded Borel sets in Y and $\varepsilon>0$ is small. This defines the mean number of particles in $\frac{1}{\varepsilon}\cdot A$ with velocities in B at time t/ε. The aim of this paper is twofold:

First we are interested in the problem of convergence to equilibrium of the time evolution $(\bar{P}_t)_t$. Then we consider the associated process of the fluctuations of the actual number $\theta_{t/\varepsilon}(\mu)\left(\frac{1}{\varepsilon}\cdot A\times B\right)$ around its mean for $\varepsilon\to 0$ (i.e., in the *hydrodynamic limit*), and after having established the existence of its limit time evolution $(\bar{Q}_t)_t$ we study its asymptotic behaviour as $t\to\infty$. To be more precise, the fluctuations are described by the following measure-valued process (which we consider as a generalized random field in $X=Y\times Y$):

$$X_t^\varepsilon\colon \mu\to\Big(f\to\varepsilon^{d/2}\cdot\Big[\sum_{(q,v)\in\theta_{t/\varepsilon}(\mu)} f(\varepsilon\cdot q,v)-\nu^1_{\bar{P}_{t/\varepsilon}}(f_\varepsilon)\Big]\Big).$$

Here $\varepsilon>0$, $f\in\mathscr{D}=\mathscr{D}(X)$[1] and f_ε is defined by $f_\varepsilon(q,v)=f(\varepsilon\cdot q,v)$. It is to be proved that for each t the laws of the fluctuations become Gaussian in the hydrodynamic limit:

$$\bar{P}_t^\varepsilon:=\bar{P}\circ(X_t^\varepsilon)^{-1}\Rightarrow\bar{Q}_t \quad \text{as } \varepsilon\to 0,$$

where $\Rightarrow$ denotes weak convergence on the conjugate $\mathscr{D}^*$ *of* $\mathscr{D}$ and $\bar{Q}_t$ is a centered Gaussian state which is translation invariant in the space variables. The time evolution $(\bar{Q}_t)_t$ is also given by the free motion in the sense that $\bar{Q}_t=\bar{Q}_0\circ\theta_t^{-1}$. Another feature of the underlying evolution $(\bar{P}_t)_t$ is also inherited: We show that $\bar{P}_t\Rightarrow P_{z\lambda\otimes\tau}$ as $t\to\infty$, where $P_{z\lambda\otimes\tau}$ is the Poisson process with intensity measure $z\lambda\otimes\tau$, determined by the constants of the free motion z = particle density and τ = velocity distribution. An ergodic theorem of this has been proved first under different conditions by Dobrushin/Suhov [6]. (See also Willms [15].) In the hydrodynamic limit this corresponds to the ergodic theorem for the fluctuation process:

$$\bar{Q}_1\Rightarrow\bar{Q} \quad \text{as } t\to\infty,$$

[1] $\mathscr{D}(X)$ denotes the Schwartz space of infinitely differentiable functions on X of compact support

where $\bar{Q}$ is a centered Gaussian state describing the equilibrium fluctuations, i.e., the fluctuations of $P_{z\lambda\otimes\tau}$. Thus we have proved for the free motion, what appears as an ad-hoc assumption in the literature (see [8, 10, 12]): fluctuations are Gaussian even out of equilibrium. A theorem of this kind has been obtained first by McKean [11] for the so-called one-dimensional 2-speed Maxwellian gas. The mathematical problem behind the results below is the problem of weak convergence of sequences of generalized random fields in X under assumptions on the asymptotic behaviour of their moments resp. cumulants. This is called method of moments and will be developed in Sect. 1 [2].

The main idea in the proof of the ergodic theorem $\bar{P}_t \Rightarrow P_{z\lambda\otimes\tau}$ as well as the existence of the Gaussian fluctuation process $(\bar{Q}_t)_t$ is the same [3]: We consider the cumulants γ_t^m of order $m \geqq 2$ of $\bar{P}_t$ resp. $\bar{P}_t^\varepsilon$. Since we assume simplicity of $\bar{P}$ (i.e., $\bar{P}$ is concentrated on the simple Radon point measures on X) and thereby of $\bar{P}_t$ resp. $\bar{P}_t^\varepsilon$, we have to consider only the restrictions $\check{\gamma}_t^m$ of γ_t^m to $\check{X}^m = \{(x_1, \ldots, x_m) \in X^m \mid j_1 \neq j_2 \Rightarrow x_{j_1} \neq x_{j_2}\}$. This reduction is explained in Sect. 2. Then we reduce $\check{\gamma}_t^m$ again, now exploiting their translation invariance in the space variables. This is the main step in the proofs of Sect. 3 and shows that the natural condition for the right asymptotic behaviour of the cumulants γ_t^m (as $t \to \infty$) is Brillinger's mixing condition (3.10). (Compare [3].)

1. The Method of Moments for Generalized Random Fields

Let V be a linear space and V_{alg}^* its algebraic dual. For $f \in V$, $\mu \in V_{\mathrm{alg}}^*$ define $\xi_f(\mu) = \mu(f)$. Let $\mathscr{B}_0(V_{\mathrm{alg}}^*) = \sigma(\xi_f;\, f \in V)$ be the smallest σ-algebra of subsets in V_{alg}^* with respect to which all the functions ξ_f, $f \in V$, are measurable.

A linear map Z from V into the set of random variables over a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a *linear process over* V (on $(\Omega, \mathscr{F}, \mathbb{P})$). A probability measure P on $(V_{\mathrm{alg}}^*, \mathscr{B}_0(V_{\mathrm{alg}}^*))$ defines a linear process $f \to \xi_f$ and therefore is also called a linear process over V. Two linear processes Z_1, Z_2 over V on probability space $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1)$, $(\Omega_2, \mathscr{F}_2, \mathbb{P}_2)$ are said to be (probabilistically) *equivalent*, if for all $k \in \mathbb{N}$, $f_1, \ldots, f_k \in V$ the corresponding k-dimensional distributions coincide, i.e.,

$$\mathbb{P}_{1,(Z_1(f_1), \ldots, Z_1(f_k))} = \mathbb{P}_{2,(Z_2(f_1), \ldots, Z_2(f_k))}.$$

A linear process Z over V on $(\Omega, \mathscr{F}, \mathbb{P})$ is called *of* k-th *order*, if for each $f_1, \ldots, f_k \in V$

$$\nu_Z^k(f_1, \ldots, f_k) = \int_\Omega Z(f_1) \cdot \ldots \cdot Z(f_k)\, d\mathbb{P}$$

exists and is finite. In this case ν_Z^k is called the *moment of* Z *of* k-th *order*. We sometimes write also $\nu_Z^k((f_i)_{i=1,\ldots,k})$. For a linear process P over V we write ν_P^k. A necessary and sufficient condition for Z to be of k-th order is

$$\int_\Omega |Z(f)|^k\, d\mathbb{P} < +\infty \qquad \text{for each } f \in V.$$

[2] see also [17]

[3] A more detailed explanation of the main ideas of the proofs can be found in [18]

Z is called *of infinite order*, if it has moments of all orders. We now prove two theorems relating the convergence of moments to the convergence of the corresponding linear process and vice versa.

