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Probability Theory and Related Fields

The Strong p-Variation of Martingales and Orthogonal Series

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Summary. Let $1 \leqq p < \infty$ and let $x=(x_n)_{n \geqq 0}$ be a sequence of scalars. The strong p-variation of x, denoted by $W_p(x)$, is defined as

$$W_p(x)=\sup\left\{\left(|x_0|^p+\sum_{k=1}^{\infty}|x_{n_k}-x_{n_{k-1}}|^p\right)^{1/p}\right\}$$

where the supremum runs over all increasing sequences of integers $0=n_0 \leqq n_1 \leqq n_2 \leqq \dots$.

Let $1 \leqq p < 2$ and let $M=(M_n)_{n \geqq 0}$ be a martingale in L_p. Our main results are as follows: If $\Sigma \mathbb{E}|M_n-M_{n-1}|^p<\infty$, then $W_p(M)$ is finite a.s. and we have

$$\mathbb{E} W_p(M)^p \leqq C(\mathbb{E}|M_0|^p+\sum_{n \geqq 1} \mathbb{E}|M_n-M_{n-1}|^p)$$

for some constant C depending only on p. On the other hand, let (φ_n) be an arbitrary orthonormal system of functions in L_2, consider $x=(x_n)_{n \geqq 0}$ in l_2 and let $S_n=\Sigma_0^n x_i \varphi_i$ and $S=(S_n)_{n \geqq 0}$. We prove that if $\Sigma|x_n|^p<\infty$ $(1 \leqq p<2)$ then $W_p(S(t))<\infty$ for a.e.t and $\|W_p(S)\|_2 \leqq C(\Sigma|x_n|^p)^{1/p}$ for some constant C. Each of these results is an extension of a result proved by Bretagnolle for sums of independent mean zero r.v.'s. The case $p>2$ in also discussed. Our proofs use the real interpolation method of Lions-Peetre. They admit extensions in the Banach space valued case, provided suitable assumptions are imposed on the Banach space.

0. Introduction

Let $1 \leqq p < \infty$ and let $x=(x_n)_{n \geqq 0}$ be a sequence of elements of a Banach space B.

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The strong p-variation of x, denoted by $W_p(x)$, is defined as follows

$$W_p(x)=\sup\{(\|x_0\|^p+\sum_{i\geqq 1}\|x_{n_i}-x_{n_{i-1}}\|^p)^{1/p}\}$$

where the supremum runs over all increasing sequences of integers $0=n_0\leqq n_1\leqq n_2\leqq\dots$.

Let $v_p=\{x\in\mathbb{R}^{\mathbb{N}}\,|\,W_p(x)<\infty\}$. More generally, let $v_p(B)=\{x\in B^{\mathbb{N}}\,|\,W_p(x)<\infty\}$.

The spaces v_p and $v_p(B)$, equipped with the norm W_p, are clearly Banach spaces.

In this paper, we study the following problem: given a sequence $X=(X_n)_{n\geqq 0}$ of random variables (r.v.'s in short), when is X a.s. in v_p? Or equivalently when is $W_p(X)$ a.s. finite?

To answer this question, we have found it useful to study another family of Banach spaces which are defined using the K-method of interpolation. We recall all the necessary facts in Sect. 1 below.

Our main results on the above problem are the following: Let $1\leqq p<2$. There is a constant C_p such that every martingale $M=(M_n)_{n\geqq 0}$ in L_p satisfies

$$\mathbb{E}\,W_p(M)^p\leqq C_p(\mathbb{E}|M_0|^p+\sum_{n\geqq 1}\mathbb{E}|M_n-M_{n-1}|^p). \tag{0.1}$$

Note that the converse inequality is trivial since for any x in $\mathbb{R}^{\mathbb{N}}$ we have

$$(|x_0|^p+\sum_{n\geqq 1}|x_n-x_{n-1}|^p)^{1/p}\leqq W_p(x).$$

Let $S_p(M)=(|M_0|^p+\sum_{n\geqq 1}|M_n-M_{n-1}|^p)^{1/p}$. More generally, for any $1\leqq r<\infty$ there is a constant C_{pr} such that every martingale M in L_r satisfies

$$\|W_p(M)\|_r\leqq C_{pr}\|S_p(M)\|_r. \tag{0.2}$$

These results extend to the martingale case a result of Bretagnolle [B] who essentially proved (0.1) and (0.2) in the case where the increments (M_n-M_{n-1}) form a sequence of independent mean zero r.v.'s. A weaker form of (0.2) was proved by Lépingle [L] using an embedding of the martingale into Brownian motion (cf. also [Br] and [S]). Essentially, Lépingle obtained (0.2) but with the left hand side replaced by $\|W_{p_1}(M)\|_r$ with $p_1>p$ and a constant depending on p_1. Related earlier results appear in [BG 1, BG 2, Mi, and M] in the particular case of stable processes or processes with stationary independent increments.

For $p\geqq 2$, these inequalities are no longer valid. We will obtain (in the case $p>2$) a new proof of the following inequality of Lépingle [L].

For $2<q<\infty$ and $1\leqq r<\infty$, there is a constant C_{qr} such that every martingale $M=(M_n)_{n\geqq 0}$ in L_r satisfies

$$\|W_q(M)\|_r\leqq C_{qr}\|\sup_n|M_n|\,\|_r. \tag{0.3}$$

Our method is quite different from Lépingle's. Actually, (0.3) and (0.2) are both derived from a single basic idea (see Lemma 2.2 below). Our method has the advantage of extending with no difficulty to the Banach space valued case. This is considered in Sect. 4 where we prove that (0.2) remains valid (with the obvious changes) for martingales with values in a Banach space B under the assumption that B is p_0-uniformly smooth for $p_0 > p$. Similarly, (0.3) extends if we assume that B is q_0-uniformly convex for some $q_0 < q$. The scalar (or Hilbert space) case corresponds to $p_0 = q_0 = 2$. One can give a rather striking interpretation of (0.2) and (0.3): for $1 \leq p < 2$, the p-variation of M can be computed – on the average – using the *finest* subsequence $n_k = k$, while for $p > 2$ it is the *coarsest* subsequence $n_0 = 0$, $n_1 = +\infty$ which plays a similar rôle.

In Sect. 3, we extend (0.1) for the partial sums of orthogonal series of the form $S_n = \Sigma_0^n \alpha_i \varphi_i$ where (φ_i) is an orthonormal sequence in L_2 and (α_i) is a sequence of coefficients. Let $S = (S_n)_{n \geqq 0}$. We prove that $\sum_0^\infty |\alpha_n|^p < \infty$ implies that $W_p(S)$ is in L_2. The case of series S_n with coefficients in a Banach space is also considered in Sect. 4. The present paper is a continuation of our earlier publication [PX].

1. Background on Interpolation Spaces

We will recall here the basic facts from interpolation theory which are used in the sequel. We refer the reader to [BL] for more information.

