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Probability Theory and Related Fields

Construction of Right Processes from Excursions

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Summary. In a previous work, the author obtained the strong Markov property of a process from conditions on its excursion process. The Ray and right properties are obtained here under similar conditions, using the Ray-Knight compactification.

1. Introduction

Let $(X_t)_{t \geqq 0}$ be a right process, with a point a which is both recurrent and regular. Let P_0^b be the law of (X_t) started at b and stopped upon hitting a. Then (X_t) has a local time (L_t) at a. Using (L_t), K. Itô showed in [8] how to obtain the *PPP* (Y_t) of excursions of (X_t) away from a. Its law is determined by its characteristic measure n. Itô obtained a list of conditions (involving (P_0^b)) that n must obey.

In [13], the author showed that under slightly modified conditions, one can reverse the procedure. That is, given (P_0^b) and n satisfying the new conditions, together with any *PPP* (Y_t) having n as its characteristic measure, one can construct a right continuous strong Markov process (X_t) having (Y_t) as its excursion process. Our object here is to show that if (P_0^b) are the transition laws of a right process or a Ray process, then (X_t) inherits the same property. Moreover, in this case, the proof of the strong Markov property of (X_t) can be simplified. We will thus actually reprove some of the results of [13], under these new hypotheses. As well, we will treat the simpler case, that a is recurrent but irregular.

Section 2 deals with the background to the problem. Section 3 covers some preliminary results, and is followed by the main argument in Sect. 4. Section 5 includes the results on the Ray property, together with an application to 'skew Brownian motion'.

2. Definitions

The following will be our standing hypotheses: E is a universally measurable subset of some compact metric space, $(\Omega, \mathscr{F}, \mathscr{F}_t^0)$ is a filtered measurable space, and (X_t^0) is a right continuous process with values in E, which is adapted to $\mathscr{F}_t^0$). The $\hat{P}^b$ are probabilities on $(\Omega, \mathscr{F})$, universally measurable in $b \in E$. As usual, we assume that

$$\mathscr{F}_{t+}^0 = \mathscr{F}_t^0 = \bigcap_\mu \mathscr{F}_t^{0,\mu}$$

where $\mathscr{F}_t^{0,\mu}$ is obtained by adding to $\mathscr{F}_t^0$ all the null sets in the $\hat{P}^\mu$-completion of $\mathscr{F}$.

We assume that (X_t^0) is a *right process*, that is; it is strong Markov (with respect to $(\mathscr{F}_t^0)$ and $(\hat{P}^b)$), $X_0^0 = b$ $\hat{P}^b$-a.s. for each b, and

(2.1) for each α-excessive function f, $f(X.)$ is $\hat{P}^\mu$-indistinguishable from an $(\mathscr{F}_t^{0,\mu})$-optional process.

[For example, this latter condition is well known to hold if $\hat{P}^b$ is Borel measurable in b].

We will assume that (X_t^0) is recurrent at a holding point a. That is, writing

$$\sigma_a = \inf\{t > 0;\ X_t^0 = a\}$$
$$(\text{and also } \tau_a = \inf\{t \geqq 0;\ X_t^0 = a\}),$$

we have that

$$\hat{P}^b(\sigma_a < \infty) = 1 \text{ for every } b,$$

and

$$\hat{P}^a(X_t^0 = a \text{ for every } t) = 1.$$

Write d for the given metric on E. Define a second metric d' by

$$d'(x, y) = \begin{cases} 1 + d(x, y), & x = a \neq y \quad \text{or} \quad y = a \neq x \\ d(x, y) & \text{otherwise.} \end{cases}$$

Notice that (X_t^0) is still a right process, in the d'-topology. Form the Ray-Knight compactification $(\bar{E}, \rho)$ of (E, d'). Thus $(\bar{E}, \rho)$ is a compact metric space containing E as a universally measurable subset, and (X_t^0) is a Ray process on $\bar{E}$.

The reason for introducing d' is that we obtain the following fact, which will be useful later; for every $\alpha > 0$,

(2.2) $x \to \hat{E}^x[e^{-\alpha\sigma_a}]$ extends to a ρ-continuous function on $\bar{E}$ (it is the α-potential of the d'-uniformly continuous function $1_{E \setminus \{a\}}$, and hence lies in the Ray cone; see Getoor [5]). (Alternatively, we could have produced $\bar{E}$ by killing X_t^0 at σ_a, deleting a from E [as it has become a branch point], forming the Ray-Knight compactification of the killed process, and then sewing a back in as the (unique) branch point to the cemetary (or as an isolated point if no such branch points exist]).

In the following, "open" or "right continuous" will always mean "d-open", or "d-right continuous". Let U be the set of right continuous u: $[0, \infty) \to E$ which are ρ-right continuous on $(0, \infty)$. Let (W_t) be the coordinate process on U. Writing $\mathscr{E}$ for the universally measurable σ-field on E, let $\mathscr{U}_t (0 \leqq t \leqq \infty)$ be the universal completion of

$$\bigvee_{\substack{s \leq t \\ s < \infty}} W_s^{-1}(\mathscr{E}).$$

Write $\mathscr{U}$ for $\mathscr{U}_\infty$. Let P_0^b be the $\hat{P}^b$-law of (X_t^0) on $(U, \mathscr{U})$. (Note that our notation is different from that used in [13]. There, the definition of U involved only right contihuity with respect to d, not ρ. Our present choice is dictated by the following technical consideration; if (X_t') is any right continuous strong Markov process with the same laws (P_0^b), then it too will be a right process [that is, satisfy (2.1)] iff it is a.s. ρ-right continuous. It is still an open question[1], whether this extra condition is necessary (see Sharpe [15]). However, by our choice of U, we obtain immediately that

(2.3) $\qquad (W_t, P_0^b)$ is a right process.)

We will use the notation σ_a, τ_a for the appropriate $(\mathscr{U}_t)$ stopping times (on U, rather than Ω). We will make use of the shift operators θ_t on U as well.