Theorem 1.1. *Let (P_n) be a sequence of linear processes over V and P a linear process over V. Suppose that for each $f \in V$ P_{n,ξ_f} converges weakly to P_{ξ_f} and that for some $k \in \mathbb{N}$*

$$(1.1) \qquad \sup_n P_n(\xi_f^{2k}) < +\infty \qquad \text{for each } f \in V.$$

Then P is of k-th *order and*

$$(1.2) \qquad v_{P_n}^k(f_1, \dots, f_k) \to v_P^k(f_1, \dots, f_k), \qquad f_1, \dots, f_k \in V.$$

Proof. Let

$$f_1, \dots, f_k \in V \quad \text{and} \quad \mu_n = P_{n,(\xi_{f_1}, \dots, \xi_{f_k})}, \qquad \mu = P_{(\xi_{f_1}, \dots, \xi_{f_k})}.$$

It is well known that weak convergence of the one-dimensional marginal distributions implies convergence of finite-dimensional marginal distributions. Thus μ_n converges weakly to μ.

Using Skorochod's representation theorem (see [1], Theorem 29.6) there exist random vectors $Y_n = (Y_n^{(1)}, \dots, Y_n^{(k)})$ and $Y = (Y^{(1)}, \dots, Y^{(k)})$ on a common probability space $(\Omega, \mathscr{F}, Q)$ s.th. Y_n has distribution μ_n, Y has distribution μ and $Y_n(\omega) \to Y(\omega)$ for each ω. Let $X_n = \prod_{j=1}^{k} Y_n^{(j)}$ and $X = \prod_{j=1}^{k} Y^{(j)}$. Since convergence with probability 1 implies convergence in distribution we have $X_n \Rightarrow X$, i.e. $Q_{X_n} \Rightarrow Q_X$. Thus if we can show that $\{X_n\}_n$ is uniformly integrable, then X is integrable with respect to Q and $Q(X_n) \to Q(X)$. In this case the theorem is proved because $v_{P_n}^k(f_1, \dots, f_k) = Q(X_n)$ and $v_P^k(f_1, \dots, f_k) = Q(X)$. But for $\alpha > 0$, $n \in \mathbb{N}$

$$\int_{|X_n| \geqq \alpha} |X_n| \, dQ \leqq \frac{1}{\alpha} \int_\Omega |X_n|^2 \, dQ = \frac{1}{\alpha} v_{P_n}^{2k}(f_1, f_1, \dots, f_k, f_k),$$

which combined with Hölder's inequality implies uniform integrability of $\{X_n\}$ in view of (1.1). □

Theorem 1.2. *Let (P_n) be a sequence of linear processes over V, each of which is of infinite order. Suppose that for each $k \in \mathbb{N}$, $f_1, \dots, f_k \in V$*

(1.3) *the limits* $v^k(f_1, \dots, f_k) = \lim_n v_{P_n}^k(f_1, \dots, f_k)$ *exist and*

$$(1.4) \qquad \sum_{n \geqq 1} \left[\sum_{i=1}^{k} v^{2n}(f_i, \overset{2n}{\dots}, f_i) \right]^{-\frac{1}{2n}} = +\infty.$$

Then there exists a unique linear process P over V of infinite order s.th. $v^k = v_P^k$, $k \in \mathbb{N}$, and P_n converges cylindrically to P, i.e.,

$$P_{n,(\xi_{f_1}, \dots, \xi_{f_k})} \Rightarrow P_{(\xi_{f_1}, \dots, \xi_{f_k})} \qquad \text{for each } k \in \mathbb{N},\ f_1, \dots, f_k \in V.$$

Here $\Rightarrow$ denotes weak convergence of probability measures in $\mathbb{R}^k$.

Proof. 1. As has been seen already in [17], combining assumption (1.3) with the method of moments in the finite-dimensional case (summarized in Lemma (1.4) of [16]) yields for given vectors $f_1, \ldots, f_k \in V$ the existence of a probability measure $\mu_{f_1,\ldots,f_k}$ on $\mathbb{R}^k$, which is a solution of the ($\mathbb{R}^k$)-moment problem corresponding to the moments

$$v^{r_1+\ldots+r_k}(f_1, \overset{r_1}{\ldots}, f_1, \ldots, f_k, \overset{r_k}{\ldots}, f_k), \qquad r_1, \ldots, r_k \in \mathbb{N}o.$$

2. From Carleman's many-dimensional uniqueness criterium (see Lemma 1.4 of [16]) we know that the probability measures $\mu_{f_1,\ldots,f_k}$ are uniquely determined by their moments on account of assumption (1.4). Therefore

$$P_{n,(\xi_{f_1},\ldots,\xi_{f_k})} \Rightarrow \mu_{f_1,\ldots,f_k}, \qquad k \in \mathbb{N},\ f_1, \ldots, f_k \in V, \tag{1.5}$$

follows from 1.3 (e.g., by Lemma 1.5 in [16]).

3. Moreover, (1.5) immediately implies that the family

$$M = \{\mu_{f_1,\ldots,f_k};\ k \in \mathbb{N}, f_1, \ldots, f_k \in V\}$$

of probability measures is consistent. Therefore, by Lenard's theorem (see e.g. [7]) M is the system of marginal distributions of a uniquely determined linear process P over V. This proves the theorem. □

In the following we consider linear processes which have continuous realizations. Let V be a linear topological space and let V^* be its conjugate equipped with the strong topology. For $f \in V$, $\mu \in V^*$ we set $\xi_f(\mu) = \mu(f)$ and denote by $\mathscr{B}_0(V^*) = \sigma(\xi_f; f \in V)$ the σ-algebra in V^* generated by the functions ξ_f, $f \in V$. A probability measure P on $(V^*, \mathscr{B}_0(V^*))$ is called a *state over* V. A sequence (P_n) of states over V converges weakly to a state P over V, and we write $P_n \Rightarrow P$, if $P_n(\varphi) \to P(\varphi)$ for each bounded continuous real function φ on V^*.

In view of the last theorem one may look for conditions on V and (P_n) which imply (1) that the limiting process P is a state over V, i.e., supported by the subspace $(V^*, \mathscr{B}_0(V^*))$ of $(V^*_{\mathrm{alg}}, \mathscr{B}_0(V^*))$, and, moreover, one may ask (2) whether cylindrical convergence of (P_n) to P implies even weak convergence $P_n \Rightarrow P$. These problems are well known and are treated in the literature.

We consider these problems in the case when $\mathscr{B}_0(V^*)$ concides with the σ-algebra $\mathscr{B}(V^*)$ of Borel subsets of V^*. This is assured in the following situation which will be assumed from now on: V is *separable*, i.e., in V there exists a dense subsequence, and V is costandard i.e., V^* is standard (with respect to strong topology). In particular this is fulfilled if V is a Fréchet-Montel space or the strict inductive limit of a sequence of Fréchet-Montel spaces[4]. Examples of such spaces are the Schwartz spaces $\mathscr{D}(\mathbb{R}^d)$, $d \in \mathbb{N}$, of infinitely differentiable functions on $\mathbb{R}^d$ of compact support or the spaces $\mathscr{S}(\mathbb{R}^d)$, $d \in \mathbb{N}$, of infinitely differentiable functions on $\mathbb{R}^d$, which together with all their partial derivatives decrease more rapidly than any negative power of $|x|$, $x \in \mathbb{R}^d$. States over $\mathscr{D}(\mathbb{R}^d)$ or $\mathscr{S}(\mathbb{R}^d)$ are called *generalized random fields in* $\mathbb{R}^d$.