Let (A_0, A_1) be a "compatible" couple of Banach spaces. By "compatible", we mean that A_0 and A_1 are continuously embedded into a large topological vector space so that we can form the sum $x_0 + x_1$ of elements x_0 in A_0 and x_1 in A_1.

For an element x in $A_0 + A_1$, the so-called K-functional is defined for $t > 0$ as follows:

$$K_t(x; A_0, A_1) = \inf\{\|x_0\|_{A_0} + t\,\|x_1\|_{A_1} \,|\, x = x_0 + x_1\}$$

We will denote it by $K_t(x)$ when there is no ambiguity. For $0 < \theta < 1$ and $1 \leqq q \leqq \infty$, the interpolation space $(A_0, A_1)_{\theta q}$ is defined as the set of all those x in $A_0 + A_1$ such that $\int_0^\infty (t^{-\theta} K_t(x))^q \frac{dt}{t} < \infty$. We equip this space with the norm

$$\|x\| = \left(\int_0^\infty (t^{-\theta} K_t(x))^q \frac{dt}{t} \right)^{1/q}.$$

We make here the usual convention: for $q = \infty$, this becomes $\sup_{t > 0} t^{-\theta} K_t(x)$.

Let $(\Omega, \mathscr{A}, \mu)$ be an arbitrary measure space and let B be a Banach space. Throughout the paper, the space $L_p(\mu, B)$ (or briefly $L_p(B)$) denotes the completion of $L_p(\mu)\otimes B$ with respect to the usual L_p-norm for B-valued functions. It is classical that if $1\leqq p_0, p_1\leqq\infty$ and if $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, $0<\theta<1$, then

$$(L_{p_0}, L_{p_1})_{\theta p}=L_p \quad \text{with equivalent norms.}$$

More generally, if (A_0, A_1) is a compatible couple, the spaces $L_{p_0}(\mu; A_0)$ and $L_{p_1}(\mu; A_1)$ also form a compatible couple. We will use the well known fact that if $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ then

$$(1.1)\qquad L_p((A_0, A_1)_{\theta p})=(L_{p_0}(A_0), L_{p_1}(A_1))_{\theta p}$$

with equivalent norms (cf. [BL] p. 123 and p. 130).

It is rather easy to check and well known that $(A_0, A_1)_{\theta p}\subset(A_0, A_1)_{\theta q}$ if $q\geqq p$ and this inclusion is bounded.

We will use several times "the interpolation theorem". This is the following fact (immediate from the definitions):

(1.2) Let (A_0, A_1) and (B_0, B_1) be compatible couples of Banach spaces, let T be a linear operator from A_0+A_1 into B_0+B_1, and assume

$$\forall x\in A_0 \ \|Tx\|_{B_0}\leqq\|x\|_{A_0} \quad \text{and} \quad \forall x\in A_1 \ \|Tx\|_{B_1}\leqq\|x\|_{A_1}.$$

Then T is a bounded operator from $(A_0, A_1)_{\theta q}$ into $(B_0, B_1)_{\theta q}$ with norm $\leqq 1$. (cf. [BL] p. 41). We will also use the following result, known as "the reiteration theorem" (cf. [BL] p. 50).

Let (A_0, A_1) be a compatible couple of Banach spaces. Consider θ_0, θ_1 in $]0, 1[$ and p_0, p_1 in $[1, \infty]$. Let $B_0=(A_0, A_1)_{\theta_0 p_0}$ and $B_1=(A_0, A_1)_{\theta_1 p_1}$. Then (B_0, B_1) naturally forms a compatible couple so that we may consider the space $(B_0, B_1)_{\theta p}$ for $0<\theta<1$ and $1\leqq p\leqq\infty$.

The reiteration theorem says that

$$(1.3)\qquad (B_0, B_1)_{\theta p}=(A_0, A_1)_{\eta p} \quad \text{for } \eta=(1-\theta)\theta_0+\theta\theta_1.$$

Moreover, we have

$$(1.4)\qquad (A_0, B_1)_{\theta p}=(A_0, A_1)_{\delta p} \quad \text{for } \delta=\theta\theta_1$$

and

$$(1.5)\qquad (B_0, A_1)_{\theta p}=(A_0, A_1)_{\omega p} \quad \text{for } \omega=(1-\theta)\theta_0+\theta.$$

The important fact about these formulas is that the preceding interpolation spaces (1.3), (1.4) or (1.5) do *not* depend on p_0 or p_1.

Of course, all the identities (1.3)–(1.5) correspond to equivalent norms on the spaces under consideration (the constants involved in these equivalences do depend on p_0, p_1, θ and p). This implies that the same formulas are true if B_0 and B_1 are only assumed to satisfy the following continuous inclusions

(1.6) $$(A_0, A_1)_{\theta_0 1} \subset B_0 \subset (A_0, A_1)_{\theta_0 \infty}$$

(1.7) $$(A_0, A_1)_{\theta_1 1} \subset B_1 \subset (A_0, A_1)_{\theta_1 \infty}.$$

Let B be a Banach space. We have defined in the introduction the spaces v_p and $v_p(B)$. For $p=1$, v_1 is the space of sequences of bounded variation and $W_1(x)=|x_0|+\sum_{n\geq 1}|x_n-x_{n-1}|$ for all x in v_1. Similarly for $v_1(B)$. As usual, we will denote by l_∞ (resp. $l_\infty(B)$) the space of all bounded sequences of scalars (resp. elements of B) with the norm $\|x\|_\infty=\sup|x_n|$ ($\sup\|x_n\|$). We will denote by c (resp. $c(B)$) the closed subspace of l_∞ (resp. c) formed by all the convergent sequences. Clearly $v_1\subset c\subset l_\infty$ so that we may view (v_1, l_∞) or (v_1, c) as a compatible couple of spaces and consider the interpolation space $(v_1, l_\infty)_{\theta q}$. For simplicity, we denote

$$A_{\theta q}=(v_1, l_\infty)_{\theta q} \quad (0<\theta<1, 1\leqq q\leqq\infty).$$

The reader can check easily that since $v_1\subset c$ and c is closed in l_∞, we also have $A_{\theta q}=(v_1, c)_{\theta q}$ with identical norms (indeed $x\in A_{\theta q}$ implies $K_t(x)\to 0$ when $t\to\infty$ hence $x\in c$).