If (X_t) is any right process agreeing with (X_t^0) for $t \leqq \sigma_a$, K. Itô showed in [8], how to obtain the excursion process (Y_t) of (X_t) from a. If (X_t) visits a at a discrete set of times, then Y_k, $(k \geqq 1)$ is the U-valued random variable consisting of the k-th excursion away from a. If, on the other hand, a is regular for itself, then there is a canonical local time at a (the CAF with 1-potential

(2.4) $\qquad \hat{E}^x[e^{-\sigma_a}])$,

and Y_t is defined to be the excursion starting when the local time is t. Thus $Y_t \in U$ for countably many t, a.s. (as usual, 'a.s.' means $\hat{P}^\mu$-a.s., for every μ), and otherwise equals some auxilliary point δ. In this case, (Y_t) is a Poisson point process (PPP). We write n for the common law of the Y_k in the first case, and the characteristic measure of the PPP in the second. The law of the extension (X_t) is characterized by n.

The reverse of this procedure was considered in [13]; that is, one wishes to start with a PPP (or a sequence of random variables) with characteristic measure (law) n, and to find conditions on n such that the resulting process (X_t) is a right process.

Itô obtained six necessary conditions on n ((i)–(vi) below). In [13], these conditions were shown to be not quite sufficient. The addition of two slightly different necessary conditions [(ii'), (vi') below] was however shown to yield the strong Markov property and the right continuity of the reconstructed process (X_t). In the present paper, we will consider the remaining of the 'hypothèses droites', namely condition (2.1). We will show that under the new conditions, (2.1) must in fact hold (Theorem (4.1)). We will translate (vi') into a statement about the Ray-Knight compactification $\bar{E}$ (Proposition (3.5)) and will discuss what else can be said when (X_t^0) is a Ray process (Corollaries (5.2), (5.3); the latter shows that (X_t) is Ray as well).

[1] No longer; in a paper to appear in Ann. Probab., the present author shows that the condition may not be dispensed with

The arguments of [13] applied to certain processes (X_t^0) not satisfying (2.1). These arguments are in some sense the correct ones, as they generalize to the situation where excursions away from a set (rather than the single point a) are considered (see [14]). However, in the present context, it turns out that the Ray-Knight compactification may be used to advantage, giving a shorter proof of the strong Markov property of (X_t). Since the present assumptions should meet the needs of all but specialists, we shall reprove the key parts of the results of [13]. We will be able to do this fairly concisely by dealing only with the essentials, leaving many technicalities (measurability arguments...) to [13].

This approach to the construction of strong Markov processes has a long history (e.g., Ikeda Watanabe [7]). See [13] for a discussion. We should however mention Kabbaj [10], and Getoor and Sharpe [6], where similar techniques to ours appear. Even more closely related are Blumenthal [1] and Rogers [11]. The latter essentially obtains our Corollary (5.3), by resolvant techniques. The former obtains a similar result when $E=[0,\infty)$, $a=0$. It should be emphasized that the general insufficiency of conditions (ii) and (vi) does not present a problem in this case; by Corollary (5.2), there are no 'counterexamples' coming from the class of Ray processes.

This work formed part of the author's Ph.D. dissertation [12], under the supervision of J.B. Walsh, to whom many thanks are due.

3. Conditions on n

In what follows, n is a positive measure on $(U,\mathscr{U})$. Itô's conditions are:

(i) n is concentrated on $\{u;0<\sigma_a(u)<\infty, u(t)=a \text{ for } t\geqq\sigma_a(u)\}$.

(ii) $n\{u;u(0)\notin V\}<\infty$ for every open neighborhood V of a.

(iii) $\int(1-e^{-\sigma_a})\,dn\leqq 1$.

(iv) $n\{u;\sigma_a(u)>t, u\in\Lambda, \theta_t(u)\in M\}=\int_{\Lambda\cap\{\sigma_a>t\}} P_0^{u(t)}(M)\,n(du)$ for $t>0$, $\Lambda\in\mathscr{U}_t$, $M\in\mathscr{U}$.

(v) $n\{u;u(0)\in B, u\in M\}=\int_{\{u;u(0)\in B\}} P_0^{u(0)}(M)\,n(du)$ for $M\in\mathscr{U}$, and $B\in\mathscr{E}$ such that $a\notin B$.

(vi) Either (a) n is a probability measure concentrated on $U^a=\{u;u(0)=a\}$ (discrete visiting case);
or (b) n is finite, $n(U^a)=0$, and $\int(1-\exp(-\sigma_a))\,dn<1$ (exponential holding case);
or (c) n is infinite and $n(U^a)=0$ or ∞ (instantaneous case).

The conditions introduced in [13] are:

(ii′) $n\{u;u \text{ leaves } V\}<\infty$ for every open neighborhood V of a.

(vi′) Either (a) n is a probability measure concentrated on U^a. If $n\geqq n'\geqq 0$ and n' satisfies (iv), then n' is a multiple of n;
or (b) as in (vi)(b);
or (c) n is infinite. If $n\geqq n'\geqq 0$ and n' satisfies (iv), then $n'(U^a)=0$ or ∞.

