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Bessel Diffusions as a One-Parameter Family of Diffusion Processes

Tokuzo Shiga and Shinzo Watanabe

0. Introduction

By a Bessel diffusion process with index α ($\alpha \geqq 0$), we mean a conservative one-dimensional diffusion process on $[0, \infty)$ determined by the local generator

$$L = \frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{\alpha-1}{x}\frac{d}{dx}\right) \tag{0.1}$$

with the point 0 as

$$\begin{array}{ll} \text{a trap} & \text{if } \alpha = 0, \\ \text{a reflecting boundary} & \text{if } 0 < \alpha < 2, \\ \text{an entrance boundary} & \text{if } 2 \leqq \alpha. \end{array} \tag{0.2}$$

When the index α is equal to the positive integer n, this diffusion is just the radial part of n-dimensional Brownian motion: i.e. if $B_1(t), B_2(t), \ldots, B_n(t)$ are n mutually independent one-dimensional Brownian motions, then

$$X^{(n)}(t) = \sqrt{B_1^2(t) + B_2^2(t) + \cdots + B_n^2(t)} \tag{0.3}$$

defines a Bessel diffusion with index n. It is clear from (0.3) that, if $X^{(n)}(t)$ and $X^{(m)}(t)$ are mutually independent Bessel diffusions of index n and m respectively where n and m are positive integers, then the process $X^{(n+m)}(t)$ defined by

$$X^{(n+m)}(t) = \sqrt{\{X^{(n)}(t)\}^2 + \{X^{(m)}(t)\}^2} \tag{0.4}$$

is a Bessel diffusion with index $n+m$.

Generally, for given two mutually independent stochastic processes $X_1(t)$ and $X_2(t)$ with probability laws $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively, if the stochastic process

$$X(t) = \sqrt{X_1^2(t) + X_2^2(t)} \tag{0.5}$$

is well defined, i.e. the probability law of $X(t)$ is the same for the same initial value $X(0)$, we denote the probability law $\mathbb{P}$ of the process $X(t)$ as

$$\mathbb{P} = \mathbb{P}_1 \circledast \mathbb{P}_2; \tag{0.6}$$

for a precise definition, see 2, Definition 2.3. If we denote by $\mathbb{P}^{(\alpha)}$ ($\alpha \geqq 0$) the probability law of a Bessel diffusion with index α, then (0.4) implies that the relation

$$\mathbb{P}^{(\alpha+\beta)} = \mathbb{P}^{(\alpha)} \circledast \mathbb{P}^{(\beta)} \tag{0.7}$$

holds when α and β are positive integers and, as we shall see in 2, *it holds for all* $\alpha, \beta \in [0, \infty)$. The relation (0.7) may be regarded as a fundamental property of the family $\{\mathbb{P}^{(\alpha)}\}$ of Bessel diffusions.

In this paper, we will determine all one parameter families of diffusions on $[0, \infty)$ having the property (0.7). The family of Bessel diffusions is one of the typical examples but besides, there is an example of such a family which contains all the radial parts of multidimensional Ornstein-Uhlenbeck Brownian motions, i.e. direct products of a one-dimensional Ornstein-Uhlenbeck Brownian motion.

In 1, we introduce a kind of convolution among stochastic processes on $\mathbb{R}^+ = [0, \infty)$ and also a notion of an *infinitely decomposable process.* We shall show that a Markov process is infinitely decomposable if and only if it is a CBI-process (continuous state branching process with immigration) in the sense of Kawazu-Watanabe [2]. Then, by the results of [2], we can determine completely the class of infinitely decomposable Markov processes.

In 2, we will remark that the study of a one parameter family of Markov processes satisfying (0.7) is reduced, by the change of coordinates $x \rightsquigarrow x^2$, to the result in 1, and will determine all such families. In particular, we see that the family of Bessel diffusions is one of typical examples.