In the case of sequences of elements of B, we again have $v_1(B)\subset c(B)\subset l_\infty(B)$. We denote

$$A_{\theta q}(B)=(v_1(B), l_\infty(B))_{\theta q}.$$

As before, this space coincides with $(v_1(B), c(B))_{\theta q}$ and the norms are identical. Now let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space and let $(\mathscr{A}_n)_{n\geq 0}$ be an increasing sequence of σ-subalgebras of $\mathscr{A}$. Let us denote simply by L_p (resp. $L_p(B)$) the space $L_p(\Omega, \mathscr{A}, \mathbb{P})$ (resp. $L_p(\Omega, \mathscr{A}, \mathbb{P}; B)$) in what follows. We will study the interpolation spaces associated to the couple $(L_1(v_1), L_\infty(l_\infty))$. We note immediately that for $0<\theta<1$ and $p=(1-\theta)^{-1}$ $\left(\text{i.e. } \frac{1}{p}=\frac{1-\theta}{1}+\frac{\theta}{\infty}\right)$, we have

(1.8) $$(L_1(v_1), L_\infty(l_\infty))_{\theta p}=L_p(A_{\theta p})$$

with equivalent norms.

Indeed this follows from (1.1).

The reason for the importance of $A_{\theta p}$ in our study of the strong p-th variation lies in the following fact:

(1.9) If $1<p<\infty$ and $1-\theta=\frac{1}{p}$, then $A_{\theta p}(B)\subset v_p(B)$ and this inclusion has norm bounded by a constant $K(p)$ depending only on p.

This is easy to prove (cf. [BP]). Indeed for any fixed sequence $0=n_0\leqq n_1\leqq\dots$ we introduce the operator $T\colon v_1(B)\to l_1(B)$ defined by

$$T(x)=(x_0, x_{n_1}-x_0, \dots, x_{n_k}-x_{n_{k-1}}, \dots).$$

This has clearly norm $\leqq 1$. On the other hand, considered as operator from $l_\infty(B)$ into $l_\infty(B)$, T has norm $\leqq 2$. Therefore it follows from the interpolation theorem (cf. (1.2) above), that T has norm $\leqq 2$ as an operator from $A_{\theta p}(B)$ into $(l_1(B), l_\infty(B))_{\theta p}$. By (1.1), this space can be identified with $l_p(B)$ with an equivalent norm. This yields (for some constant $K(p)$)

$$(\|x_0\|^p + \Sigma \|x_{n_k} - x_{n_{k-1}}\|^p)^{1/p} \leqq K(p) \|x\|_{A_{\theta q}(B)},$$

and (1.9) clearly follows from this.

2. Martingales

Our main result is the following statement which was proved in [B] for sums of independent mean zero r.v.'s. A similar but weaker inequality appears in [L].

Theorem 2.1. *Assume* $1 \leqq p < 2$. (i) *There is a constant* C_p *such that all martingales* $M = (M_n)_{n \geqq 0}$ *in* L_p *satisfy (with the convention* $M_{-1} \equiv 0$)

$$\mathbb{E}\, W_p(M)^p \leqq C_p \mathbb{E}\Big(\sum_{n \geqq 0} |M_n - M_{n-1}|^p\Big).$$

(ii) *More generally, if* $1 \leqq r < \infty$, *there is a constant* C_{pr} *such that every martingale* $M = (M_n)_{n \geqq 0}$ *in* L_r *satisfies*

$$\| W_p(M)\|_r \leqq C_{pr} \Big\| \Big(\sum_{n \geqq 0} |M_n - M_{n-1}|^p\Big)^{1/p}\Big\|_r.$$

Throughout the sequel, we will set by convention $M_{-1} = 0$ whenever $M = (M_n)_{n \geqq 0}$ is a martingale. All the r.v.'s are assumed to be defined on a given probability space $(\Omega, \mathscr{A}, \mathbb{P})$. We will need the following key lemma.

Lemma 2.2. *For any martingale* M *in* L_2, *we have*

$$\|M\|_{(L_1(v_1), L_\infty(l_\infty))_{\frac{1}{2}\infty}} \leqq 2\Big(\sum_{n \geqq 0} \mathbb{E}|M_n - M_{n-1}|^2\Big)^{1/2}.$$

Note that by orthogonality we have

$$\sum_{n \geqq 0} \mathbb{E}|M_n - M_{n-1}|^2 = \sup_{n \geqq 0} \mathbb{E}|M_n|^2. \tag{2.1}$$

Proof of Lemma 2.2. Given a sequence of r.v.'s $X = (X_n)_{n \geqq 0}$, we denote simply by $K_t(X)$ the K_t-norm of X with respect to the couple $(L_1(v_1), L_\infty(l_\infty))$. Explicitly, we have

$$K_t(X) = \inf\Big\{\|X_0^0\|_1 + \sum_{n \geqq 1} \|X_n^0 - X_{n-1}^0\|_1 + t \sup_n \|X_n^1\|_\infty\Big\} \tag{2.2}$$

where the infimum runs over sequences of r.v.'s X^0 and X^1 such that $X_n = X_n^0 + X_n^1$ for all $n \geqq 0$.

Let (M_n) be a martingale, relative to an increasing sequence of σ-algebras $(\mathscr{A}_n)_{n\geqq 0}$, and let $0\leqq T_0\leqq T_1\leqq\dots$ be a sequence of stopping times (relative to $(\mathscr{A}_n)_{n\geqq 0}$) with values in $\mathbb{N}\cup\{\infty\}$. We assume that (M_n) is bounded in L_2, hence M_n converges a.s. (and in L_2) to a limit denoted by M_∞ which is in L_2. Moreover, we have $M_n=\mathbb{E}(M_\infty|\mathscr{A}_n)$ and $M_T=\mathbb{E}(M_\infty|\mathscr{A}_T)$ for any stopping time T with values in $\mathbb{N}\cup\{\infty\}$. (For more details, cf. e.g., [N]). Therefore, the sequence $(M_{T_k})_{k\geqq 0}$ is a martingale, and (2.1) implies

$$(2.3)\quad \mathbb{E}(|M_{T_0}|^2+\sum_{k\geqq 1}|M_{T_k}-M_{T_{k-1}}|^2)\leqq\sup\mathbb{E}|M_{T_k}|^2$$

$$\leqq\mathbb{E}|M_\infty|^2=\sum_{n\geqq 0}\mathbb{E}|M_n-M_{n-1}|^2.$$

To prove Lemma 2.2, we may assume for simplicity that $\|M_\infty\|_2\leqq 1$. Then we define by induction starting with $T_0=\inf\{n\geqq 0, |M_n|>t^{-1/2}\}$,

$$T_1=\inf\{n\geqq T_0, |M_n-M_{T_0}|>t^{-1/2}\}$$
$$\vdots$$
$$T_k=\inf\{n\geqq T_{k-1}, |M_n-M_{T_{k-1}}|>t^{-1/2}\}$$

and so on.