We will also make use of the following conditions:

(vi″a) There is a point $c \in \bar{E}$ such that

$$n(W_0 = a, \rho - \lim_{t \downarrow 0} W_t \neq c) = 0$$

(vi″c) $n(W_0 = a, \rho - \lim_{t \downarrow 0} W_t \neq a) = 0.$

We start with a technical lemma:

(3.1) **Lemma.** *Let n satisfy* (i), (iv), *and* (v), *and suppose that $n(\sigma_a > \varepsilon) < \infty$ for every $\varepsilon > 0$.*

(a) *Let R be a $(\mathcal{U}_{t+})$ stopping time such that $n(U^a, R = 0) = 0$. Then the coordinate process $(W_t, \mathcal{U}_t, P_0^b)$ is strong Markov at R under n.*

(b) *$n((W_t)$ leaves $V) < \infty$ for every ρ-open neighborhood V of a.*

Proof. By (9.4) of Getoor [5], together with (iv) and (v), we have (for f bounded and measurable), that $U_0^\alpha f(W.)$ is n-a.s. right continuous on $(0, \infty)$, and is n-a.s. right continuous at 0 on $U \setminus U^a$ (here U_0^α is the resolvant of (P_0^b)). Thus, it is n-a.s. right continuous at R, so that (a) follows as in (8.11) of Blumenthal and Getoor [2]. To show (b), we have by (2.2) that

$$V_\varepsilon = \{b \in \bar{E}; E_0^b[1 - e^{-\sigma_a}] < \varepsilon\}, \qquad \varepsilon > 0$$

defines a nested family of ρ-open neighborhoods of a. Since the resolvant of (X_t^0) separates points of $\bar{E}$, we have that

$$P_0^b(U^a) = 0 \qquad \text{for } b \neq a, \tag{3.2}$$

hence their intersection is $\{a\}$. Thus if V' is any other ρ-open neighborhood of a,

$$\{V'\} \cup \{\bar{E} \setminus \bar{V}_\varepsilon; \varepsilon > 0\}$$

forms a ρ-open cover of $\bar{E}$. Since $\bar{E}$ is compact, this shows that the V_ε form a base of ρ-open neighborhoods of a. Thus, it suffices to show that

$$n((W_t) \text{ leaves } V_\varepsilon) < \infty$$

for each $\varepsilon > 0$. Fix $\varepsilon > 0$, and let

$$\tau(u) = \inf\{t \geqq 0; u(t) \notin V_\varepsilon\}.$$

Then $W_\tau \notin V_\varepsilon$ on $\{\tau < \infty\}$, since V_ε is ρ-open. Also, for $b \in E \setminus V_\varepsilon$ and $v > 0$ we have that

$$\varepsilon \leqq E_0^b(1 - e^{-\sigma_a}) \leqq 1 - e^{-v} + P_0^b(\sigma_a \geqq v),$$

so that by part (a),

$$\begin{aligned} \infty > n(\sigma_a \geqq v) &\geqq n(\tau < \infty, \sigma_a \geqq v + \tau) \\ &= \int_{\{\tau < \infty\}} P_0^{u(\tau(u))}(\sigma_a \geqq v) n(du) \\ &\geqq (\varepsilon - (1 - e^{-v})) n(\tau < \infty). \end{aligned}$$

Choosing v small, we obtain that $n(\tau < \infty) < \infty$. □

Next, we relate (vi″) to (vi′):

(3.3) **Proposition.** *Let n be as in Lemma* (3.1). *Then*

(a) $n(\rho-\lim_{t\downarrow 0} W_t$ *does not exist*$)=0$

(b) *Suppose that n is a probability measure concentrated on* U^a. *Then* (vi′a) *and* (vi″a) *are equivalent.*

(c) *Suppose that* $n(U^a)>0$. *Then* (vi′c) *and* (vi″c) *are equivalent.*

Proof. By (v),

$$n(W_0 \neq a, \rho-\lim_{t\downarrow 0} W_t \neq W_0)=0,$$

so that without loss of generality, we may assume that $n(U\setminus U^a)=0$.

(a) Let V be a ρ-open neighborhood of a and let

$$B=\{\text{for every } v>0,\ W_s\notin V \text{ for some } s\in(0,v)\}.$$

Then by Lemma (3.1)(b),

$$\mu_t(A)=n(W_t\in A, B)$$

forms a bounded system of entrance laws for (P_0^b), so that since (W_t) is Ray in the ρ-topology, there is a finite positive measure μ_0 on $\bar{E}$ with

$$\mu_t(A)=P_0^{\mu_0}(W_t\in A), \qquad t>0.$$

Thus

$$n|_B=P_0^{\mu_0} \quad \text{on} \quad \sigma(W_t; t>0,$$

so that

$$n(B, \rho-\lim_{t\downarrow 0} W_t \text{ does not exist})=0.$$

Now let $V=V_k$ range through a countable base of ρ-open neighborhoods of a. Then

$$U\setminus\bigcup_k B_k=\{\rho-\lim_{t\downarrow 0} W_t=a\},$$

showing (a).

(b) If (vi″a) holds for n, then whenever $n\geqq n'\geqq 0$, it also holds for n'. Thus if n' satisfies (iv) as well, we have that

$$n'=n'(U)P_0^c=n'(U)n.$$

Conversely, as above, there is a probability μ_0 on $\bar{E}$ such that

$$n=P_0^{\mu_0} \quad \text{on} \quad \sigma(W_t; t>0). \tag{3.4}$$

In fact, μ_0 is the distribution of

$$\rho-\lim_{t\downarrow 0} W_t,$$

hence (vi′a) shows that μ_0 takes on only the values 0 and 1. Thus it is concentrated on some point c, giving (vi″a).

(c) Assume condition (vi′c) holds, and let V_k be a countable base of ρ-open neighborhoods of a. Then

$$A_k = \{W_0 = a, W_{t_j} \notin V_k \text{ for some sequence } t_j \downarrow 0\} \in \mathscr{U}_{0+},$$

so that, taking n' to be n restricted to A_k we see that $n(A_k)=0$. Taking the union over k yields (vi″c). Conversely, assume (vi″c). If (vi′c) fails, then there is a finite nonzero measure n' concentrated on U^a, which satisfies (iv). Thus

$$\mu_t(A) = n'(W_t \in A)$$

defines a bounded system of entrance laws for (P_0^b).

As before, there is a μ_0 on $\bar{E}$ with

$$n' = P_0^{\mu_0} \quad \text{on} \quad \sigma(W_t; t>0).$$

By (vi″c), μ_0 is concentrated on a, which is absurd (recall that $n' \leqq n$, and n satisfies (i)). □

Now let (X_t) be a right process under the $\hat{P}^b$, with $X_t = X_t^0$ for $t \leqq \sigma_a$. Then a.s. (X_t) has an infinite lifetime, and in [8], K. Itô constructed its excursion measure n. He showed (in a slightly different context) that n satisfies (i)–(vi). In [13] it was shown that the stronger conditions (ii′), (vi′) hold as well. The main goal of this section will be to rederive this result (Proposition (3.5)). The simplification arises in that in the present situation, we have the Ray Knight compactification at our disposal.