[4] This situation was considered in [17]

The situation where V is separable and costandard is particularly nice because states over V are Radon measures on $(V^*, \mathscr{B}(V^*))$ ([7], Theorem I.3.2).

To answer problem (1) posed above we use Minlos' theorem which states that a linear process P over V is probabilistically equivalent to a state over V if we assume also that (α) V is a locally convex space which is conuclear (i.e., V^* is nuclear) and quasi-complete, (β) P is p-continuous for some $p>0$, i.e., there exists p and $c>0$ and a continuous semi-norm q on V s.th.

(1.6) $$P(|\xi_f|^p)^{1/p} \leqq c \cdot q(f), \quad f \in V. \text{ (See [13])}$$

Therefore one condition on (P_n) which in the situation of Theorem 1.1 ensures (1.6) for the limiting state P is the following:

(1.7) there exists $p>0$ s.th. (P_n) is *uniformly p-continuous*, i.e., there exists $c>0$ and a continuous semi-norm q on V s.th. for each n.

$$P_n(|\xi_f|^p)^{1/p} \leqq c \cdot q(f), \quad f \in V.$$

For the second problem we refer to a theorem of Boulicaut ([2], Theorem 4.5) which states that each separable nuclear locally convex space E has the property in question, namely that cylindrical convergence of a sequence of Radon measures on E implies weak convergence.

To summarize we have found the following generalization of Theorem 2.1 in [17]:

Theorem 1.3. *Let V be a separable, quasi-complete locally convex space which is conuclear and costandard. Let (P_n) be a sequence of states over V uniformly p-continuous for some $p>0$* s.th. (1.3) *and* (1.4) *are satisfied. Then there exists a unique state P over V of infinite order* s.th. $v^k = v_P^k$, $k \in \mathbb{N}$, *and* $P_n \Rightarrow P$.

Remark 1.4. Important examples of spaces V satisfying the assumptions of the theorem are the spaces of distributions over $\mathbb{R}^d$: $\mathscr{D}(\mathbb{R}^d)$, $\mathscr{S}(\mathbb{R}^d)$. This follows from well-known results on topological vector spaces.

We now discuss the problem of weak convergence to a *Gaussian state*. By this we mean a state over V s.th. for each $f \in V$ the random variable ξ_f is Gaussian.

Corollary 1.5. *Let V be a space satisfying the assumptions of Theorem 1.3. Let (P_n) be a sequence of states over V satisfying* (1.3) *and* (1.7). *Assume furthermore that the limiting moments v^n, $n \in \mathbb{N}$, appearing in* (1.3) *obey*

(1.8) $$v^k(f_1, \ldots, f_k) = \sum_{J=\{J_1, \ldots, J_l\}} \prod_{i=1}^{l} v^{\# J_i}((f_j)_{j \in J_i})$$

($k \in \mathbb{N}, f_1, \ldots, f_k \in V$), where $\# J_i$ denotes the number of elements in J_i, and J is summed over all pair partitions of $\{1, \ldots, k\}$[5]. *Then there exists a unique Gaussian state P over V with first moment v^1 and second moment v^2* s.th. $P_n \Rightarrow P$.

[5] By a pair partition we mean a partition $\{J_1, \ldots, J_l\}$ of $\{1, \ldots, k\}$ into disjoint subsets J_i, s.th., in the case when k is even, each J_i has two elements, and in the case when k is odd, exactly one J_i has a single element and the rest have two elements

The proof is the same as in the special situation of [17], Cor. 1.1.

We now reformulate this result in terms of cumulants which will prove more useful than moments in the situations considered in Sect. 3. The *cumulant* γ_P^k of a linear process P over V of k-th order is given by

$$\gamma_P^k(f_1, \dots, f_k) = \sum_{J=\{J_1, \dots, J_l\}} (-1)^{l-1} \cdot (l-1)! \prod_{i=1}^{l} v_P^{\# J_i}((f_j)_{j \in J_i}) \tag{1.9}$$

$(f_1, \dots, f_k \in V)$, where J is summed over all partitions of $\{1, \dots, k\}$ into disjoint non-empty subsets. The moments of a linear process P over V of k-th order can be expressed by its cumulants $\gamma_P^1, \dots, \gamma_P^k$ in the following way:

$$v_P^k(f_1, \dots, f_k) = \sum_{J=\{J_1, \dots, J_l\}} \prod_{i=1}^{l} \gamma_P^{\# J_i}((f_j)_{j \in J_i}) \tag{1.10}$$

$(f_1, \dots, f_k \in V)$ where J is summed over all partition of $\{1, \dots, k\}$ into disjoint non-empty subsets. Thus Corollary 1.5 can be reformulated in the following way:

Corollary 1.6. *Let V be as above and (P_n) a sequence of states over V uniformly p-continuous for some $p>0$. If furthermore for each $k \in \mathbb{N}$, $f_1, \dots, f_k \in V$ the limits $\gamma^k(f_1, \dots, f_k) = \lim_{n \to \infty} \gamma_{P_n}^k(f_1, \dots, f_k)$ exist* s.th. *$\gamma^k(f_1, \dots, f_k) = 0$ for $k \geqq 3$ and $k=1$, then there exists a unique centered Gaussian state P over V with covariance γ^2* s.th. $P_n \Rightarrow P$.

Combining Theorem 1.1 and 1.3 we obtain:

Corollary 1.7. *Let V be as in the Theorem* 1.3 *and (P_n) a sequence of states over V uniformly p-continuous for some $p>0$* s.th.

$$\sup_n P_n(\xi_f^{2k}) < +\infty \qquad (f \in V,\ k \in \mathbb{N}). \tag{1.11}$$

If furthermore P_{n, ξ_f} converges weakly to a Gaussian distribution with mean value $v^1(f)$ and variance $v^2(f,f)$ s.th. $v^1: V \to \mathbb{R}$ is linear and $v^2: V \to \mathbb{V}$ is bilinear, symmetric, and nonnegative definite, then there exists a unique Gaussian state P over V s.th. $v_P^1 = v^1$, $v_P^2 = v^2$ and $P_n \Rightarrow P$ and

$$v_{P_n}^k(f_1, \dots, f_k) \to v_P^k(f_1, \dots, f_k), \qquad f_1, \dots, f_k \in V,\ k \in \mathbb{N}. \tag{1.12}$$

Proof. It is well known (see e.g. [4]) that there exists a unique linear Gaussian process P over V s.th. $v_P^1 = v^1$ and $v_P^2 = v^2$. On account of Theorem 1.1 in view of (1.11), the convergence of all moments of P_n to the corresponding moments of P follows. Since P is Gaussian it satisfies Carleman's uniqueness condition (1.4). Thus we are in the situation of Theorem 1.3 and the theorem is proved. □

We now give a condition for relative compactness of sets of states taking into account that V^* is also *regular*, i.e., for each closed subset $F \subset V^*$ and each $x \notin F$ there exists disjoint neighborhoods of F and x.

Corollary 1.8. *Let V be as in Theorem* 1.8. *If $\mathscr{P}$ is a family of states over V satisfying*

(1.13) *there exists $c>0$ and a continuous semi-norm q on V* s.th. *for each k, $P\in\mathscr{P}, f\in V$*

$$P(|\xi_f|^k)^{1/k}\leqq c\cdot q(f),$$

then $\mathscr{P}$ is relatively compact with respect to weak topology.