As usual, we make the convention $\inf\emptyset=+\infty$, i.e., we set $T_k=+\infty$ on the set where

$$\sup_{n\geqq T_{k-1}}|M_n-M_{T_{k-1}}|\leqq t^{-1/2}.$$

Clearly $\{T_k\}$ is increasing sequence of stopping times so that (2.3) holds. We note that if $T_0(\omega)<\infty$ then $|M_{T_0(\omega)}(\omega)|\geqq t^{-1/2}$ and

$$(2.4)\qquad \text{if } T_k(\omega)<\infty \quad\text{and}\quad k\geqq 1 \quad\text{then } (M_{T_k}-M_{T_{k-1}})(\omega)\geqq t^{-1/2}.$$

Moreover, we have for all $k\geqq 0$

$$(2.5)\qquad \sup_{T_k\leqq n<T_{k+1}}|M_n-M_{T_k}|\leqq t^{-1/2}\text{ a.s.}\quad\text{and}\quad\sup_{n<T_0}|M_n|\leqq t^{-1/2}.$$

Hence, we can write $M_n=X_n^0+X_n^1$, with X^0, X^1 defined as follows

$$X_n^0=\sum_{k\geqq 0}1_{\{T_k\leqq n<T_{k+1}\}}M_{T_k}$$

$$X_n^1=\sum_{k\geqq 0}1_{\{T_k\leqq n<T_{k+1}\}}(M_n-M_{T_k})+1_{\{n<T_0\}}M_n$$

By (2.5) we have $\|\sup|X_n^1|\|_\infty\leqq t^{-1/2}$.

On the other hand,

$$\text{let } \Delta_0=|M_{T_0}| \quad\text{and}\quad \Delta_k=|M_{T_k}-M_{T_{k-1}}|\quad\text{for}\quad k\geqq 1.$$

We have

(2.6) $$|X_0^0|+\sum_{n\geq 1}|X_n^0-X_{n-1}^0|=1_{\{T_0=0\}}\Delta_0+\sum_{k\geq 1}\Delta_k 1_{\{T_k<\infty\}}.$$

This can be estimated as follows. We have by (2.4)

(2.7) $$t^{-1/2}(1_{\{T_0=0\}}+\sum_{k\geq 1}1_{\{T_k<\infty\}})\leq\Delta_0 1_{\{T_0=0\}}+\sum_{k\geq 1}\Delta_k 1_{\{T_k<\infty\}}.$$

Let $N=1_{\{T_0=0\}}+\sum_{k\geq 1}1_{\{T_k<\infty\}}$. By Cauchy-Schwarz, (2.7) implies

(2.8) $$N t^{-1/2}\leq N^{1/2}(|\Delta_0|^2+\sum_{k\geq 1}|\Delta_k|^2)^{1/2}$$

Clearly N is finite a.s. (since M_n converges a.s.) and (2.8) implies

$$N^{1/2}\leq t^{1/2}(|\Delta_0|^2+\Sigma|\Delta_k|^2)^{1/2}$$

hence by (2.3)

(2.9) $$(\mathbb{E}N)^{1/2}\leq t^{1/2}\|M_\infty\|_2\leq t^{1/2}.$$

Now going back to (2.6) we find again by Cauchy-Schwarz

$$\begin{aligned}\mathbb{E}(|X_0^0|+\sum_{n\geq 1}|X_n^0-X_{n-1}^0|)&\leq(\mathbb{E}N)^{1/2}\|(|\Delta_0|^2+\Sigma|\Delta_k|^2)^{1/2}\|_2\\&\leq t^{1/2},\end{aligned}$$

hence by (2.3), (2.9) and (2.2), this yields $K_t(M)\leq 2t^{1/2}$ so that

$$\|M\|_{[L_1(v_1),L_\infty(l_\infty)]_{\frac{1}{2}\infty}}\leq 2.$$

By homogeneity, this completes the proof of Lemma 2.2.

Proof of Theorem 2.1. Let $(\mathcal{A}_n)_{n\geq 0}$ be a fixed increasing sequence of σ-subalgebras of $\mathcal{A}$. All martingales below will be with respect to $(\mathcal{A}_n)_{n\geq 0}$. For $1\leq p\leq\infty$, we will denote by D_p the subspace of $l_p(L_p)$ formed of all the sequences $\varphi=(\varphi_n)_{n\geq 0}$ such that φ_n is $\mathcal{A}_n$-measurable for all $n\geq 0$ and $\mathbb{E}(\varphi_n|\mathcal{A}_{n-1})=0$ for all $n\geq 1$. We first claim that if $1\leq p_0, p_1\leq\infty$ and if $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ then

(2.10) $$D_p=(D_{p_0},D_{p_1})_{\theta p}.$$

This is quite easy to check using a known argument. Indeed, to check this, we first note that by (1.1) we have

(2.11) $$(L_{p_0}(l_{p_0}),L_{p_1}(l_1))_{\theta p}=L_p(l_p),$$

with equivalent norms.

We can clearly identify isometrically $L_p(l_p)$ and $l_p(L_p)$. There is a projection $P: L_p(l_p) \to D_p$ defined by

$$\forall X = (X_n)_{n \geqq 0} \in L_p(l_p) \qquad P(X) = (\varphi_n)_{n \geqq 0}$$

with

$$\varphi_0 = \mathbb{E}(X_0 | \mathscr{A}_0) \quad \text{and} \quad \varphi_n = \mathbb{E}(X_n | \mathscr{A}_n) - \mathbb{E}(X_n | \mathscr{A}_{n-1}).$$

Clearly, P is a bounded projection onto D_p and

$$\|P(X)\|_{D_p} \leqq 2 \|X\|_{L_p(l_p)},$$

and consequently $\|P(X)\|_{(D_{p_0}, D_{p_1})_{\theta p}} \leqq 2 \|X\|_{(L_{p_0}(l_{p_0}), L_{p_1}(l_{p_1}))_{\theta p}}$. By (2.11), this implies that for some constant $C = C(p_0, p_1, \theta)$

$$\|P(X)\|_{(D_{p_0}, D_{p_1})_{\theta p}} \leqq C \|X\|_{L_p(l_p)}.$$

Applying this for X in D_p, we find

$$\|X\|_{(D_{p_0}, D_{p_1})_{\theta p}} \leqq C \|X\|_{D_p}. \tag{2.12}$$

On the other hand, we have trivially

$$\|X\|_{L_{p_i}(l_{p_i})} \leqq \|X\|_{D_{p_i}} \quad \text{for } i = 0, 1$$

hence by interpolation

$$\|X\|_{L_p(l_p)} \leqq C' \|X\|_{(D_{p_0}, D_{p_1})_{\theta p}} \tag{2.13}$$

for some constant $C' = C'(p_0, p_1, \theta)$.

Combining (2.12) and (2.13), we find the above claim (2.10).

We can now complete the proof of Theorem 2.1 (i).