(3.5) **Proposition.** *Let (X_t) be a right process under the $\hat{P}^b$, with $X_t = X_t^0$ for $t \leqq \tau_a$. Then the excursion measure n of (X_t) satisfies* (ii′) *and* (vi′).

Proof. Condition (ii′) clearly holds. We know, a priori, that (X_t) is ρ-right continuous at all times t with $X_t \neq a$, a.s. Thus Lemma (3.1) part (b), and Proposition (3.3)(a) show that

$$\bar{X}_t = \rho - \lim_{s \downarrow t} X_s$$

is well defined for every t, a.s.

Consider first the discrete visiting case. As in Proposition (3.3)(b), there is a probability μ_0 on $\bar{E}$ such that (3.4) holds. Thus

$$\hat{P}^a(\bar{X}_0 \in C) = \mu_0(C), \tag{3.6}$$

for C any measurable subset of $\bar{E}$. If μ_0 is not concentrated on some single point c, then there is a $C \subset \bar{E}$ with

$$0 < \mu_0(C) < 1.$$

Let

$$T = \inf\{t \geqq 0; X_t = a, \bar{X}_t \in C\}.$$

Then T is a.s. finite, and $X_T = a$, $\bar{X}_T \in C$ a.s. By (3.6), this contradicts the strong Markov property of (X_t) at T. Thus (vi″a) holds, and hence so does (vi′a).

Now suppose that a is regular for itself. In n is finite then Itô showed that (vi b) holds, hence we may assume both that n is infinite and that $n(U^a)>0$. If (vi″c) were to fail, we could use (b) of Lemma (3.1) to find a measurable $C\subset\bar{E}$ with $a\notin C$ and

$$0<n(U^a,\rho-\lim_{t\downarrow 0} W_t\in C)<\infty.$$

Thus

$$T=\inf\{t\geqq 0; X_t=a, \bar{X}_t\in C\}$$

is a.s. finite, and $X_T=a$, $\bar{X}_T\in C$ a.s. Therefore T is a stopping time at which excursions in U^a start. By the regularity of a, this contradicts the strong Markov property. □

4. Hypothèses Droites for (X_t)

We will now consider the 'converse' of Proposition (3.5). That is, we will start with n, and show that (X_t) is a right process. We will do this in a pathwise manner, hence will in fact start with the 'excursion process' (for the construction of a PPP from its characteristic measure, see Itô [8]).

The following will be the conditions in force throughout the rest of this section: Recall that (X_t^0) is a right process. Let n satisfy (i), (ii′), (iii), (iv), (v), and (vi′).

In case (vi′b) or (vi′c), let

$$(Y_t,\mathscr{F}_t)_{t>0}$$

be a PPP under each $\hat{P}^b$, with characteristic measure n (not dependant on b). That is, for any A with $n(A)<\infty$, the process

$$t\to N((0,t]\times A, Y)=\#\{s\in(0,t]; Y_s\in A\}$$

is adapted to $(\mathscr{F}_t)$, and is a Poisson process with intensity $n(A)$ and independent increments relative to $(\mathscr{F}_t)$, under each $\hat{P}^b$. [Put another way, $N((0,t]\times A,Y)-tn(A)$ is an $(\mathscr{F}_t)$-martingale under each $\hat{P}^b$.] We assume that $(\mathscr{F}_t)$ is right continuous, and that

$$\mathscr{F}_\infty^0\subset\mathscr{F}_{0-}\subset\mathscr{F}_\infty\subset\mathscr{F}.$$

For convenience, write $Y_0=\delta$. [This notation is different from that of [13]; there Y_0 was taken to be the process (X_t^0).] Set

$$m=1-\int(1-e^{-\sigma_a})\,dn.$$

(This is the appropriate 'delay coefficient', given by the normalization entailed in (2.4)),

$$S^-(s)=\begin{cases}0, & s=0\\ \sigma_a(X^0_\cdot)+ms+\sum\limits_{\substack{r<s\\ Y_r\neq\delta}}\sigma_a(Y_r), & s>0\end{cases}$$

$$S^+(s)=S^-(s+)$$

$$L_t = \inf\{s; S^+(s) > t\}$$

$$X_t = \begin{cases} X_t^0, & t \leqq \sigma_a(X_\cdot^0) \\ Y_{L_t}(t - S^-(L_t)), & Y_{L_t} \neq \delta \\ a, & \text{otherwise} \end{cases}$$

$$\mathscr{G}_t^0 = \mathscr{F}_t^0 \vee \mathscr{F}_{L_t-} \vee \sigma(k_{t-S^-(L_t)}(Y_{L_t})) \vee \sigma(X_t)$$

$$\mathscr{G}_t = \bigcap_{s>t} \mathscr{G}_s^0$$

[as usual, $k_r(u)$ denotes u killed at time r].

Similarly, in case (vi′a), let $\mathscr{F}_k)_{k \geqq 0}$ be a filtration, with

$$\mathscr{F}_\infty^0 \subset \mathscr{F}_{0-1} \subset \mathscr{F}_\infty \subset \mathscr{F},$$

and let $Y_k \in \mathscr{F}_k$, $k \geqq 1$ be random variables, independent of $\mathscr{F}_{k-1}$ and with distribution n under each $\hat{P}^b$. Again, take $Y_0 = \delta$. Set

$$S^-(k) = \begin{cases} 0, & k = 0 \\ \sigma_a(X_\cdot^0) + \sum_{j<k} \sigma_a(Y_j), & k \geqq 1 \end{cases}$$

$$S^+(k) = S^-(k+1)$$

$$L_t = \inf\{k; S^+(k) > t\}$$

$$X_t = \begin{cases} X_t^0, & t < \sigma_a(X^0) \\ Y_{L_t}(t - S^-(L_t)), & \text{otherwise} \end{cases}$$

$$\mathscr{G}_t^0 = \mathscr{F}_t^0 \vee \mathscr{F}_{L_t-1} \vee \sigma(k_{t-S^-(L_t)}(Y_{L_t})) \vee \sigma(X_t)$$

$$\mathscr{G}_t = \bigcap_{s>t} \mathscr{G}_s^0.$$

Our main result is the following.