Proof. It is well known that the space of all states over V with weak topology is standard regular since V^* is standard regular. Thus $\mathscr{P}$ is relatively compact if and only if each sequence (P_n) of elements of $\mathscr{P}$ contains a convergent subsequence. (See [7].) Thus we consider a sequence (P_n) in $\mathscr{P}$. Using Hölder's inequality, the assumption (1.13) implies

(1.14) there exists $c>0$ and a continuous semi-norm q on v s.th. for each $k\in\mathbb{N}, f_1, \dots, f_k\in V$

$$\sup_n |v^k_{P_n}(f_1, \dots, f_n)|\leqq c^k\cdot\prod_{j=1}^k q(f_j).$$

Separability of V combined with Cantor's diagonal procedure and (1.14) gives a subsequence $(P_{n_l})_l$ of (P_n) s.th. for each $k, f_1, \dots, f_k\in V$ the limits

$$v^k(f_1, \dots, f_k)=\lim_l v^k_{P_{n_l}}(f_1, \dots, f_k)$$

exist and the same bound as in (1.14) hold in the limit. Therefore Carleman's uniqueness criterium (1.4) holds for the limit moments. Thus we are in the situation of theorem (1.3) and the results follows. □

2. Cumulant Measures of Simple Point Processes

We now consider special linear processes called simple point processes and give a detailed analysis of their cumulants, based on ideas and results of Brillinger [3] and Krickeberg [9], whose notations we use throughout.

Let X be a locally compact space with a countable base. Denote by $\mathscr{K}(X)$ the set of all continuous functions f on X with compact carrier. $\mathscr{B}(X)$ resp. $\mathscr{B}_0(X)$ denotes the class of all Borel resp. relatively compact Borel subsets of X and $\mathscr{M}^{\boldsymbol{\cdot}}(X)$ is the set of all simple Radon point measures in X. In $\mathscr{M}^{\boldsymbol{\cdot}}(X)$ we consider the vague topology. A probability measure P on the σ-algebra $\mathscr{B}(\mathscr{M}^{\boldsymbol{\cdot}}(X))$ of Borel subsets of X is called a *simple point process in* X.

Let now P be a simple point process in X. We assume that P is of k-th order for some integer k and that P has correlation functions up to order k. By this we mean that

(2.1) $$\dot{v}^l_P\ll\rho^l, \quad l=1, \dots, k,$$

for some diffuse Radon measure ρ in X. Here ρ^l denotes the k-th power of ρ and $\mathring{\nu}_P^l$ denotes the restriction of ν_P^l to

$$\mathring{X}^l = \{(x_1, \ldots, x_l) \in X^l \mid j_1 \neq j_2 \Rightarrow x_{j_1} \neq x_{j_2}\}.$$

The results below are based on the following

Lemma 2.1 (Krickeberg [9]). *For each partition $J = \{J_1, \ldots, J_m\}$ of $\{1, \ldots, k\}$ and Borel set C in $D_J = \{(x_1, \ldots, x_k) \in X^k \mid \forall j = 1, \ldots, m \forall i, i' \in J_j\ x_i = x_{i'}\}$ the simplicity of P implies*

$$\nu_P^k(C) = \nu_P^m(\Pi_J C). \tag{2.2}$$

Here Π_J denotes the projection of D_J onto the space X^m defined by $\Pi_J(x_1, \ldots, x_k) = (y_1, \ldots, y_m)$, where $y_j = x_i$ for all $i \in J_j$.

We show now that Lemma 2.1 has an analog for cumulants.

For a partition J of $\{1, \ldots, k\}$ we define the set E_J of all $(x_1, \ldots, x_k) \in D_J$ s.th. $i \in J_j$, $i' \in J_{j'}$ and $j \neq j'$ imply $x_i \neq x_{i'}$.

Lemma 2.2. *Let P be a simple point process in X satisfying the assumptions above and $J = \{J_1, \ldots, J_m\}$ a partition of $\{1, \ldots, k\}$. Then*

$$\gamma_P^k(C) = \gamma_P^m(\Pi_J C)^6, \quad C \in \mathscr{B}(E_J). \tag{2.3}$$

Proof. To simplify the notations we consider the case where $j < j'$ and $i \in J_j$, $i' \in J_{j'}$ imply $i < i'$; the general case can be reduced to this one by a permutation of the axis of X^k. Furthermore, to prove (2.3) we have to consider only sets C of the form

$$C = E_J \cap (C_1^{k_1} \times \ldots \times C_m^{k_m}),$$

where $C_1, \ldots, C_m \in \mathscr{B}_0(X)$ are pairwise disjoint and $k_i = \# J_i$, $i = 1, \ldots, m$. Observe that $C = \tilde{C}_1 \times \ldots \times \tilde{C}_m$ with $\tilde{C}_i = \{(z, \ldots, z) \in X^{k_i} \mid z \in C_i\}$ and that $\Pi_J C = C_1 \times \ldots \times C_m$. Now

$$\gamma_P^k(C) = \sum_{K = \{K_1, \ldots, K_l\}} (-1)^{l-1} \cdot (l-1)! \cdot \int_{X^k} 1_C((x_j)_{j \in K_1}, \ldots, (x_j)_{j \in K_l})\, \nu_P^{\# K_1}(d(x_j)_{j \in K_1}) \ldots \nu_P^{\# K_l}(d(x_j)_{j \in K_l}), \tag{2.4}$$

where K is summed over all partitions of $\{1, \ldots, k\}$.

Consider two different kinds of summands:

Case 1. If J is the subpartition of K given by the partition $\mathscr{L} = \{\mathscr{L}_1, \ldots, \mathscr{L}_l\}$ of $\{1, \ldots, m\}$, i.e.,

$$K_i = \bigcup_{j \in \mathscr{L}_i} J_i, \quad i = 1, \ldots, l,$$

then the corresponding integral in (2.4) equals

[6] Note that in the case of a point process ν_P^k and γ_P^k may be considered as a (signed) measure on X^k (see [9])

$$\nu_P^{\# K_1}(\bigtimes_{j\in\mathscr{L}_1} \tilde{C}_j)\cdot\ldots\cdot\nu_P^{\# K_l}(\bigtimes_{j\in\mathscr{L}_l} \tilde{C}_j),$$

which by Lemma 2.1 equals

$$\nu_P^{\#\mathscr{L}_1}(\bigtimes_{j\in\mathscr{L}_1} C_j)\ldots\nu_P^{\#\mathscr{L}_l}(\bigtimes_{j\in\mathscr{L}_l} C_j)$$

Case 2. Suppose J is not a subpartition of K, i.e., there exists K_i which is no union of certain J_j. We show that the integral corresponding to K vanishes. Consider the smallest set $\mathscr{L}\subset\{1, \ldots, m\}$ which satisfies

$$K_i\subset\bigcup_{j\in\mathscr{L}} J_j.$$

In this case the integrand of the corresponding integral in (2.4) contains the factor

$$\begin{aligned}&\prod_{j\in\mathscr{L}} 1_{\tilde{C}_j}((x_n)_{n\in J_j})\\ &=\prod_{\substack{j\in\mathscr{L}\\ J_j\cap K_i^C\neq\emptyset}}[1_{\{x_j\}^{\# K_i\cap J_j}}((x_n)_{n\in K_i\cap J_j})\cdot 1_{C_j'}((x_n)_{n\in J_j\setminus K_i})]\\ &\cdot\prod_{\substack{j\in\mathscr{L}\\ J_j\subset K_i}} 1_{\tilde{C}_j}((x_n)_{n\in J_j}).\end{aligned}$$

Here $C_j'=\{(z, \ldots, z)\in X^{\# J_j\setminus K_i}: z\in C_j\}$ and x_j is any element of $(x_n)_{n\in J_j\setminus K_i}$.