Let us denote by T the operator which associates to any φ in D_1 the martingale $(M_n)_{n \geqq 0}$ defined by $M_n = \sum_{i \leqq n} \varphi_i$. Clearly $\|T(\varphi)\|_{L_1(v_1)} \leqq \|\varphi\|_{D_1}$. On the other hand, Lemma 2.2 implies that T is bounded from D_2 into $B_1 = (L_1(v_1), L_\infty(l_\infty))_{\frac{1}{2}\infty}$, with norm $\leqq 2$. Therefore if $1 < p < 2$ the interpolation theorem (1.2) implies that T is bounded from $(D_1, D_2)_{\theta p}$ into $(L_1(v_1), B_1)_{\theta p}$. By the reiteration theorem (cf. (1.4) above) we have $(L_1(v_1), B_1)_{\theta p} = (L_1(v_1), L_\infty(l_\infty))_{\delta p}$ with $\delta = \theta/2$. Now if θ is chosen so that $\frac{1}{p} = 1 - \delta$, we have by (1.1) and (1.9)

$$(L_1(v_1), L_\infty(l_\infty))_{\delta p} = L_p(A_{\delta p}) \subset L_p(v_p).$$

On the other hand, by (2.10) we have $\left(\text{since } \frac{1-\theta}{1} + \frac{\theta}{2} = \frac{1}{p}\right) (D_1, D_2)_{\theta p} = \mathrm{D}_p$. Recapitulating, we find a constant $C = C(p)$ depending only on $1 \leqq p < 2$ such that for all φ in D_p we have

$$\|T(\varphi)\|_{L_p(v_p)} \leqq C \|\varphi\|_{D_p}.$$

This establishes the first part of Theorem 2.1.

The second part follows from the standard arguments used to prove the Burkholder-Davis-Gundy inequalities. We skip the details and refer the reader instead to [Bu], [BDG], [G] or [LLP].

The next result is an immediate consequence of Theorem 2.1.

Corollary 2.3. *Let $1 \leqq p < 2$. Let $M = (M_t)_{t \geqq 0}$ be a martingale in L_p. Assume that the paths of M are right continuous and admit left limits and that the continuous part of M is 0. Let*

$$W_p(M) = \sup_{0 = t_0 \leqq t_1 \leqq \ldots} (|M_0|^p + \sum_{i \geqq 1} |M_{t_i} - M_{t_{i-1}}|^p)^{1/p}$$

and

$$S_p(M) = (\sum_{t \in [0, \infty[} |M_t - M_{t^-}|^p)^{1/p}.$$

Then, for all $1 \leqq r < \infty$, we have for any martingale M in L_r

$$\|W_p(M)\|_r \leqq C_{pr} \|S_p(M)\|_r. \tag{2.14}$$

Remark. There are also inequalities similar to Theorem 2.1 (ii) or (2.14) with a "moderate" Orlicz function space instead of L_r, cf. [Bu, BDG].

Our method gives (with almost no extra effort) a new proof of the following result of Lépingle [L].

Theorem 2.4. *Assume $2 < p < \infty$ and $1 \leqq r < \infty$. Then there is a constant C_{pr} such that every martingale $M = (M_n)_{n \geqq 0}$ in L_r satisfies*

$$\|W_p(M)\|_r \leqq C_{pr} \|\sup_n |M_n|\,\|_r.$$

Proof. We first consider the particular case $r = p$. With the above notation, consider the operator

$$S: L_\infty \to L_\infty(l_\infty)$$

defined for φ in L_∞ by $S(\varphi) = (\mathbb{E}(\varphi | \mathscr{A}_n))_{n \geqq 0}$. Clearly $\|S\| \leqq 1$. Let $B_0 = (L_1(v_1), L_\infty(l_\infty))_{\frac{1}{2}\infty}$. By Lemma 2.2, S is bounded from L_2 into B_0. By (1.2), S must be bounded from $(L_2, L_\infty)_{\theta p}$ into $(B_0, L_\infty(l_\infty))_{\theta p}$ $(0 < \theta < 1,\ 1 \leqq p \leqq \infty)$. Now assume that $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{\infty}$. Then, by (1.1), $(L_2, L_\infty)_{\theta p} = L_p$. Moreover, by (1.5) (reiteration) $(B_0, L_\infty(l_\infty))_{\theta p} = (L_1(v_1), L_\infty(l_\infty))_{\omega p}$ for $\omega = \frac{1-\theta}{2} + \theta = \frac{1+\theta}{2}$. Note that $\frac{1-\omega}{1} + \frac{\omega}{\infty} = \frac{1}{p}$, hence by (1.1), the last equality implies $(B_0, L_\infty(l_\infty))_{\theta p} = L_p(A_{\omega p})$. Recapitulating, we find that S is bounded from L_p into $L_p(A_{\omega p})$ with norm $\leqq C_1(p)$ for some constant $C_1(p)$ depending only on p.

Let $\varphi \in L_p$ and let $M_n = \mathbb{E}(\varphi | \mathscr{A}_n)$. Applying (1.9) again we conclude that

$$\|M\|_{L_p(v_p)} \leqq K(p) \|M\|_{L_p(A_{\omega p})} \leqq K(p) C_1(p) \|\varphi\|_p.$$

This proves Theorem 2.4 in the case $r=p$.

The general case follows from this by a standard argument (cf. [Bu]) and we leave the details to the reader.

Remark. Of course, there is also a version of Theorem 2.4 in the case of a continuous parameter martingale $(M_t)_{t>0}$.

Remark. One can easily derive from Theorem 2.1 and 2.4 the following analogous "almost sure" statements.

Let (M_n) be a martingale such that $\mathbb{E}\sup_{n\geq 1}|M_n-M_{n-1}|<\infty$.

If $1\leqq p<2$, then $\{W_p(M)<\infty\}\overset{\text{a.s.}}{=}\{\sum_{n\geq 1}|M_n-M_{n-1}|^p<\infty\}$.

Moreover if $2<p<\infty$, then $\{W_p(M)<\infty\}\overset{\text{a.s.}}{=}\{\sup|M_n|<\infty\}$.

Indeed, this follows directly from Theorems 2.1 and 2.4 and a classical stopping time argument.

Remark. The almost sure finiteness of the strong p-variation has been also studied for a general Gaussian process in [JM] and also in [PX]. However, no simple necessary and sufficient condition is known in that case.

Note that we prove a little bit more than is stated in Theorems 2.1 and 2.4, namely we prove all these results with $A_{\theta p}$ $(\theta=1-1/p)$ instead of v_p. The space $A_{\theta p}$ is studied in detail in [PX], using some ideas from [BP]. To describe this space, we introduce more notation.

Let B be a Banach space. For each integer k and each x in $B^{\mathbb{N}}$, let

$$W_p^k(x)=\sup\{(\|x_0\|^p+\sum_{1\leqq i\leqq k}\|x_{n_i}-x_{n_{i-1}}\|^p)^{1/p}\}$$

where the supremum runs over all k-tuples of integers $0=n_0\leqq n_1\leqq\ldots\leqq n_k$. Note the obvious identity $W_p(x)=\sup_k W_p^k(x)$. The next lemma is crucial in the paper [PX], it follows immediately from [BP].