In 3, we will apply the fact that the family of Bessel diffusions satisfies (0.7) to obtain an invariance of Bessel diffusions under the inversion of time. Also, we apply this invariance to obtain some results on the local behavior of sample functions of Bessel diffusions and related diffusions.

1. Infinitely Decomposable Markov Process and CBI-Process

Let W be the path space with state space $\mathbb{R}^+ = [0, \infty)$:

$$W = \{w\colon [0, \infty) \to \mathbb{R}^+ = [0, \infty), \text{ right continuous, } \exists \text{ left limits}\}$$

and $\mathscr{B}(W)$ be the σ-field on W generated by Borel cylinder sets. We denote $X(t, w)(= X_t(w)) = w(t)$ for $w \in W$ and $t \geqq 0$. Let $\mathbb{P} = \{P_x, x \in \mathbb{R}^+\}$ be a system of probability measures on $(W, \mathscr{B}(W))$ such that

$$x \rightsquigarrow P_x(B) \quad \text{is Borel measurable in } x \in \mathbb{R}^+ \text{ for every } B \in \mathscr{B}(W), \tag{1.1}$$

$$P_x(X(0) = x) = 1 \qquad \text{for every } x \in \mathbb{R}^+ \tag{1.2}$$

We denote by $\mathscr{P}(W)$ the set of all such systems $\mathbb{P}$.

Definition 1.1. $\mathscr{P}_M(W) = \{\mathbb{P} \in \mathscr{P}(W);\ \mathbb{P}$ has the time homogeneous strong Markov property$\}$, $\mathscr{P}_D(W) = \{\mathbb{P} \in \mathscr{P}(W);\ \mathbb{P}$ is a time homogeneous diffusion, i.e. $\mathbb{P} \in \mathscr{P}_M(W)$ and $P_x(W_c) = 1, \forall x \in \mathbb{R}^+\}$ where $W_c = \{w\colon [0, \infty) \to \mathbb{R}^+, \text{ continuous}\}$.

In the following $\mathbb{P} \in \mathscr{P}_M(W)$ is called a *Markov process* and $\mathbb{P} \in \mathscr{P}_D(W)$ a *diffusion process.*

Definition 1.2. Let $\mathbb{P}, \mathbb{Q}, \mathbb{R} \in \mathscr{P}(W)$. We define

$$\mathbb{R} = \mathbb{P} * \mathbb{Q} \tag{1.3}$$

if and only if

$$\text{for every } x, y \in \mathbb{R}^+, \qquad R_{x+y} = \phi(P_x \times Q_y) \tag{1.4}$$

where ϕ is the mapping $W \times W \to W$ defined by

$$\phi(w_1, w_2) = w_1 + w_2. \tag{1.5}$$

Thus $\mathbb{R}$ is the probability law of the stochastic process which is the sum of two mutually independent stochastic processes governed by laws $\mathbb{P}$ and $\mathbb{Q}$ respectively. Note that $\mathbb{P} * \mathbb{Q}$ can not be defined always: in order that it is well defined, it is necessary and sufficient that

$$\phi(P_x \times Q_y) = \phi(P_{x'} \times Q_{y'}) \quad \text{if} \quad x + y = x' + y'. \tag{1.6}$$

Lemma 1.1. *Let $\mathbb{P}_1$ and $\mathbb{P}_2 \in \mathscr{P}(W)$. In order that $\mathbb{P}_1 * \mathbb{P}_2$ is well defined it is necessary and sufficient that, for every $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $0 < t_1 < t_2 < \cdots < t_n$ and $\lambda_k > 0$, $k = 1, 2, \ldots, n$, there exist $\psi(\mathbf{t}, \boldsymbol{\lambda}) > 0$ and $\varphi_i(\mathbf{t}, \boldsymbol{\lambda}) > 0$ $(i = 1, 2)$ such that*

$$E_x^{(i)}(e^{-(\boldsymbol{\lambda}, X(\mathbf{t}))}) = e^{-x\psi(\mathbf{t}, \boldsymbol{\lambda})} \varphi_i(\mathbf{t}, \boldsymbol{\lambda}), \qquad i = 1, 2 \tag{1.7}$$

for every $x \in \mathbb{R}^+$, where $(\boldsymbol{\lambda}, X(\mathbf{t})) = \lambda_1 X(t_1) + \lambda_2 X(t_2) + \cdots + \lambda_n X(t_n)$. Here $E_x^{(i)}$ stands for the expectation with respect to $\mathbb{P}_i$.