Thus, using Fubini's theorem, the corresponding integral in (2.4) contains the factor

$$\nu_P^{K_i}(\bigtimes_{\substack{j\in\mathscr{L}\\ J_j\cap K_i^C\neq\emptyset}}\{x_j\}^{\# K_i\cap J_j}\times\bigtimes_{\substack{j\in\mathscr{L}\\ J_j\subset K_i}}\tilde{C}_j)=\nu_P^{\#\mathscr{L}}(\bigtimes_{\substack{j\in\mathscr{L}\\ J_j\cap K_i^C\neq\emptyset}}\{x_j\}\times\bigtimes_{\substack{j\in\mathscr{L}\\ J_j\subset K_i}} C_j)$$

In the case $x_j\in C_j$, which we have to consider only, the integrand A in this last integral is a subset of $\bigtimes_{j\in\mathscr{L}} C_j$ and thus of $\hat{X}^{\# J}$, which satisfies $\rho^{\#\mathscr{L}}(A)=0$ on account of ρ being diffuse. Since we assumed also that P has correlation functions up to order K we obtain $\nu_P^{\#\mathscr{L}}(A)=0$.

To summarize we have shown that

$$\gamma_P^k(C)=\sum_{\mathscr{L}=\{\mathscr{L}_1,\ldots,\mathscr{L}_l\}}(-1)^{l-1}(l-1)!\prod_{i=1}^{l}\nu_P^{\#\mathscr{L}_i}(\bigtimes_{j\in\mathscr{L}_i} C_j)$$

where $\mathscr{L}$ is summed over all partitions of $\{1, \ldots, m\}$.

On the other hand the rhs equals $\gamma_P^m(C_1\times\ldots\times C_m)=\gamma_P^m(\Pi_J C)$. This proves the lemma. □

Observe that $\cup_J E_J=X^k$, and the set E_J are mutually disjoint. Thus in the situation above we can decompose γ_P^k in the following way:

For $A \in \mathscr{B}(X^k)$ Lemma 2.2 implies

$$\begin{aligned}\gamma_P^k(A) &= \sum_J \gamma_P^k(A \cap E_J) \\ &= \sum_J \gamma_P^{\# J}(\check{X}^{\# J} \cap \Pi_J A)\end{aligned}$$

We denote by $\check{\gamma}_P^{\# J}$ the restriction of $\gamma_P^{\#}$ to $\check{X}^{\# J}$.

If we now express the $\check{\gamma}_P^{\# J}$ in terms of the moments of P and use that P has correlation functions up to order k, we immediately get:

Corollary 2.2 (Brillinger [3], Theorem 3.2). *Let ρ be a diffuse Radon measure in X and P a simple point process in X of k-th order* s.th.

$$\check{\gamma}_P^l \ll \rho^l \quad \text{with density } \rho_P^l, \quad l = 1, \dots, k.$$

Then for $A \in B(X^k)$

$$\gamma_P^k(A) = \sum_{J = \{J_1, \dots, J_m\}} \int_{\Pi_J A} r_P^m(x_1, \dots, x_m)\, \rho(dx_1) \dots \rho(dx_m) \tag{2.5}$$

where J is summed over all partitions of $\{1, \dots, k\}$ and

$$r_P^m(x_1, \dots, x_m) = \sum_{K = \{K_1, \dots, K_l\}} (-1)^{l-1} \cdot (l-1)! \cdot \prod_{i=1}^{l} \rho_P^{\# K_i}((x_j)_{j \in K_i}) \tag{2.6}$$

In (2.6) K is summed over all partitions of $\{1, \dots, m\}$.

3. Convergence to Equilibrium and Fluctuations of the Free Motion in the Hydrodynamic Limit

We now use the methods developed above for the study of the time-asymptotic behaviour of the free motion and its fluctuations. Let $Y = \mathbb{R}^d$ and denote by λ the Lebesgue measure in Y. On the space $X = Y^2 = Y \times Y$ we consider the family of homeomorphisms θ_t defined by

$$\theta_t(q, v) = (q + tv, v), \quad (q, v) \in X, \; t \in \mathbb{R}. \tag{3.1}$$

$(\theta_t)_{t \in \mathbb{R}}$ is called *free motion in X*. Let $\rho = \lambda \otimes \lambda$ denote the Lebesgue measure in X.

We first study the time evolution of a large class of initial states under the free motion and prove an ergodic theorem. The main assumption for these initial states is a condition concerning the correlation decay in the space variables due to Brillinger [3]. We shall see later that this condition also guarantees the existence of the associated fluctuation process in the so-called hydrodynamic limit of this time evolution.

Denote by $p_i: X \to Y$, $i = 1, 2$, the projection onto the first resp. second coordinate, and consider the following two families of translations:

(3.2) $T_q: X \to X, \quad T_q(q', v') = (q' - q, v'), \quad (q', v') \in X, \; q \in Y;$

(3.3) $t_q: Y \to Y, \quad t_q(q') = q' - q, \quad q, q' \in Y.$

We now consider an initial point process $\bar{P}$ in X which satisfies the following conditions:

(3.4) $\bar{P}$ is simple;

(3.5) $\bar{P}$ has moments and correlation functions of all orders;

(3.6) $\bar{P}$ is stationary up to infinite order, i.e., for each k $\nu^k_{\bar{P}}$ is $(T^k_q)_q$-invariant, where $T^k_q(q_1, v_1, \ldots, q_k, v_k) = (T_q(q_1, v_1), \ldots, T_q(q_k, v_k))$;

(3.7) there exists a non-negative function $g \in \mathscr{L}_1(\lambda)$ s.th. for each k $\rho^k_{\bar{P}} \leqq \bigotimes_{j=1}^{k} g \circ p_2$ ρ^k-a.e.

Before we formulate a further condition for $\bar{P}$ we note some consequences of (3.4)–(3.7) for the time evolutions $(\bar{P}_t)_t$ in X and $(P_t)_t$ in Y, given by $P_t = \bar{P}_t \circ p_1^{-1}$ and $\bar{P}_t = \bar{P} \circ \theta_t^{-1}$. Observe that in general P_t is not a point process in Y. But in the situation considered here each P_t is even simple:

Simplicity of $\bar{P}$ implies simplicity of each $\bar{P}_t$, $t \in \mathbb{R}$. Thus given a bounded Borel subset A of Y we see that $\nu^k_{P_t}(A^k) = \nu^k_{\bar{P}_t}(p_1^{-1}(A)^k)$ is a sum of integrals of the form

$$\int_{p_1^{-1}(A)^l} \rho^l_{\bar{P}_t} \, d\rho^l = \int_{p_1^{-1}(A)^l} \rho^l_{\bar{P}} \circ \theta^l_t \, d\rho^l, \qquad l = 1, \ldots, k,$$

which are all $\leqq \lambda(A)^l \cdot \lambda(g)^l < +\infty$ on account of (3.7).