(4.1) **Theorem.** *Under the above conditions, $(X_t, \mathscr{G}_t, \hat{P}^b)$ is a right process. (Y_t) is its excursion process.*

We will basically give the proof as a sequence of lemmas. The first should soothe the reader's anxieties concerning the existence and measurability of the above objects. It combines various results from [13]; the reader should look there for proofs.

(4.2) **Lemma.** (a) *In case* (vi′b) *or* (vi′c), *the following hold:*

$(S^-(s))$ is a.s. finite, left continuous, and strictly increasing. It is $(\mathscr{F}_s)$-predictable, and

$S^-(L_t) \leqq t \leqq S^+(L_t)$ for every t a.s. Each L_t is an $(\mathscr{F}_s)$ stopping time, and

$$\mathscr{F}_{L_t-} \subset \mathscr{G}_t \subset \mathscr{F}_{L_t}.$$

(b) *In case* (vi′a),

$(S^-(k))$ is $(\mathscr{F}_k)$-predictable. It is finite and satisfies

$S^-(L_t) \leqq t < S^+(L_t)$ for every t, a.s. Each L_t is an $(\mathscr{F}_k)$-stopping time, and

$$\mathscr{F}_{L_t-1} \subset \mathscr{G}_t \subset \mathscr{F}_{L_t}.$$

(c) (X_t^0) *is simple Markov with respect to* $(\mathscr{G}_t^0)$

(d) (X_t) *is a.s. right continuous, and is adapted to* $(\mathscr{G}_t)$.

Remark. Of course, (c) could be strengthened, but this is all we will need.

This Lemma uses only (i), (ii′), and (iii). In [13], the reader will find a more precise statement of the relationships between the objects under consideration. It is actually shown there, that $X_{\cdot}$ is a measurable function

$$F(X_{\cdot}^0, Y_{\cdot}),$$

for F defined on the product of U with a suitable space of 'point functions'. To make this work, an appropriate null set is discarded. In the present situation, this set is null with respect to each $\hat{P}^\mu$, so that no problem arises. The modification in the definition of $(U, \mathscr{U})$ used here also causes no problem.]

The following is a special case of the key lemma (Lemma 7) of [13]. As remarked there, a similar result may be found in Getoor and Sharpe [6].

(4.3) **Lemma.** *Assume that we are in case* (vi′b) *or* (vi′c). *Fix* t, *and let*

$$R(\omega, u) = \begin{cases} t - S^-(L_t)(\omega), & \text{if } 0 < t - S^-(L_t)(\omega) \leqq \sigma_a(u) \text{ and } L_t(\omega) \neq 0 \\ \infty, & \text{otherwise.} \end{cases}$$

Then

(a) $S^-(L_t) < t$ *a.s. on* $\{Y_{L_t} \neq \delta\}$.

(b) *For any* b, *and any* $A \in \mathscr{U}$ *we have*

$$\hat{P}^b(Y_{L_t} \in A \mid \mathscr{F}_{L_t-})(\omega) = \frac{n\{u; u \in A, R(\omega, u) < \infty\}}{n\{u; R(\omega, u) < \infty\}} \quad \text{for } \hat{P}^b\text{-a.e. } \omega$$

(*with the convention* $0/0 = 0$)

Proof. (a) Since S^- is strictly increasing, L_t is accessible on $\{t = S^-(L_t) > 0\}$. Also, S^- is a subordinator, hence quasi-left continuous. Thus $S^+(L_t) = S^-(L_t)$ a.s. on $\{t = S^-(L_t) > 0\}$, showing (a).

(b) The following formula is well known; let b be arbitrary, and let $f(u, s, \omega)$ be positive and measurable with respect to the product of $\mathscr{U}$ and the σ-field of $(\mathscr{G}_t)$ predictable subsets of $[0, \infty) \times \Omega$. Then

$$\hat{E}^b\Big[\sum_{\substack{s>0 \\ Y_s \neq \delta}} f(Y_s, s)\Big] = \hat{E}^b\left[\int_0^\infty \int f(u, s)\, n(du)\, ds\right]. \tag{4.4}$$

(The reader unfamiliar with dual predictable projections (see Dellacherie [3]) may check this by observing that monotone convergence and linearity allow one to take f of the form $Z(s, \omega) 1_A(u)$, where $n(A) < \infty$ and Z is bounded and predictable. In this case (4.4) follows, as the stochastic integral of Z with respect to the martingale

$$N(Y, (0, s] \times A) - n(A)s$$

is also a martingale, hence has expectation zero.] Now let $B \in \mathscr{F}_r$.

$$\hat{E}^b[B, L_t > r, Y_{L_t} \in A] = \hat{E}^b[B, \sum_{\substack{s \in (r, L_t] \\ Y_s \neq \delta}} 1_A(Y_s) 1_{[0,\infty)}(S^-(s) + \sigma_a(Y_s) - t)]$$

$$= \hat{E}^b \left[B, \int_r^{L_t} n\{u; u \in A, \sigma_a(u) \geqq t - S^-(s)\} \, ds \right]$$

$$= \hat{E}^b \left[B, \int_r^{L_t} \frac{n\{u; u \in A, \sigma_a(u) \geqq t - S^-(s)\}}{n\{u; \sigma_a(u) \geqq t - S^-(s)\}} n\{u; \sigma_a(u) \geqq t - S^-(s)\} \, ds \right]$$

$$= \hat{E}^b \left[B, \sum_{\substack{s \in (r, L_t] \\ Y_s \neq \delta}} \frac{n(A, \sigma_a(u) \geqq t - S^-(s))}{n(\sigma_a(u) \geqq t - S^-(s))} 1_{[0,\infty)}(S^-(s) + \sigma_a(Y_s) - t) \right]$$

$$= \hat{E}^b \left[B, L_t > r, Y_{L_t} \neq \delta, \frac{n(A, \sigma_a \geqq t - S^-(L_t))}{n(\sigma_a \geqq t - S^-(L_t))} \right]$$

$$= \hat{E}^b \left[B, L_t > r, \frac{n(A, R < \infty)}{n(R < \infty)} \right] \quad \text{by part (a).}$$