Lemma 2.5. *Let B be a Banach space. Let $1\leqq p_0<\infty$. There is a constant $C>0$ such that for every x in $B^{\mathbb{N}}$, and every $k\geqq 1$*

$$\text{(2.15)}\quad \text{if } t^{p_0}\in[k,k+1]\qquad C^{-1}W_{p_0}^k(x)\leqq K_t(x;v_{p_0}(B),l_\infty(B))\leqq C\,W_{p_0}^k(x).$$

Moreover, we have for $\theta_0=1-1/p_0$,

$$\text{(2.16)}\qquad (v_1(B),l_\infty(B))_{\theta_0 p_0}\subset v_{p_0}(B)\subset(v_1(B),l_\infty(B))_{\theta_0\infty}.$$

Note. The inequality (2.15) can be proved similarly as the above Lemma 2.2. Recall the notation $A_{\theta q}(B)=(v_1(B),l_\infty(B))_{\theta q}$.

By the reiteration theorem (1.5), (2.16) implies that if

$$\text{(2.17)}\qquad \theta_0=1-1/p_0,\quad \theta_0<\theta<1,\quad 1\leqq q\leqq\infty$$

then $$A_{\theta q}(B)=(v_{p_0}(B), l_\infty(B))_{\eta q} \quad \text{with} \quad 1-\theta=(1-\eta)/p_0.$$

This implies, using (2.15), that the following is an equivalent norm on $A_{\theta q}(B)$

$$(\sum_{k\geq 1}(W_{p_0}^k(x))^q k^{-1-\eta q/p_0})^{1/q}. \tag{2.18}$$

In particular, if $\frac{1}{p}=1-\theta$, we find (for $1\leq p_0<p$)

$$\|x\|_{A_{\theta p}(B)}\approx(\sum_{k\geq 1}(k^{-1/p_0}W_{p_0}^k(x))^p)^{1/p}.$$

Finally, let us state a result from [PX] which we use in the next section (cf. [PX] Theorem 3 and Lemma 10).

Theorem 2.6. *Let B be a Banach space, $0<\theta<1$, $1\leq q\leq\infty$.*

Let $A_{\theta q}(B)=(v_1(B), l_\infty(B)_{\theta q}$ as above.

Let $p_\theta=(1-\theta)^{-1}$ $\left(\text{so that } \frac{1}{p_\theta}=\frac{1-\theta}{1}+\frac{\theta}{\infty}\right)$.

Let (Ω, μ) be any measure space. Let us denote simply $L_p(B)$ instead of $L_p(\Omega, \mu; B)$. Then:

(i) *If $p<p_\theta$ and $p\leq q$, we have*

$$L_p(A_{\theta q}(B))\subset A_{\theta q}(L_p(B)). \tag{2.19}$$

(ii) *If $r>p_\theta$ and $r\geq q$, we have*

$$A_{\theta q}(L_r(B))\subset L_r(A_{\theta q}(B)). \tag{2.20}$$

(iii) *If $p_\theta<t\leq\infty$ and $s=\inf(p_\theta, q)$, we have*

$$l_s(A_{\theta q}(B))\subset A_{\theta q}(l_t(B)). \tag{2.21}$$

Moreover, all these inclusions are bounded operators.

Remark. Note that (2.19) follows easily from (2.18). Then (2.20) follows by duality. Indeed, it is proved in [PX] that if $0<\theta<1$, $1\leq q<\infty$, $\frac{1}{q}+\frac{1}{q'}=1$, the dual of $A_{\theta q}(B)$ can be identified with $A_{1-\theta q'}(B^*)$ in a natural way. We refer to [PX] for more details on the spaces $A_{\theta q}(B)$.

3. Orthonormal Series

We will obtain here a generalization of the Bretagnolle result in a different direction. Let $\alpha=(\alpha_n)_{n\geq 0}$ be in l_2 and let (φ_n) be an orthonormal sequence of functions in $L_2(\Omega, \mu)$. (Here (Ω, μ) is an arbitrary measure space). Let $S_n=\sum_{i\leq n}\alpha_i\varphi_i$ and let $S=(S_n)_{n\geq 0}$.

Theorem 3.1. *With this notation, if $1 \leqq p < 2$, there is a constant K'_p (depending only on p) such that for any α in l_p*

$$\left(\int W_p(S)^2\, d\mu\right)^{1/2} \leqq K'_p (\Sigma |\alpha_n|^p)^{1/p}.$$

Remark. This applies in particular to Fourier series. If $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^p < \infty$ $(1 < p < 2)$ and if $S_n = \Sigma_0^n \hat{f}(k) e^{ikt}$ then $S(t) = (S_n(t))_{n \geqq 0}$ belongs to v_p for a.e. t and $W_p(S)$ is in $L_2(dt)$ (and similarly, of course, with n running over the negative integers).

We do not know a reference even for the trigonometric case, although we feel that this should be known. Note however that (at least intuitively) the case of $\varphi_n = e^{int}$ contains the Bretagnolle result: The latter corresponds roughly to the very particular case of a lacunary sequence such as $\varphi_n(t) = e^{i 2^n t}$. The above Theorem 3.1 is somewhat implicit in our previous work [PX], but is not stated there.

Proof of Theorem 3.1. We will use Theorem 2.6 with $1 < p_\theta < 2$. Let $p = p_\theta$ and $0 < \theta < \frac{1}{2}$.

By (2.21) with $s = p = q$ and $t = 2$, we have

$$l_p(A_{\theta p}) \subset A_{\theta p}(l_2) \tag{3.1}$$

Now, let $T\colon l_2 \to L_2$ be the operator defined by

$$\forall \alpha \in l_2 \qquad T(\alpha) = \sum_1^\infty \alpha_i \varphi_i.$$

Clearly $\|T\| = 1$. This operator T induces an operator $\tilde{T}$ from $(l_2)^{\mathbb{N}}$ into $(L_2)^{\mathbb{N}}$ simply by setting

$$\forall x = (x_n)_{n \geqq 0} \in (l_2)^{\mathbb{N}} \qquad \tilde{T}(x) = (T(x_n))_{n \geqq 0}.$$

Obviously, $\tilde{T}$ is bounded from $l_\infty(l_2)$ into $l_\infty(L_2)$, but also from $v_1(l_2)$ into $v_1(L_2)$. By the interpolation theorem, $\tilde{T}$ is bounded from $A_{\theta p}(l_2)$ into $A_{\theta p}(L_2)$. But by (2.20), $A_{\theta p}(L_2) \subset L_2(A_{\theta p})$. Therefore, by (3.1) there is a constant C such that

$$\forall x = (x_n)_{n \geqq 0} \in (A_{\theta p})^{\mathbb{N}} \qquad \|\tilde{T}(x)\|_{L_2(A_{\theta p})} \leqq C \|x\|_{l_p(A_{\theta p})}. \tag{3.2}$$

Here $\tilde{T}(x)$ is identical to the function $t \to \Sigma \varphi_n(t) x_n$ considered as an element of $L_2(A_{\theta p})$.

