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CORRECTIONS TO "ADMISSIBILITY OF ESTIMATORS OF THE PROBABILITY OF UNOBSERVED OUTCOMES"

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In the above titled paper (this Annals Vol. 42, No. 4 (1990), pp. 623–636), the following is a correction of the proof of Theorem 2.3.

THEOREM 2.3. *For $k = \infty$ Good's estimator and Robbins' estimator are admissible.*

PROOF. We prove the result for Good's estimator and remark that the proof for Robbins' estimator is similar. Suppose G is not admissible. Then there exists an estimator $\delta(\boldsymbol{X}^n)$ which is better. That is,

$$\sum_{\boldsymbol{x}^n}(\boldsymbol{G}(\boldsymbol{x}^n) - U_n(\boldsymbol{x}^n, \boldsymbol{p}))^2 f(\boldsymbol{x}^n; \boldsymbol{p}) \geq \sum_{\boldsymbol{x}^n}(\delta(\boldsymbol{x}^n) - U_n(\boldsymbol{x}^n, \boldsymbol{p}))^2 f(\boldsymbol{x}^n; \boldsymbol{p}) \tag{2.6}$$

where $f(\boldsymbol{x}^n; \boldsymbol{p}) = (n!/\prod_i x_i^n!)\prod_i^\infty p_i^{x_i^n}$. Since (2.6) must be true for all $\boldsymbol{p}$ our approach is to iteratively examine (2.6) for particular choices of $\boldsymbol{p}$. We will show that the validity of (2.6) for each particular $\boldsymbol{p}$ implies the equality of G and δ for certain sample points. Also as we consider all our $\boldsymbol{p}$ choices we will cover all sample points.

Let $\boldsymbol{p}(\theta, r) = (1 - \theta, \theta/r, \dots, \theta/r, 0, \dots)$ be a parameter point and define

$$T_j = \left\{\boldsymbol{x}^n : x_1^n = n - j,\ x_i^n = 0 \text{ or } 1,\ i = 2, \dots, r + 1,\ \sum_1^\infty x_i^n = n\right\}, \quad j = 0, 1, \dots, n.$$

For any sample point $\boldsymbol{u}^n \in T_j$

$$P\{\boldsymbol{X}^n = \boldsymbol{u}^n\} = (n!/(n - j)!)(1 - \theta)^{n-j}(\theta/r)^j. \tag{2.7}$$

Since T_j contains $\binom{r}{j}$ points we have

$$P(T_j) = (n!/(n - j)!)\binom{r}{j}(1 - \theta)^{n-j}(\theta/r)^j \to \binom{n}{j}\theta^j(1 - \theta)^{n-j}, \quad \text{as } r \to \infty. \tag{2.8}$$

Thus

$$P\left(\bigcup_{j=0}^{n} T_j\right) \to 1 \quad \text{as} \quad r \to \infty. \tag{2.9}$$

Hence (2.6) and (2.9) imply (when $r \to \infty$) that

$$\begin{aligned}\lim_{r\to\infty} \sum_{j=0}^{n} \sum_{\boldsymbol{x}^n \in T_j} (G(\boldsymbol{x}^n) - U_n(\boldsymbol{x}^n, \boldsymbol{p}(\theta, r)))^2 p(\boldsymbol{X}^n = \boldsymbol{x}^n) \\ \geq \lim_{r\to\infty} \sum_{j=0}^{n} \sum_{\boldsymbol{x}^n \in T_j} (\delta(\boldsymbol{x}^n) - U_n(\boldsymbol{x}^n, \boldsymbol{p}(\theta, r)))^2 P(\boldsymbol{X}^n = \boldsymbol{x}^n).\end{aligned} \tag{2.10}$$

It follows from (2.7) that if (2.10) holds for some function $\delta(\boldsymbol{x}^n)$, (2.10) will also hold for some function of the form

$$\delta^*(\boldsymbol{x}^n) = \begin{cases} \gamma(j) & \text{if } \boldsymbol{x}^n \in T_j \\ \delta(\boldsymbol{x}^n) & \text{otherwise.} \end{cases}$$