This suffices. □

The discrete time version is

(4.5) **Lemma.** *Assume we are in case* (vi'a). *Fix t and let*

$$R(\omega, u) = \begin{cases} t - S^-(L_t)(\omega), & \text{if } t - S^-(L_t)(\omega) \leqq \sigma_a(u) \quad \text{and} \quad L_t(\omega) \neq 0 \\ \infty, & \text{otherwise.} \end{cases}$$

Then for any b, and any $A \in \mathscr{U}$ we have

$$\hat{P}^b(Y_{L_t} \in A \mid \mathscr{F}_{L_t - 1}) = \frac{n(A, R < \infty)}{n(R < \infty)} \quad \hat{P}^b\text{-a.s.} \quad (\textit{with } 0/0 = 0).$$

Proof. The proof is essentially the same as that of (b) above, with (4.4) replaced by the trivial formula

$$\hat{E}^b[\sum_{k>0} f(Y_k, k)] = \hat{E}^b[\sum_{k>0} \int f(u, k) n(du)]. \quad \square$$

In case (vi'a), let c be the element of $\bar{E}$ given by (vi''a), and set

$$\bar{X}_t = \begin{cases} c, & X_t = a \\ X_t, & \text{otherwise.} \end{cases}$$

In case (vi'b) or (vi'c), let $\bar{X}_t = X_t$.

(4.6) **Lemma.** (a) *(X_t) is simple Markov with respect to $(\mathscr{G}_t^0)$.*

(b) *$(\bar{X}_t, \mathscr{G}_t, \hat{P}^b)$ is a Ray process.*

Proof. To prove (a), consider first case (vi'b) and (vi'c). By monotone class arguments, (b) of Lemma (4.3) extends immediately to the following: let $f(u, \omega)$ be bounded and

$$\mathscr{U} \otimes \mathscr{F}_{L_t -} \text{ measurable;}$$

then

$$(4.7) \qquad \hat{E}^b[f(Y_{L_t})] = \int \frac{1}{n(R(\cdot,\omega)<\infty)} \int_{\{R(\cdot,\omega)<\infty\}} f(u,\omega) n(du) \hat{P}^b(d\omega).$$

For $A, B \in \mathscr{U}$ and $C \in \mathscr{F}_{L_t-}$ we therefore have that

$$(4.8) \qquad P^b(Y_{L_t}(\cdot + t - S^-(L_t)) \in A, k_{t-S^-(L_t)}(Y_{L_t}) \in B, C) = \hat{E}^b\left[C, \frac{n(\theta_R^{-1} A, k_R^{-1} B, R<\infty)}{n(R<\infty)}\right].$$

By definition, R is nonzero, so that by (iv),

$$n(\theta_{R(\cdot,\omega)}^{-1} A, k_{R(\cdot,\omega)}^{-1} B, R(\cdot,\omega)<\infty) = \int_{\{R(\cdot,\omega)<\infty\} \cap k_{R(\cdot,\omega)}^{-1} B} P_0^{u(R(u,\omega))}(A) n(du).$$

Thus by (4.7) again, (4.8) equals

$$\hat{E}^b[C, k_{R(Y_{L_t})}(Y_{L_t}) \in B, P_0^{Y_{L_t}(R(Y_{L_t}))}(A)] = \hat{E}^b[C, k_{t-S^-(L_t)}(Y_{L_t}) \in B, P_0^{X_t}(A), L_t>0] \quad \text{(by Lemma (4.3)(a)).}$$

By definition of $(\mathscr{G}_t^0)$, we obtain that

$$(4.9) \qquad \hat{P}^b(Y_{L_t}(\cdot + t - S^-(L_t)) \in A \mid \mathscr{G}_t^0) = P_0^{X_t}(A) = \hat{P}^{X_t}(X_\cdot^0 \in A) \quad \hat{P}^b\text{-a.s. on } \{L_t>0\}.$$

Recall that following Lemma (4.2) we observed that the construction of (X_t) could be formalized as

$$X_\cdot = F(X_\cdot^0, Y_\cdot).$$

for F an appropriate measurable function. Also, by construction,

$$X_{t+\cdot} = F(Y_{L_t}(\cdot + t - S^-(L_t)), Y_{L_t+\cdot}) \quad \text{on} \quad \{L_t>0\}.$$

The strong Markov property of Poisson point processes (see Itô [8]) shows that

$$(Y_{L_t+s})_{s>0} \text{ is independent of } \mathscr{F}_{L_t}.$$

Since

$$Y_{L_t}(\cdot + t - S^-(L_t)) \in \mathscr{F}_{L_t}, \quad \text{and} \quad \mathscr{G}_t^0 \subset \mathscr{F}_{L_t},$$

it follows that

$$(4.10) \qquad \hat{P}^b(X_{t+\cdot} \in A \mid \mathscr{G}_t^0) = \hat{P}^{X_t}(A)$$

$\hat{P}^b$-a.s. on $\{L_t>0\}$. On $\{L_t=0\}$ we argue similarly, using Lemma (4.2)(c) and the fact that

$$X_{t+\cdot} = F(X_{t+\cdot}^0, Y_\cdot) \quad \text{on} \quad \{L_t=0\}.$$

This yields (a) in case (vi′b) and (vi′c).