Proof. For $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, set

$$f_i(x) = E_x^{(i)}(e^{-(\boldsymbol{\lambda}, X(\mathbf{t}))}), \qquad i = 1, 2.$$

In order that $\mathbb{P}_3 = \mathbb{P}_1 * \mathbb{P}_2$ is well defined, it is necessary and sufficient that

$$f_1(x) f_2(y) = f_3(x + y) \qquad \forall x, y \in \mathbb{R}^+, \tag{1.8}$$

where $f_3(x) = E_x^{(3)}(e^{-(\boldsymbol{\lambda}, X(\mathbf{t}))})$. It is quite easy to see that (1.8) is equivalent to

$$f_i(x) = a_i e^{-bx} \qquad i = 1, 2, 3, \tag{1.9}$$

for some $a_i > 0$ and $b > 0$ with $a_1 a_2 = a_3$.

Definition 1.3. $\mathbb{P} \in \mathscr{P}(W)$ is *infinitely decomposable* (i.d.) if for every $n = 1, 2, \ldots$, there exists $\mathbb{P}^{(n)} \in \mathscr{P}(W)$ such that

$$\mathbb{P} = \underbrace{\mathbb{P}^{(n)} * \mathbb{P}^{(n)} * \cdots * \mathbb{P}^{(n)}}_{n}. \tag{1.10}$$

Thus, by the above lemma, if $\mathbb{P}$ is i.d., then, for every $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $0 < t_1 < t_2 < \cdots < t_n$ and $\lambda_i > 0$, $i = 1, 2, \ldots, n$, there exists $\psi(\mathbf{t}, \boldsymbol{\lambda}) > 0$ and $\varphi(\mathbf{t}, \boldsymbol{\lambda}) > 0$ such that

$$E_x(e^{-(\boldsymbol{\lambda}, X(\mathbf{t}))}) = e^{-x\psi(\mathbf{t}, \boldsymbol{\lambda})} \varphi(\mathbf{t}, \boldsymbol{\lambda}) \qquad \forall x \in \mathbb{R}^+. \tag{1.11}$$

Generally, it is not true that $\mathbb{P} \in \mathscr{P}(W)$ having the property (1.11) is i.d. but this is true for $\mathbb{P} \in \mathscr{P}_M(W)$ as we shall see. Following [2], we give

Definition 1.4. $\mathbb{P} \in \mathscr{P}_M(W)$ is a *CB-process* if and only if, for every $\lambda > 0$ and $t > 0$, there exists $\psi(t, \lambda) > 0$ such that

$$E_x(e^{-\lambda X_t}) = e^{-\psi(t, \lambda) x}, \qquad \forall x \in \mathbb{R}^+. \tag{1.12}$$

$\mathbb{P} \in \mathscr{P}_M(W)$ is a *CBI-process* if and only if, for every $\lambda > 0$ and $t > 0$, there exist $\psi(t, \lambda) > 0$ and $\varphi(t, \lambda) > 0$ such that

$$E_x(e^{-\lambda X_t}) = e^{-\psi(t,\lambda)x}\,\varphi(t,\lambda), \qquad \forall\, x \in \mathbb{R}^+. \tag{1.13}$$