Therefore each P_t is of infinite order. Combining this with the fact that each $\bar{P}_t$ is simple implies simplicity of each P_t (see [16], last remark). Since $\bar{P}$ is of infinite order each $\bar{P}_t$ is of infinite order.

Furthermore, each $\bar{P}_t$ resp. P_t is stationary up to infinite order with respect to $(T_q)_q$ resp. $(t_q)_q$. This follows from the stationary of $\bar{P}$ and the fact that $T_q \circ \theta_t = \theta_t \circ T_q$, $t \in \mathbb{R}$, $q \in Y$. Finally we remark that each $\bar{P}_t$ has correlation functions of all orders with respect to ρ resp. λ, namely

(3.8) $\rho^k_{\bar{P}_t}(q_1, v_1, \ldots, q_k, v_k) = \rho^k_{\bar{P}}(q_1 - t v_1, v_1, \ldots, q_k - t v_k, v_k)$ resp.

(3.9) $\rho^k_{P_t}(q_1, \ldots, q_k) = \int_{Y^k} \rho^k_{\bar{P}}(q_1 - t v_1, v_1, \ldots, q_k - t v_k, v_k) \, \lambda(dv_1) \ldots \lambda(dv_k).$

We finally assume the basic condition of correlation decay in the space variables of the initial state P_0:

(3.10) (Brillinger [3]) the reduced factorial cumulant measures $\check{K}^m_{P_0}$, $m = 2, 3, \ldots$, of P_0 [7] are of bounded variation, i.e., for $m \geqq 2$

$$\int_{Y^{m-1}} |r^m_{P_0}(q_1, \ldots, q_{m-1}, 0)| \, \lambda(dq_1) \ldots \lambda(dq_{m-1}) < +\infty.$$

[7] reduced with respect to $(t_q)_q$, using as reference measure λ

We now show that this condition in the situation above guarantees an ergodic theorem for $(\bar{P}_t)_t$ as well as $(P_t)_t$: Consider for $k=1,2,\ldots, f_1,\ldots,f_k \in \mathscr{K}(X)$, $f := f_1 \otimes \ldots \otimes f_k$

$$\check{\nu}^k_{\bar{P}_t}(f) = \sum_{J=\{J_1,\ldots,J_m\}} \prod_{i=1}^{m} \check{\gamma}^{\# J_i}_{\bar{P}_t}((f_j)_{j \in J_i}). \tag{3.11}$$

We show that $\check{\nu}^k_{\bar{P}_t}(f) \to (z\lambda \otimes \tau)^k(f)$ as $t \to \infty$, where $z = \int_Y \rho^1_{\bar{P}}(0, v)\, \lambda(dv)$ and $\tau = \frac{1}{z} \cdot \rho^1_{\bar{P}}(0, \cdot) \cdot \lambda$. Then by the method of moments we know that $\bar{P}_t \Rightarrow P_{z\lambda\otimes\tau}$ [8] as $t \to \infty$ (see [16], Theorem 4.1 for a proof). Now the summand corresponding to $J = \{\{1\}, \ldots, \{k\}\}$ on the rhs of (3.11) is given by

$$\nu^1_{\bar{P}_t}(f_1) \cdot \ldots \cdot \nu^1_{\bar{P}_t}(f_k) = (z\lambda \otimes \tau)^k(f),$$

using the $(T_q)_q$-invariance of $\nu^1_{\bar{P}_t}$. We thus have to show for $m \geqq 2$, $f \in \mathscr{K}(X^m)$ that

$$\check{\gamma}^m_{\bar{P}_t}(f) \to 0 \quad \text{as } t \to \infty.$$

But by Corollary 2.2

$$\check{\gamma}^m_{\bar{P}_t}(f) = \int_{X^m} f(x_1, \ldots, x_m) \cdot r^m_{\bar{P}_t}(x_1, \ldots, x_m)\, \rho(dx_1) \ldots \rho(dx_m), \tag{3.12}$$

where

$$r^m_{\bar{P}_t}(x_1, \ldots, x_m) = r^m_{\bar{P}}(q_1 - t v_1, v_1, \ldots, q_m - t v_m, v_m).$$

Here and in the sequel we often write $x_i = (q_i, v_i)$.

Now by the stationarity of $\bar{P}$ one can choose a version $r^m_{\bar{P}}$ which is invariant under $(T^m_q)_q$, where $T^m_q = (T_q, \underset{m}{\ldots}, T_q)$. Thus

$$\begin{aligned} \check{\gamma}^m_{\bar{P}_t}(f) = & \int_{X^m} f(x_1, \ldots, x_m) \cdot r^m_{\bar{P}}(q_1 - t(v_1 - v_m) - q_m, v_1, \ldots, q_{m-1} - t(v_{m-1} - v_m) \\ & - q_m, v_{m-1}, 0, v_m) \quad \rho(dx_1) \ldots \rho(dx_m) \\ = & \int_{Y^m} \Big[\int_{Y^m} f(q_1 + t(v_1 - v_m) + q_m, v_1, \ldots, q_{m-1} + t(v_{m-1} - v_m) \\ & + q_m, v_{m-1}, q_m, v_m)\, r^m_{\bar{P}}(x_1, \ldots, x_{m-1}, 0, v_m)\, \lambda(dv_1) \ldots \lambda(dv_m) \Big]\, \lambda(dq_1) \ldots \lambda(dq_m) \end{aligned} \tag{3.13}$$

Observe now that Brillinger's mixing condition implies that the absolute value of $[\ldots]$ in (3.13) is dominated by a λ^m-integrable function of the form

$$\begin{aligned} & \|f\|_\infty \cdot 1_K(q_m) \cdot \Big| \int_{Y^m} r^m_{\bar{P}}(q_1, v_1, \ldots, q_{m-1}, v_{m-1}, 0, v_m)\, \lambda(dv_1) \ldots \lambda(dv_m) \Big| \\ & \qquad = \|f\|_\infty \cdot 1_K(q_m) \cdot |r^m_{P_0}(q_1, \ldots, q_{m-1}, 0)| \end{aligned}$$

for some $K \subset Y$ compact (uniformly in t).

[8] Here as usual $P_{z\lambda\otimes\tau}$ denotes the Poisson process in X with intensity measure $z \cdot \lambda \otimes \tau$

On the other hand [...] in (3.13) a.s. converges to 0 as $t\to\infty$ on account of Lebesgue's theorem combined with (3.7). Thus

$$\gamma^m_{\bar{P}_t}(f)\to 0 \quad \text{as } t\to\infty.$$

To summarize we have shown the following:

Theorem 3.1. *Let $\bar{P}$ be a point process in X satisfying* (3.4)–(3.7) *together with* (3.10).