Now consider case (vi'a). For $A, B \in \mathscr{U}$ and $C \in \mathscr{F}_{L_t-1}$, we still have that (4.8) holds. We may argue as before (using (iv)), that (4.9) holds $\hat{P}^b$-a.s. on

$$\{S^-(L_t) < t, L_t \neq 0\}.$$

As above, this yields (4.10) $\hat{P}^b$-a.s. on this set. Similarly, we use Lemma (4.2)(c) to obtain (4.10) on $\{L_t = 0\}$. Finally, consider

$$\{S^-(L_t) = t, L_t > 0\} = \{X_t = a\}.$$

Since this set belongs to $\mathscr{F}_{L_t-1}$, we have from Lemma (4.5) that for $A \in \mathscr{U}$, $C \in \mathscr{F}_{L_t-1}$,

$$\hat{P}^b(Y_{L_t} \in A, X_t = a, C) \\ = n(A)\hat{P}^b(X_t = a, C).$$

That is,

$$\hat{P}^b(Y_{L_t} \in A \,|\, \mathscr{G}_t^0) = n(A) \quad \hat{P}^b\text{-a.s.} \quad \text{on} \quad \{X_t = a\}.$$

Since by construction,

$$F(X_\cdot^0, Y_\cdot) = F(Y_1, Y_{1+\cdot}) \quad \hat{P}^a\text{-a.s.},$$

we argue as before, obtaining (4.10) on $\{X_t = a\}$, and hence completing the proof of (a).

Because the correspondence between X_t and $\bar{X}_t$ is one-to-one, it follows from part (a) that $(\bar{X}_t)$ is simple Markov with respect to $(\mathscr{G}_t^0)$. Lemma (3.1)(b) and Proposition (3.3)(a), together with (vi'b) and (vi'') show that $(\bar{X}_t)$ is a.s. ρ-right continuous. We therefore need only show that the resolvant $\bar{U}^\alpha$ of $(\bar{X}_t)$ preserves the ρ-continuous functions on $\bar{E}$.

Since $Y_\cdot$ is independent of $\mathscr{F}_\infty^0$, we have that

$$\bar{U}^\alpha f(b) = \hat{E}^b\left[\int_0^{\tau_a(X_\cdot^0)} e^{-\alpha t} f(X_t^0)\,dt\right] + \hat{E}^b[e^{-\alpha\tau_a(X_\cdot^0)}]\,\bar{U}^\alpha f(a).$$

Also, writing U_0 for the resolvant of (X_t^0),

$$U_0^\alpha f(b) = \hat{E}^b\left[\int_0^{\tau_a(X_\cdot^0)} e^{-\alpha t} f(X_t^0)\,dt\right] + \hat{E}^b[e^{-\alpha\tau_a(X_\cdot^0)}]\,\frac{f(a)}{\alpha}.$$

The latter is by definition ρ-continuous whenever f is, so that by (2.2), the same will be true for $\bar{U}^\alpha f$, showing (b). □

Proof of Theorem (4.1). By (a) of Lemma (4.6), we need only show that if f is α-excessive for (X_t), then $t \to f(X_t)$ is a.s. right continuous; in fact, we need only show this for f of the form $U^\alpha g$, where (U^α) is the resolvant of (X_t) (see Getoor [5], (9.4) and (9.6)). Extend g to $\bar{g}$ on $\bar{E}$, and let $\bar{f} = \bar{U}^\alpha \bar{g}$. By (b) of Lemma (4.6), (and (5.8) of Getoor [5]) $t \to \bar{f}(\bar{X}_t)$ is a.s. right continuous.

In case (b) or (c) of (vi'), we have $\bar{X}_\cdot = X_\cdot$, so $\bar{f} = f$ on E, hence the conclusion is shown.

In case (vi′a), $\bar{X}_t = X_t$ for all but countably many t, so that $\bar{f} = f$ on E once more. Since $f(a) = \bar{f}(c)$, we have that

$$f(X_t) = \bar{f}(\bar{X}_t) \quad \text{for every } t,$$

which again yields the conclusion of the theorem. □

It seems worthwhile to point out where the above proof fails, if instead of (vi′), we assume only (vi). Part (a) of Lemma (4.6) still works, and if we define

$$\bar{X}_t = \rho - \lim_{s \downarrow t} X_s$$

(c.f. Proposition (3.5)) then $(\bar{X}_t)$ is a.s. ρ-right continuous. By Lemma (4.3)(a) we have $X_t = \bar{X}_t$ a.s., for each t. Thus the proof of (b) of Lemma (4.6) still applies.

The difference comes when we apply Lemma (4.6); we may not be able to recover the hypothèses droites of (X_t) from those of $(\bar{X}_t)$, since we may not have that $\bar{X}_t = X_t$ for every t, a.s. In fact, with our new definition of $(\bar{X}_t)$, there may be more than one point a' in $\bar{E}$ for which we can have

$$\bar{X}_t = a' \quad \text{and} \quad X_t = a.$$

The strong Markov property fails at stopping times at which this occurs. The reader may wish to examine example 2 of [13] in this light.

5. Further Results

In our haste to cut a straightforward path through the proof of Theorem (4.1), we have avoided pointing out two further consequences of our arguments. We'll start by rectifying this situation.

A natural question is that of what happens to our results when we start with stronger assumptions on n? In particular, what can we say when (X_t^0) is moreover a Ray process (hence E is compact)? We will (for the moment) retain our earlier assumptions, so that by normality (X_t^0) has no branch points. If, in addition, we assume that for each α,

(5.1) $$x \to \hat{E}^x[e^{-\alpha\sigma_a}] \quad \text{is continuous}$$

(or, what is the same thing, that (X_t^0) killed at σ_a is a Ray process) then the ρ and d topologies coincide, so that U consists merely of the d-right continuous functions.

(5.2) **Corollary.** *Let* (X_t^0) *be Ray, and satisfy* (5.1). *Let n satisfy the conditions of Lemma* (3.1). *Then n satisfies* (ii′). *If* $n(U^a) > 0$ *then n also satisfies* (vi′c). [*In particular, there are no 'discrete visiting extensions', as in* (vi′a).]