This latter fact follows by virtue of sufficiency of j on T_j. Furthermore,

$$\lim_{r\to\infty} U_n(\boldsymbol{x}^n, \boldsymbol{p}(\theta, r)) = \begin{cases} \theta & \text{if } \boldsymbol{x}^n \in T_j, j = 0, 1, \ldots, n-1 \\ 1 & \text{if } \boldsymbol{x}^n \in T_n \end{cases}$$

which is constant (as are $G(\boldsymbol{x}^n)$ and $\delta^*(\boldsymbol{x}^n)$) on each T_j. Hence (2.10) reduces to

$$\begin{aligned}\sum_{j=0}^{n-2} (j/n - \theta)^2 \binom{n}{j} \theta^j (1-\theta)^{n-j} + (1-\theta)^2 \binom{n}{n-1} \theta^{n-1}(1-\theta) \\ \geq \sum_{j=0}^{n-1} (\gamma(j) - \theta)^2 \binom{n}{j} \theta^j (1-\theta)^{n-j} + (\gamma(n) - 1)^2 \theta^n.\end{aligned} \tag{2.11}$$

Taking $\theta \to 1$ in (2.11) implies $\gamma(n) = 1$. Furthermore, set $\gamma(n) = 1$, divide both sides of (2.11) by $(1-\theta)$, then let $\theta \to 1$ again to find $\gamma(n-1) = 1$. Now with $\gamma(n) = \gamma(n-1) = 1$, rewrite (2.11) as

$$\sum_{j=0}^{n-2} ((j/n) - \theta)^2 \binom{n}{j} \theta^j (1-\theta)^{n-j} \geq \sum_{j=0}^{n-2} (\gamma(j) - \theta)^2 \binom{n}{j} \theta^j (1-\theta)^{n-j}. \tag{2.12}$$

Divide both sides of (2.12) by $\theta(1-\theta)$ and recognize that for $j = 0, 1, \ldots, n-2$, (j/n) is proper Bayes against a uniform prior which implies that for these sample points $\gamma(j) = j/n$.

In later stages consider the sequences of parameter points

$$\left(\frac{1-\theta}{k}, \ldots, \frac{1-\theta}{k}, \frac{\theta}{r}, \ldots, \frac{\theta}{r}, 0, \ldots\right)$$

and sets

$$T_j(w_1, \ldots, w_k) = \Bigg\{ \boldsymbol{x}^n : x_i^n = w_i,\ i = 1, \ldots, k,$$

$$x_i^n = 0 \text{ or } 1,\ i = k+1, \ldots, k+r,\ \sum_1^\infty x_i^n = n \Bigg\}$$

where $w_1 + \cdots + w_k = n - j$. For $\boldsymbol{u}^n = (u_1, u_2, \ldots) \in T_j(w_1, \ldots, w_k)$

$$P(\boldsymbol{X}^n = \boldsymbol{u}^n) = \frac{n!}{w_1! \cdots w_k!} \left(\frac{1-\theta}{k} \right)^{\sum_1^k w_i} \left(\frac{\theta}{r} \right)^j.$$

Since $T_j(w_1, \ldots, w_k)$ contains $\binom{r}{j}$ points

$$P(T_j(w_1, \ldots, w_k)) \to_{r \to \infty} \frac{n!}{w_1! \cdots w_k! j!} \left(\frac{1-\theta}{k} \right)^{n-j} \theta^j.$$

Again we find that j is a sufficient statistic for θ so that to beat Good's estimator we need only consider estimators depending on j.

Let $\mathcal{T} = \{T_j(w_1, \ldots, w_k) : w_i \neq 0, w_i \neq 1, i = 1, \ldots, k\}$. Note that any estimator which is at least as good as Good's would have already been (at an earlier stage) shown to be identical to $G(\boldsymbol{x}^n)$ on all $T_j(w_1, \ldots, w_k) \notin \mathcal{T}$. Furthermore, $G(\boldsymbol{x}^n) = j/n$ on all $T_j(w_1, \ldots, w_k) \in \mathcal{T}$. Thus (2.6) reduces to

$$\sum_{\mathcal{T}} (j/n - \theta)^2 \frac{n!}{x_1! \cdots x_k! j! k^{n-j}} \theta^j (1-\theta)^{n-j}$$
$$\geq \sum_{\mathcal{T}} (\gamma(j) - \theta)^2 \frac{n!}{x_1! \cdots x_k! j! k^{n-j}} \theta^j (1-\theta)^{n-j}.$$

Again using the Bayes argument we have $\gamma(j) = j/n$. The only sample points unaccounted for are those which are permutations of sample points already considered. These permutations would follow the same pattern for suitably permuted parameter points. Finally note that the number stages $k \leq n/2$. $\square$