Then

$$\bar{P}_t \Rightarrow P_{z\lambda\otimes\tau} \quad \textit{as } t\to\infty. \tag{3.15}$$

From this theorem we obtain the following ergodic theorem for the time evolution (P_t) of simple point processes:

Theorem 3.2. *In the situation of the theorem*

$$P_t \Rightarrow P_{z\lambda} \quad \textit{as } t\to\infty. \tag{3.16}$$

We now consider the time evolution $(\bar{P}_t)_t$ in the *hydrodynamical limit*. By this we understand the following limit: Consider the measure-valued process given for $f\in\mathscr{D}(X)$, $\mu\in\mathscr{M}^{\cdot}=\mathscr{M}^{\cdot}(X)$, $\varepsilon>0$, on $(\mathscr{M}^{\cdot}, \mathscr{B}(\mathscr{M}^{\cdot}), \bar{P})$ by

$$X^\varepsilon_t\colon \mu\to\Big(f\to\varepsilon^{d/2}\cdot\Big[\sum_{(q,v)\in\theta_{t/\varepsilon}(\mu)} f(\varepsilon\cdot q, v) - v^1_{\bar{P}_{t/\varepsilon}}(f_\varepsilon)\Big]\Big) \tag{3.17}$$

Here $f_\varepsilon\in\mathscr{D}(X)$ is defined by $f_\varepsilon(q,v):=f(\varepsilon\cdot q, v)$.

Let $\bar{P}^\varepsilon_t$ be the image of $\bar{P}$ under X^ε_t. We are interested in the limiting behaviour of $\bar{P}^\varepsilon_t$ as $\varepsilon\to 0$.

Theorem 3.3. *Let $\bar{P}$ be a point process in X satisfying* (3.4)–(3.7) *together with* (3.10). *Then for each t there exists a centered Gaussian state $\bar{Q}_t$ over $\mathscr{D}(X)$ with covariance*

$$\gamma^2_{\bar{Q}_t}(f)=(z\lambda\otimes\tau)_0(f)+\int_{X^2} f(q_2+tv_1, v_1, q_2+tv_2, v_2) \cdot r^2_{\bar{P}}(q_1, v_1, 0, v_2)\,\lambda(dv_1)\,\lambda(dv_2)\,\lambda(dq_1)\,\lambda(dq_2) \tag{3.18}$$

($f\in\mathscr{D}(X^2)$), s.th. $\bar{P}^\varepsilon_t\Rightarrow\bar{Q}_t$ as $\varepsilon\to 0$. Here $(z\lambda\otimes\tau)_0$ denote the image of $z\lambda\otimes\tau$ with respect to $x\to(x,x)$.

Proof. We shall use Corollary 1.6. Using a well-known property of cumulants we obtain for $k\geqq 2$, $\varepsilon>0$, $t\in\mathbb{R}$ that $\gamma^k_{\bar{P}^\varepsilon_t}=\varepsilon^{k\cdot\frac{d}{2}}\cdot\gamma^k_{\bar{Q}^\varepsilon_t}$, where $\bar{Q}^\varepsilon_t$ is the image of $\bar{P}$ under $\mu\to\big(f\to\sum_{(q,v)\in\theta_{t/\varepsilon}(\mu)} f(\varepsilon\cdot q, v)\big)$.

Note that each $\bar{Q}^\varepsilon_t$ is simple and has moments and correlation functions of all orders with respect to ρ, namely

$$\rho^k_{\bar{Q}^\varepsilon_t}(q_1, v_1, \dots, q_k, v_k)=\varepsilon^{-k\cdot d}\cdot\rho^k_{\bar{P}}\left(\frac{1}{\varepsilon}(q_1-tv_1), v_1, \dots, \frac{1}{\varepsilon}(q_k-tv_k), v_k\right).$$

Thus by Corollary 2.2.

(3.19) $$\gamma^k_{Q^\varepsilon_t}(f)=\sum_{J=\{J_1,\dots,J_m\}}\varepsilon^{-m\cdot d}\int_{X^m} f\circ\Pi_J^{-1}(x_1,\dots,x_m)\cdot r^m_{\bar P}\left(\frac{1}{\varepsilon}(q_1-tv_1),v_1,\dots,\frac{1}{\varepsilon}(q_m-tv_m),v_m\right)\quad \rho(dx_1)\dots\rho(dx_m)$$

($f\in\mathscr{D}(X^k)$). Proceeding exactly as in the proof of the ergodic theorem, by using stationarity of $\bar P$ up to infinite order, we see that this equals

$$\sum_J\varepsilon^{-m\cdot d}\int_{X^m} f\circ\Pi_J^{-1}(x_1,\dots,x_m)\cdot r^m_{\bar P}\left(\frac{1}{\varepsilon}(q_1-t(v_1-v_m)-q_m),v_1,\dots,\frac{1}{\varepsilon}(q_{m-1}-t(v_{m-1}-v_m)-q_m),v_{m-1},0,v_m\right)\quad \rho(dx_1)\dots\rho(dx_m)$$

which in turn equals

$$\sum_J\varepsilon^{-m\cdot d}\cdot\varepsilon^{(m-1)\cdot d}\int_{X^m} f\circ\Pi_J^{-1}(\varepsilon\cdot q_1+t(v_1-v_m)+q_m,v_1,\dots,\varepsilon\cdot q_{m-1}+t(v_{m-1}-v_m)+q_m,v_{m-1},q_m,v_m)\,r^m_{\bar P}(q_1,v_1,\dots,q_{m-1},v_{m-1},0,v_m)\,\rho(dx_1)\dots\rho(dx_m)$$

To summarize, we have shown that for each $f\in\mathscr{D}(X^k)$

(3.20) $$\gamma^k_{Q^\varepsilon_t}(f)=\varepsilon^{-d}\cdot\sum_{J=\{J_1,\dots,J_m\}}\int_{Y^m}\Big[\int_{Y^m} f\circ\Pi_J^{-1}(\varepsilon\cdot q_1+t(v_1-v_m)+q_m,v_1,\dots,\varepsilon\cdot q_{m-1}+t(v_{m-1}-v_m)+q_m,v_{m-1},q_m,v_m)\,r^m_{\bar P}(q_1,v_1,\dots,q_{m-1},v_{m-1},0,v_m)\cdot\lambda(dv_1)\dots\lambda(dv_m)\Big]\,\lambda(dq_1)\dots\lambda(dq_m)$$

Arguing as above in the proof of the ergodic theorem, by Brillinger's mixing condition the terms [...] in (3.20) satisfy

$$|[\dots]|\leqq\|f\|_\infty\cdot 1_K(q_m)\cdot|r^m_{P_0}(q_1,\dots,q_{m-1},0)|$$

for some $K\subset Y$ compact, and the rhs of this inequality is λ^m-integrable. On the other hand by Lebesgue's theorem combined with (3.7) [...] converges as $\varepsilon\to 0$ to

$$\int_{Y^m} f\circ\Pi_J^{-1}(q_m+t(v_1-v_m),v_1,\dots,q_m+t(v_{m-1}-v_m),v_{m-1},q_m,v_m)\cdot r^m_{\bar P}(q_1,v_1,\dots,q_{m-1},v_{m-1},0,v_m)\,\lambda(dv_1)\dots\lambda(dv_m).$$