Hence, by (1.9), we have

$$\|\Sigma \varphi_n(t) x_n\|_{L_2(v_p)} \leqq C'_p (\Sigma \|x_n\|^p_{A_{\theta p}})^{1/p} \tag{3.3}$$

with C'_p depending only on p.

(Note that by (3.2) if x is in $l_p(A_{\theta p})$, the series $\Sigma \varphi_n x_n$ converges in $L_2(A_{\theta p})$ and a fortiori in $L_2(v_p)$.) Now let (e_n) be the canonical basis of $\mathbb{R}^{\mathbb{N}}$. Let $\delta_i = \sum_{n \geqq i} e_n$. We take $x_i = \alpha_i \delta_i$. Then $\tilde{T}(x) = \Sigma \alpha_i \delta_i \varphi_i = \Sigma e_n S_n$.

Note that $\{\delta_i\}$ is bounded in v_1, hence, a fortiori, it is bounded in $A_{\theta p}$. Therefore (3.3) implies

$$(\int \|\{S_n\}\|_{v_p}^2 \, d\mu)^{1/2} \leqq K'_p(\Sigma |\alpha_i|^p)^{1/p}$$

for some constant K'_p. q.e.d.

4. The Banach Space Valued Case

Let $1 \leqq p \leqq 2$.

We will say that a Banach space B is p-smoothable if there is an equivalent norm $|||\ |||$ on B and a constant C such that

$$\forall x, y \in B \qquad 2^{-1}(|||x+y|||^p + |||x-y|||^p) \leqq |||x|||^p + C|||y|||^p.$$

It is known that this holds iff the modulus of smoothness $\rho(t)$ of $(B, |||\ |||)$ is $O(t^p)$ when $t \to 0$. Moreover, it is known that B is p-smoothable iff there is a constant C such that all B-valued martingales $(M_n)_{n \geqq 0}$ satisfy

$$\sup_{n \geqq 0} \mathbb{E}\|M_n\|^p \leqq C(\mathbb{E}\|M_o\|^p + \sum_{n \geqq 1} \mathbb{E}\|M_n - M_{n-1}\|^p). \tag{4.1}$$

We refer the reader to [P2] for more details.

The Banach space version of Theorem 2.1 is the following.

Theorem 4.1. *Let $1 < p_0 \leqq 2$. Assume that B is a p_0-smoothable Banach space. Then for each $1 < p < p_0$ and each $1 \leqq r < \infty$ there is a constant C_{pr} such that all B-valued martingales $M = (M_n)_{n \geqq 0}$ in $L_r(B)$ satisfy*

$$\|W_p(M)\|_r \leqq C_{pr} \|(\|M_0\|^p + \Sigma \|M_n - M_{n-1}\|^p)^{1/p}\|_r. \tag{4.2}$$

Proof. As earlier, it is enough to check this for $r = p$. Note that if $T_0 \leqq T_1 \leqq \ldots$ is an increasing sequence of stopping times and if $M_n = \mathbb{E}(M_\infty | \mathscr{A}_n)$ with M_∞ in $Lp_0(B)$ then the B-valued version of (2.3) is simply

$$\mathbb{E}\|M_{T_k} - M_{T_{k-1}}\|^{p_0} \leqq C\mathbb{E} \sum_{T_{k-1} < n \leqq T_k} \|M_n - M_{n-1}\|^{p_0}. \tag{4.3}$$

Indeed this immediately follows from (4.1) (taking $p = p_0$) applied to the martingales $(M_{T_k \wedge n} - M_{T_{k-1} \wedge n})$.

Once (4.3) is clear, the rest of the proof of Theorem 2.1 remains valid with routine changes.

Let $2 \leqq q < \infty$.

We will say that a Banach space B is q-convexifiable if there is an equivalent norm $|||\ |||$ on B a constant $C > 0$ such that

$$\forall x, y \in B \qquad |||x|||^q + C|||y|||^q \leqq 2^{-1}(|||x+y|||^q + |||x-y|||^q).$$

It is known that this holds iff the modulus of uniform convexity $\delta(\varepsilon)$ of $(B, |||\ |||)$ satisfies a lower estimate $\delta(\varepsilon) \geqq K\varepsilon^q$ for some $K>0$. Moreover, it is known that B is q-convexifiable iff there is a constant C such that all B-valued martingales $(M_n)_{n\geqq 0}$ satisfy

$$\mathbb{E}\|M_o\|^q + \sum_{n\geqq 1} \mathbb{E}\|M_n - M_{n-1}\|^q \leqq C \sup_{n\geqq 0} \mathbb{E}\|M_n\|^q. \tag{4.4}$$

We refer again to [P2] for more information. It is rather easy to show that B is p-smoothable iff B^* is q-convexifiable for $\frac{1}{p}+\frac{1}{q}=1$. The B-valued version of Theorem 2.4 is as follows.

Theorem 4.2. *Let $2 \leqq q_0 < \infty$ and $1 \leqq r < \infty$.*

Let B be a q_0-convexifiable Banach space. Assume $q_0 < p < \infty$. Then there is a constant C_{pr} such that all B-valued martingales $(M_n)_{n\geqq 0}$ in $L_r(B)$ satisfy

$$\|W_p(M)\|_r \leqq C_{pr} \|\sup_{n\geqq 0} \|M_n\|\,\|_r. \tag{4.5}$$

Proof. Let (T_k) be as above. Then (4.4) for $q=q_0$ implies

$$\sum_{k\geqq 1} \mathbb{E}\|M_{T_k} - M_{T_{k-1}}\|^{q_0} \leqq C \sup_{n\geqq 0} \mathbb{E}\|M_n\|^{q_0}$$

Using this inequality, the argument for Theorem 2.4 remains valid with routine changes. We leave the details to the reader.

In the Banach space case, it is known that martingale inequalities such as (4.1) or (4.4) are stronger than the same inequalities restricted to sums of independent mean zero variables. The latter sums are better understood within the framework of the notions of type and cotype. Let (ε_n) be an i.i.d. sequence of r.v.'s on $(\Omega, \mathscr{A}, \mathbb{P})$ with $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$.

Let $1 \leqq p \leqq 2 \leqq q < \infty$.

A Banach space B is called of type p (resp. cotype q) if there is a constant C such that for all finite sequences (x_i) in B we have

$$\|\sum \varepsilon_i x_i\|_{L_2(B)} \leqq C (\sum \|x_i\|^p)^{1/p}$$

(resp. $(\sum \|x_i\|^q)^{1/q} \leqq C \|\sum \varepsilon_i x_i\|_{L_2(B)}$).