Proof. The first statement is merely (b) of Lemma (3.1). For the second, suppose that $n(U^a) > 0$. If n is finite, or (vi′c) fails, then there is a finite non zero measure n' concentrated on U^a and satisfying (iv), so that

$$\mu_t(A) = n'(W_t \in A)$$

defines a bounded system of entrance laws for (P_0^b). Because (X_t^0) is Ray, there is a finite measure μ_0 on E such that $\mu_t = P_0^{\mu_0}(W_t \in \cdot)$. Thus $n' = P_0^{\mu_0}$, which is impossible, as n' is concentrated on U^a. Thus (vi′c) holds (and (vi′a) is vacuous). □

As a consequence of this, and of (b) of Lemma (4.6) we have

(5.3) **Corollary.** *Let* (X_t^0) *be Ray, and satisfy* (5.1). *Let* n *satisfy* (i), (iii), (iv), (v). *If* $n(U^a)=0$ *and* n *is finite, assume also that* (*as in* (vi b))

$$\int (1-e^{-\sigma_a})\,dn < 1.$$

Then there is an $(X_t, \mathcal{G}_t)$ *as in Theorem* (4.1), *which is a Ray process.* □

That is, if we start with a Ray process, we end up with a Ray process.

In fact, we could obtain slightly more. So far we have dealt with right processes. When dealing with Ray processes, however, it is useful to allow branch points; that is, to not insist on having $X_t^0 = b$, $\hat{P}^b$-a.s. One can easily check that Corollary (5.2) and Corollary (5.3) hold in this situation as well, provided we assume also that

$$P_0^b(U^a) < 1 \qquad \text{for each } b \neq a. \tag{5.4}$$

This condition replaces the use of (3.2) in the proof of Lemma (3.1). The following example shows that we can't eliminate (5.4);

(5.5) *Example.* Let $E=[0,1]\times\{0,1\}$, $a=(0,0)$. Make a absorbing, and on $[0,1]\times\{1\}$ let (P_0^b) correspond to uniform motion to the left, with $(0,1)$ a branch point to a. On $(0,1]\times\{0\}$, let (P_0^b) correspond to uniform motion to the left, except that there is a jump from $(v,0)$ to $(v,1)$ at rate $g(v)$, where $g(v)\to\infty$ fast enough as $v\downarrow 0$ so that

$$P_0^b((W_t) \text{ hits } [0,1]\times\{1\}) = 1$$

for every $b \in E\setminus\{a\}$. Then we can find μ on $E\setminus\{a\}$ such that $n = P_0^\mu$ satisfies (i), (ii), (iii), (iv), (v), and (vi′), but not (ii′), even though (P_0^b) is Ray.

Our final result of this type will involve a condition weaker than the Ray property; the reason we built ρ-right continuity into the definition of U was to obtain that $(\bar{X}_t)$ was a.s. ρ-right continuous. If we had taken U to consist of all d-right continuous paths, then in general we don't know whether $(X_t^0, \hat{P}^b)$ being a right process implies that the coordinate process (W_t, P_0^b) will be one as well[2]. With this new simpler U, one condition that will guarantee this (see Sharpe [15]) is that all α-excessive functions f be *nearly Borel.* That is, for each μ, there are Borel functions f_1, f_2 with $f_1 \leqq f \leqq f_2$, and

$$\hat{P}^\mu(f_1(X_t^0) \neq f_2(X_t^0) \text{ for some } t) = 0.$$

We will show that if the α-excessive functions for (X_t^0) are nearly Borel, then so are those for (X_t);

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2 See the remark preceding (2.3)

(5.6) **Corollary.** *Assume the conditions of Theorem* (4.1) (*with our new* U). *Assume that all* α*-excessive functions for* (X_t^0) *are nearly Borel. Then* $(X_t, \mathscr{G}_t, \hat{P}^b)$ *is a right process, with nearly Borel* α*-excessive functions.*

Proof. As indicated above, the nearly Borel assumption guarantees the ρ-right continuity of $(\bar{X}_t)$, so that the proof of Theorem (4.1) applies to show that $(X_t, \mathscr{G}_t, \hat{P}^b)$ is a right process.

Now let f be α-excessive for (X_t), for some $\alpha > 0$. Then

$$f^0 = f \cdot 1_{E \setminus \{a\}}$$

is α-excessive for (X_t^0), hence is nearly Borel. Using (iv) and (v) we can find Borel functions f^1 and f^2 with $f^1 \leqq f^0 \leqq f^2$ and

$$n(f^1(W_t) \neq f^2(W_t) \text{ for some } t \geqq 0) = 0.$$

It follows that f^0 is nearly Borel for (X_t), and hence that so is f. □

We conclude with a typical application of our results. In general, our conditions are hard to check, unless we are modifying a process already known to be strong Markov. In this case, necessity implies that our conditions hold for the old process, and if the conditions are preserved under the transformation in question, Theorem (4.1) will apply.

(5.7) *Example* (Skew Brownian motion). Let n be the characteristic measure of the *PPP* of excursions of Brownian motion on $\mathbb{R}$ from 0, and let P_0^b correspond to Brownian motion started at b, and absorbed at 0. For $\alpha \in [0, 1]$, let

$$n^+ = 2n|_{\{W_t \geqq 0 \text{ for every } t\}}$$
$$n^- = 2n|_{\{W_t \leqq 0 \text{ for every } t\}}$$
$$n_\alpha = \alpha n^+ + (1-\alpha) n^-.$$

Then the conditions of Theorem (4.1), for n_α, follow easily from those for n, hence the process constructed is a right process. It is easily seen to be the 'skew Brownian motion' examined in Walsh [16]. This process was introduced by Itô and McKean [9]. They gave a construction in terms of excursions; the methods of Theorem (4.1) are exactly what one needs to show the strong Markov property of the process they construct (see also the epilogue to [16]).

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