Thus, if a Markov process $\mathbb{P}$ is i.d., then it is a CBI-process. Conversely, a CBI-process is i.d. To show this, we recall some results in [2]: there is a one to one correspondence between a CBI-process $\mathbb{P}$ and a system of generating functions $[R, F]$. R and F are functions of the form

$$R(\lambda) = -a\lambda^2 + b\lambda - \int_0^\infty (e^{-\lambda u} - 1 + \lambda u(1+u^2)^{-1})\, n_1(du), \tag{1.14}$$

$$F(\lambda) = c\lambda - \int_0^\infty (e^{-\lambda u} - 1)\, n_2(du) \tag{1.15}$$

where $a \geqq 0$, $c \geqq 0$ and $\int_0^\infty u^2(1+u^2)^{-1}\, n_1(du) + \int_0^\infty u(1+u)^{-1}\, n_2(du) < \infty$ such that

$$\int_{0+} [R(\lambda) \vee 0]^{-1}\, d\lambda = \infty. \tag{1.16}$$

Note that, by the above definition of a CBI-process, we assume always that it is right continuous and conservative. Finally

$$E_x(e^{-\lambda X_t}) = e^{-x\psi(t,\lambda)}\,\varphi(t,\lambda) \tag{1.17}$$

where $\psi(t, \lambda)$ is the solution of

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= R(\psi) \\ \psi|_{t=0} &= \lambda \end{aligned} \tag{1.18}$$

and

$$\varphi(t,\lambda) = \exp\left[-\int_0^t F(\psi(s,\lambda))\, ds\right]. \tag{1.19}$$

By the Markov property, for $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $0 < t_1 < t_2 < \cdots < t_n$ and $\lambda_i > 0$, $i = 1, 2, \ldots, n$,

$$E_x(e^{-(\boldsymbol{\lambda}, X(\mathbf{t}))}) = e^{-x\psi(t_1, \tilde{\lambda}_1)} \prod_{i=1}^n \varphi(t_k - t_{k-1}, \tilde{\lambda}_k) \tag{1.20}$$

where $\tilde{\lambda}_k$, $k = 1, 2, \ldots, n$ are determined by

$$\begin{aligned} \tilde{\lambda}_n &= \lambda_n, \\ \tilde{\lambda}_{k-1} &= \lambda_{k-1} + \psi(t_k - t_{k-1}, \tilde{\lambda}_k). \end{aligned} \tag{1.21}$$

Now, a CBI-process is i.d. since, if $\mathbb{P}$ corresponds to $[R, F]$ then

$$\mathbb{P} = \underbrace{\mathbb{P}^{(n)} * \mathbb{P}^{(n)} * \cdots * \mathbb{P}^{(n)}}_{n}$$

where $\mathbb{P}^{(n)}$ is the CBI-process which corresponds to $[R, n^{-1} F]$. Also, we have the following

Lemma 1.2. *Let* $\mathbb{P}_i$ $(i=1,2)$ *be CBI-processes corresponding to* $[R, F_i]$ $(i=1,2)$ *respectively. Then,* $\mathbb{P}=\mathbb{P}_1 * \mathbb{P}_2$ *is well defined and it is the CBI-process corresponding to* $[R, F_1+F_2]$.

We can now determine a one-parameter family $\{\mathbb{P}^{(\alpha)}\}_{\alpha\geqq 0}$ of Markov processes which satisfies the relation

$$\mathbb{P}^{(\alpha+\beta)}=\mathbb{P}^{(\alpha)} * \mathbb{P}^{(\beta)}, \qquad \alpha, \beta\in[0, \infty). \tag{1.22}$$

Theorem 1.1. *There is a one to one correspondence between a one-parameter family* $\{\mathbb{P}^{(\alpha)}\}_{\alpha\geqq 0}\subset\mathscr{P}_M(W)$ *satisfying* (1.22) *and a system of generating functions* $[R, F]$ *given in the form* (1.14) *and* (1.15) *satisfying* (1.16); $\mathbb{P}^{(\alpha)}$ *is the CBI-process corresponding to* $[R, \alpha F]$.

Proof. If $\mathbb{P}^{(\alpha)}$ is the CBI-process corresponding to $[R, \alpha F]$ then the family $\{\mathbb{P}^{(\alpha)}\}_{\alpha\geqq 0}$ satisfies (1.22) by Lemma 1.2.