Equivalently, this holds iff there is a constant C such that all B-valued martingales $M = (M_n)_{n\geqq 0}$ with independent increments (i.e., $(M_n - M_{n-1})_{n\geqq 0}$ is an independent sequence) satisfy (4.1) or (4.2) (resp. (4.4)). We refer the reader to [MP], [HJP], [P1] for more information on these notions.

We state next a result already observed in [PX].

Theorem 4.3. *Let* $1 < p_0 \leqq 2$ (resp. $2 \leqq p_0 < \infty$). *Assume that B is of type p_0* (resp. *cotype q_0*) *and let* $1 < p < p_0$ (resp. $q_0 < q < \infty$). *Then for all $1 \leqq r < \infty$ there is* *a constant C_{pr} such that all B-valued martingales $(M_n)_{n\geqq 0}$ with independent increments satisfy* (4.2) (resp. (4.5)). (*Moreover, if the increments are independent and symmetric, the result is valid also when* $0<r<1$).

Proof. Let $x=(x_n)_{n\geqq 0}$ be in $l_{p_0}(B)$, let $S_n(x)=\Sigma_1^n \varepsilon_i x_i$ and let $S(x)=(S_n(x))_{n\geqq 0}$.

Let $1<p<p_0$, $\frac{1}{p}=\frac{1-\theta}{1}+\frac{\theta}{p_0}$, $0<\theta<1$, and let $0<r<\infty$. We first claim that S is a bounded linear operator from $l_p(B)$ into $L_r(v_p(B))$.

To check this, we may assume (without loss of generality) that $r\geqq 2$. Then, S is bounded from $l_{p_0}(B)$ into $v_{p_0}(L_r(B))$ (since B is of type p_0) and from $l_1(B)$ into $v_1(L_r(B))$ (trivially). Hence, by (1.1) and (1.2), S is bounded from $l_p(B)$ into $(v_1(L_r(B)), v_{p_0}(L_r(B)))_{\theta p}$. By (2.16) and the reiteration theorem (1.4), the latter space coincides with $A_{\delta p}(L_r(B))$ for $\delta=1-1/p$. Moreover, by (2.20) and (1.9) $A_{\delta p}(L_r(B))\subset L_r(A_{\delta p}(B))\subset L_r(v_p(B))$ (since $r\geqq 2$).

Recapitulating, we find that S is bounded from $l_p(B)$ into $L_r(v_p(B))$, as announced above.

Let $(d_n)_{n\geqq 0}$ be a sequence of independent and symmetric B-valued r.v.'s. By the preceding claim, we have for any fixed ω'

$$\| W_p(\{\Sigma_1^n \varepsilon_i(\omega)\, d_i(\omega')\})\|_{L^r(d\mathbb{P}(\omega))} \leqq C(\Sigma \|d_n(\omega')\|^p)^{1/p}$$

for some constant C.

Taking the L_r-norm with respect to ω' of both sides of this inequality and observing that (d_i) and $(\varepsilon_i(\omega)\, d_i(\omega'))$ have the same distribution, we obtain

$$\| W_p(\{\Sigma_1^n d_i\})\|_r \leqq C\, \|(\Sigma \|d_n\|^p)^{1/p}\|_r. \tag{4.6}$$

When d_i is only assumed mean zero (instead of symmetric), we can prove (4.6) for $1\leqq r<\infty$ by a classical symmetrization argument. Thus, in the case of type, the proof is complete.

Now assume that B is of cotype q_0.

Let $0<r<\infty$ and let $x=(x_n)_{n\geqq 0}$ be a sequence in B such that $S_n(x)=\Sigma_1^n \varepsilon_i x_i$ converges in $L_r(B)$.

We will use a result of Kahane which ensures that all the L_r-norms are equivalent on sums of the form S_n (cf. [LT] p. 74).

In particular this implies that there is a constant C such that

$$\|S(x)\|_{v_{q_0}(L_r(B))} \leqq C\, \|\Sigma_1^\infty \varepsilon_n x_n\|_{L_r(B)}.$$

(Note that this is obvious for $r=q_0$, hence by Kahane's result it is true is general).

Now if $q>q_0$ and $\theta=1-1/q$, then (by Lemma 2.5) $v_{q_0}(L_r(B))\subset A_{\theta q}(L_r(B))$ hence if $r>q$ by (2.20) and (1.9) we have

$$\|S(x)\|_{L_r(v_q(B))} \leqq C'\, \|\Sigma_1^\infty \varepsilon_n x_n\|_{L_r(B)}$$

for some constant C'.

By Kahane's result this must be valid for all $0<r<\infty$. We then extend this inequality easily for sums of independent symmetric (or mean zero) r.v.'s by the same argument as above. q.e.d.

We can give also a Banach space version of the results of Sect. 3. We will say that a Banach space B is of H-type p if there is a constant C such that for all orthonormal sequences (φ_n) in L_2 and all finite sequences (x_n) in B we have

$$\|\sum \varphi_n x_n\|_{L_2(B)} \leqq C(\sum \|x_n\|^p)^{1/p}.$$

We can then state:

Theorem 4.4. *Let $1<p_0\leqq 2$. Assume that B is of H-type p_0. Then for $1\leqq p<p_0$ there is a constant C such that for all (φ_n) as above and all sequences (x_n) in B, the partial sums $S_n=\sum_{k=1}^{n} \varphi_k x_k$ satisfy the inequality*

$$\|W_p(\{S_n\})\|_2 \leqq C(\Sigma \|x_n\|^p)^{1/p}.$$

This can be proved exactly as we did in the preceding proof with (ε_n) in the place of (φ_n).

Remarks. (i) A Banach space is of H-type 2 iff it is isomorphic to a Hilbert space (cf. [K]).

(ii) The notion of H-type p can be connected easily to the geometry of Banach spaces. Indeed, for any n dimensional spaces E, F let

$$d(E,F)=\inf\{\|T\|\,\|T^{-1}\|\,|\,T\colon E\to F \text{ invertible}\}.$$

For an infinite dimensional Banach space X, let

$$d_n(X)=\sup\{d(E,l_2^n)\,|\,E\subset X \dim E=n\}.$$

Then if X is of H-type p, there is a $K>0$ such that $d_n(X)\leqq K n^{\frac{1}{p}-\frac{1}{2}}$ for all n (This follows from a result of [TJ]). Conversely, it is rather easy to prove that if $d_n(X)\leqq K n^{\frac{1}{p}-\frac{1}{2}}$ for all n, then X is of H-type p_1 for all $p_1<p$. Thus

$$\sup\{p\,|\,X \text{ is } H \text{ type } p\}=\left(\limsup_{n\to\infty} \frac{\operatorname{Log} d_n(X)}{\operatorname{Log} n}+\frac{1}{2}\right)^{-1}.$$

(iii) The above inequality (3.2) shows that if $1<p<2$ and $\theta=1-1/p$ then $A_{\theta p}$ is of H-type p. (In fact, every space of cotype 2 and of type p is of H-type p).

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