Thus we obtain for $\varepsilon\to 0$

(3.21) $$\gamma^k_{P^\varepsilon_t}(f)\to 0\qquad\text{if } k>2$$

and

(3.22) $$\gamma^2_{P^\varepsilon_t}(f)\to z\cdot\int_{Y^2} f(x,x)\,\lambda(dq)\,\tau(dv)+\int_{Y^2}\int_{Y^2} f(q_2+t(v_1-v_2),v_1,q_2,v_2)\cdot r^2_{\bar P}(q_1,v_1,0,v_2)\,\lambda(dv_1)\,\lambda(dv_2)\,\lambda(dq_1)\,\lambda(dq_2).$$

To complete the proof of the first part we show that each family $(\bar{P}_t^\varepsilon)_\varepsilon$ is uniformly 2-continuous or equivalently that the family $f \to X_t^\varepsilon(f, \cdot)$, $\varepsilon > 0$, of linear mappings from $\mathscr{D}(X)$ in $L^2(\bar{P})$ is uniformly continuous. It is sufficient to show that for each $C \subset X$ compact the family $f \to X_t^\varepsilon(f, \cdot)$, $\varepsilon > 0$, of linear mappings from $\mathscr{D}_C(X)$ in $L^2(\bar{P})$ is uniformly continuous. Here $\mathscr{D}_C(X)$ are the functions in $\mathscr{D}(X)$ with compact support in C. Now there exists $K \subset Y$ compact s.th. for $f \in \mathscr{D}_C(X)$ by the argument above

$$\begin{aligned}\bar{P}(|X_t^\varepsilon(f, \cdot)|^2) &= \varepsilon^d \cdot \gamma_{\bar{Q}_t^\varepsilon}^2(f \otimes f)\\ &\leqq z \cdot \int_X f(x, x)\, \lambda(dq)\, \tau(dv) + \|f\|_\infty^2 \cdot \lambda(K) \cdot \int_Y |r_{P_0}^2(q_1, 0)|\, \lambda(dq_1)\\ &\leqq (z \cdot \rho(C) + \lambda(K) \cdot \int_Y |r_{P_0}^2(q_1, 0)|\, \lambda(dq_1)) \cdot \|f\|_\infty^2 .\end{aligned}$$

Thus by Corollary 1.6 for each t there exists a centered Gaussian state $\bar{Q}_t$ over $\mathscr{D}(X)$ with covariance $\gamma_{\bar{Q}_t}^2$ given by the limit in (3.22) s.th. for each t $\bar{P}_t^\varepsilon \Rightarrow \bar{Q}_t$ as $\varepsilon \to 0$. □

Theorem 3.3 implies the following structure of the Gaussian time evolution $(\bar{Q}_t)_t$ which is inherited from the underlying time evolution $(\bar{P}_t)_t$: $(\bar{Q}_t)_t$ is guided by the free motion in the sense that $\bar{Q}_t = \bar{Q}_0 \circ \theta_t^{-1}$, $t \in \mathbb{R}$. This means, that the law which governs the fluctuations at time t (in the hydrodynamic limit) is given by the initial law developed under the free motion. Moreover, an ergodic theorem is true for $(\bar{Q}_t)_t$. To be more precise, we have

Corollary 3.4. *In the situation of Theorem* 3.3 *we have* $\bar{Q}_t = \bar{Q}_0 \circ \theta_t^{-1}$ *for each* $t \in \mathbb{R}$. *Furthermore* $\bar{Q}_t \Rightarrow \bar{Q}$ *as* $t \to \infty$, *where* $\bar{Q}$ *is a centered Gaussian state over* $\mathscr{D}(X)$ *with covariance* $\gamma_{\bar{Q}}^2 = (z\lambda \otimes \tau)_0$.

Proof. The ergodic theorem immediately follows from (3.18) combined with Corollary 1.6 using the arguments above. For the first assertion note that the theorem shows that for $k \neq 2$, $f \in \mathscr{D}(X^k)$

$$\gamma_{\bar{Q}_t}^k(f) = \gamma_{\bar{Q}_0}^k(f \circ \theta_t^k) = 0,$$

and if $k = 2$, we have from (3.18) that

$$\gamma_{\bar{Q}_t}^2(f) = \gamma_{\bar{Q}_t}^2(f \circ \theta_t^2).$$

On the other hand $\gamma_{\bar{Q}_0}^k(f \circ \theta_t^k) = \gamma_{\bar{Q}_0 \circ \theta_t^{-1}}^k(f)$ for each k. Therefore $\bar{Q}_t$ and $\bar{Q}_0 \circ \theta_t^{-1}$ have the same cumulants and consequently the same moments. But since Gaussian states are determined by their moments (see [17] e.g.), we see that $\bar{Q}_t = \bar{Q}_0 \circ \theta_t^{-1}$, $t \in \mathbb{R}$. □

Corollary 3.5. *The stationarity up to infinite order of each* $\bar{P}_t$ *implies translation invariance in the space variables of each* $\bar{Q}_t$, *i.e.,* $\bar{Q}_t \circ T_q^{-1} = \bar{Q}_t$, *and thus also of* $\bar{Q}$. *This follows in the same way from the theorem as Corollary* 3.4.

Remark 3.6. If in Theorem 3.3 we take an equilibrium state as initial state, for example a Poisson process $P_{z \cdot \lambda \otimes g \cdot \lambda}$, where $z > 0$ and $g \geqq 0$, $\lambda(g) = 1$, then for each $t\bar{P}_t^\varepsilon \Rightarrow \bar{Q}$ as $\varepsilon \to 0$, where $\bar{Q}$ is the centered Gaussian state over $\mathscr{D}(X)$ with covariance $\gamma_{\bar{Q}}^2 = (z\lambda \otimes g\lambda)_0$. Thus $\bar{Q}$ describes the law of the equilibrium fluctuations. Therefore Corollary 3.4 says that $\bar{Q}_t$ converges weakly to that centered Gaussian equilibrium state $\bar{Q}$, which is determined by the invariants of the underlying free motion, namely by the particle density and the probability distribution of the single particle velocity.

Remark 3.7. If we only look at the fluctuations process of the particle number, given by the image P_t^ε of $\bar{P}$ under the mapping

$$\mu \to (f \to \varepsilon^{d/2} \cdot [\sum_{q \in p_1 \circ \theta_{t/\varepsilon}(\mu)} f(e \cdot q) - \varepsilon^d \cdot z \cdot \lambda(f)])$$

($\mu \in \mathscr{M}^{\cdot}(X)$, $f \in \mathscr{D}(Y)$), we obtain from Theorem 3.3 the existence of a time evolution $(Q_t)_t$ of translation invariant centered Gaussian states over $\mathscr{D}(Y)$ with covariance

$$\gamma_{Q_t}^2(f) = z \cdot \lambda_0(f) + \int_{X^2} f(q_2 + t v_1, q_2 + t v_2)\, r_{\bar{P}}^2(q_1, v_1, 0, v_2) \cdot \lambda(dv_1)\, \lambda(dv_2)\, \lambda(dq_1)\, \lambda(dq_2) \tag{3.23}$$

($f \in \mathscr{D}(Y^2)$), s.th. $P_t^\varepsilon \Rightarrow Q_t$ as $\varepsilon \to 0$. Here $\lambda_0(f) = \int_Y f(q, q)\, \lambda(dq)$. Furthermore, $Q_t \Rightarrow Q$ as $t \to \infty$, where Q is a translation invariant centered Gaussian state over $\mathscr{D}(Y)$ with covariance $\gamma_Q^2 = z \cdot \lambda_0$.

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