Conversely, if a family $\{\mathbb{P}^{(\alpha)}\}_{\alpha\geqq 0}$ satisfies (1.22), then clearly $\mathbb{P}^{(\alpha)}$ is an i.d. Markov process and hence, it is a CBI-process. Let $\mathbb{P}^{(\alpha)}$ correspond to $[R_\alpha, F_\alpha]$. Since $\mathbb{P}^{(\alpha+\beta)}=\mathbb{P}^{(\alpha)} * \mathbb{P}^{(\beta)}$ is well defined, R_α is independent of α by Lemma 1.1 and also $F_\alpha+F_\beta=F_{\alpha+\beta}$. Thus, $F_\alpha=\alpha F_1$.

In particular, a one-parameter family $\{\mathbb{P}^{(\alpha)}\}$ of diffusions satisfying the relation (1.22) is determined as follows:

Theorem 1.2. *There is a one-to-one correspondence between a one parameter family* $\{\mathbb{P}^{(\alpha)}\}_{\alpha\geqq 0}$ *of diffusions satisfying* (1.22) *and a set of three real constants* a, b *and* c *such that* $a\geqq 0$, $c\geqq 0$; $\mathbb{P}^{(\alpha)}$ *is the CBI-process corresponding to* $[R, \alpha F]$ *where* $R(\lambda)=-a\lambda^2+b\lambda$ *and* $F(\lambda)=c\lambda$.

The diffusion $\mathbb{P}^{(\alpha)}$ is characterized as the diffusion generated by the differential operator

$$L_\alpha=a x \frac{d^2}{dx^2}+(b x+c\alpha)\frac{d}{dx} \tag{1.23}$$

with the domain

$$\mathscr{D}(L)=\mathbb{C}_0^2[0, \infty), \tag{1.24}$$

$\mathbb{C}_0^2[0, \infty)$ is the space of all twice continuously differentiable functions on $[0, \infty)$ with compact support.

Also, we can characterize $\mathbb{P}^{(\alpha)}=\{P_x^{(\alpha)}\}_{x\in\mathbb{R}^+}$ as follows: $P_x^{(\alpha)}$ is the probability law of a solution of the following stochastic differential equation

$$\begin{aligned} dx_t&=\sqrt{2a}\,(x_t^+)^{\frac{1}{2}}\, dB_t+(b\, x_t+c\,\alpha)\, dt\\ x_0&=x, \end{aligned} \tag{1.25}$$

where $x^+=x\vee 0$. It is well known that the pathwise uniqueness holds for solutions of (1.25) (cf. e.g. [6]).

Using the fact that $P_x^{(\alpha)}$ is the law of a solution of the Eq. (1.25), we can give the following direct proof that $\mathbb{P}^{(\alpha)}=\{P_x^{(\alpha)}\}$ satisfies (1.22). Assume $a>0$ and $c>0$. Let $B_t^{(1)}$ and $B_t^{(2)}$ be mutually independent Brownian motions and consider the following two stochastic differential equations:

$$\begin{aligned} dx_t^{(1)} &= \sqrt{2a}\,(x_t^{(1)+})^{\frac{1}{2}}\, dB_t^{(1)} + (b\, x_t^{(1)} + c\,\alpha)\, dt, \\ x_0^{(1)} &= x, \end{aligned} \tag{1.26}$$

$$\begin{aligned} dx_t^{(2)} &= \sqrt{2a}\,(x_t^{(2)+})^{\frac{1}{2}}\, dB_t^{(2)} + (b\, x_t^{(2)} + c\,\beta)\, dt, \\ x_0^{(2)} &= y. \end{aligned} \tag{1.27}$$

$x_t^{(1)}$ and $x_t^{(2)}$ are functionals of $\{B_t^{(1)}\}$ and $\{B_t^{(2)}\}$ respectively and hence they are mutually independent. Now, set

$$x_t = x_t^{(1)} + x_t^{(2)} \quad \text{and} \quad d\tilde{B}_t = \frac{\{x_t^{(1)+}\}^{\frac{1}{2}}\, dB_t^{(1)} + \{x^{(2)+}\}^{\frac{1}{2}}\, dB_t^{(2)}}{(x_t^+)^{\frac{1}{2}}}.$$

Then $\tilde{B}_t$ is a one dimensional Brownian motion, and it is known that $P\{x_t^{(i)} > 0\} = 1$ for all $t > 0$, $i = 1, 2$. Moreover

$$\begin{aligned} dx_t &= \sqrt{2a}\,(x_t^+)^{\frac{1}{2}}\, d\tilde{B}_t + \{b\, x_t + c(\alpha+\beta)\}\, dt, \\ x_0 &= x + y. \end{aligned} \tag{1.28}$$

Thus, the probability law of x_t is $P_{x+y}^{(\alpha+\beta)}$, that is,

$$\mathbb{P}^{(\alpha+\beta)} = \mathbb{P}^{(\alpha)} * \mathbb{P}^{(\beta)}.$$

2. Bessel Diffusions as One Parameter Family of Diffusions

Definition 2.1. τ is a mapping $W \to W$ defined by $(\tau w)(t) = w^2(t)$.

Definition 2.2. Let $\mathbb{P}$, $\tilde{\mathbb{P}} \in \mathscr{P}(W)$. We define $\tilde{\mathbb{P}} = \tau \cdot \mathbb{P}$ if for every $x \in \mathbb{R}^+$, $\tilde{P}_x = \tau \cdot P_{\sqrt{x}}$.

Definition 2.3. Let $\mathbb{P}, \mathbb{Q}, \mathbb{R} \in \mathscr{P}(W)$. We define $\mathbb{R} = \mathbb{P} \circledast \mathbb{Q}$ if, for $\tilde{\mathbb{P}} = \tau \cdot \mathbb{P}$, $\tilde{\mathbb{Q}} = \tau \cdot \mathbb{Q}$ and $\tilde{\mathbb{R}} = \tau \cdot \mathbb{R}$, the relation $\tilde{\mathbb{R}} = \tilde{\mathbb{P}} * \tilde{\mathbb{Q}}$ holds.

Thus $\mathbb{R}$ is the probability law of the stochastic process $X(t) = \sqrt{X_1^2(t) + X^2(t)}$, where $X_1(t)$ and $X_2(t)$ are mutually independent stochastic processes governed by the laws $\mathbb{P}$ and $\mathbb{Q}$ respectively, when $X(t)$ is well defined, that is, the law of $X(t)$ is the same for the same initial value.

Now we want to determine all one-parameter families $\{\mathbb{P}^{(\alpha)}\}_{\alpha \geqq 0}$ of Markov processes or diffusions which satisfy

$$\mathbb{P}^{(\alpha+\beta)} = \mathbb{P}^{(\alpha)} \circledast \mathbb{P}^{(\beta)} \qquad \forall\, \alpha, \beta \in [0, \infty). \tag{2.1}$$

Since $\tilde{\mathbb{P}}^{(\alpha)} = \tau \cdot \mathbb{P}^{(\alpha)}$ satisfies (1.22), $\tilde{\mathbb{P}}^{(\alpha)}$ is a CBI process corresponding to $[R, \alpha F]$ for some system of generating functions $[R, F]$. Thus, we can determine all $\{\mathbb{P}^{(\alpha)}\}_{\alpha \geqq 0} \subset \mathscr{P}_M(W)$ which satisfy (2.1). In particular, for the case of diffusions, we have the following

Theorem 2.1. *There is a one-to-one correspondence between a one parameter family $\{\mathbb{P}^{(\alpha)}\}_{\alpha \geqq 0}$ of diffusions which satisfies* (2.1) *and a set of three real constants a, b and c such that $a \geqq 0$, $c \geqq 0$; $\mathbb{P}^{(\alpha)}$ is a diffusion process determined by the local* *generator*

$$L_\alpha = \frac{1}{4} a \frac{d^2}{dx^2} + \frac{1}{2} b x \frac{d}{dx} + \frac{1}{4x} (2c\alpha - a) \frac{d}{dx} \tag{2.2}$$

with the point 0 *as*

$$\begin{aligned} &\textit{a trap if} && c\alpha = 0. \\ &\textit{a reflecting or entrance boundary if } && c\alpha > 0. \end{aligned} \tag{2.3}$$

Proof. There is a one-to-one correspondence between a family of diffusions satisfying (1.22) and a family of diffusions satisfying (2.1) by the mapping τ. Then, the theorem follows at once from Theorem 1.2 if we note that if $X^{(\alpha)}(t)$ is the diffusion given by (1.23) and (1.24) then $Y^{(\alpha)}(t) = \sqrt{X^{(\alpha)}(t)}$ is the diffusion given by (2.2) and (2.3).

For simplicity, we set $a=2$ and $c=1$. Then

$$L_\alpha = \frac{1}{2} \left(\frac{d^2}{dx^2} + b x \frac{d}{dx} + \frac{\alpha - 1}{x} \frac{d}{dx} \right), \quad \alpha \geqq 0 \tag{2.4}$$

and thus we have the family of Bessel diffusions when $b=0$. When $b<0$, $\mathbb{P}^{(n)}$, $(n=1, 2, \ldots)$ is just the radial part of an n-dimensional Ornstein-Uhlenbeck Brownian motion, i.e. the n-fold direct product of a one-dimensional Ornstein-Uhlenbeck Brownian motion. Indeed, the generator of an n-dimensional Ornstein-Uhlenbeck Brownian motion is given by

$$\frac{1}{2} \left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + b \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right) \tag{2.5}$$

and its radial part is just L_n.

3. An Invariance of Bessel Diffusions under the Inversion of Time and Its Applications

We will now apply the property (2.1) of the family of Bessel diffusions to prove the following

Theorem 3.1. *Let* $\mathbb{P}^{(\alpha)} = \{P_x^{(\alpha)}\}$ *be a Bessel diffusion of index* α, *i.e., a diffusion on* $[0, \infty)$ *given by* (0.1) *and* (0.2). *Then the processes* $\{X(t)\}_{t \geqq 0}$ *and* $\left\{ t X\left(\frac{1}{t}\right) \right\}_{t \geqq 0}$ *are equivalent with respect to* $P_0^{(\alpha)}$.

Proof. The case when $\alpha = 0$ is trivial and we assume $\alpha > 0$. Let α be a positive integer n. If $B_1(t)$, $B_2(t)$, ..., $B_n(t)$ are n-independent copies of one-dimensional Brownian motions $(B_i(0)=0)$, then the process $X(t)$ defined by

$$X(t) = \sqrt{B_1^2(t) + B_2^2(t) + \cdots + B_n^2(t)} \tag{3.1}$$

has the law $P_0^{(n)}$. As is well known, for a one-dimensional Brownian motion $B(t)$ $(B(0)=0)$, the processes $\{B(t)\}$ and $\left\{ t B\left(\frac{1}{t}\right) \right\}$ are equivalent: indeed, both are centered Gaussian processes with the same covariance $\min(t,s)$. Since

$$tX\left(\frac{1}{t}\right)=\sqrt{\left\{tB_1\left(\frac{1}{t}\right)\right\}^2+\left\{tB_2\left(\frac{1}{t}\right)\right\}^2+\cdots+\left\{tB_n\left(\frac{1}{t}\right)\right\}^2} \tag{3.2}$$

the assertion of the theorem clearly holds when $\alpha=n$.

Now let n and m be two positive integers and $Y_1(t), Y_2(t), \ldots, Y_m(t)$ be m-independent copies of a process with the law $P_0^{(n/m)}$. Then, since

$$\mathbb{P}^{(n)}=\underbrace{\mathbb{P}^{(n/m)}\circledast\mathbb{P}^{(n/m)}\circledast\cdots\circledast\mathbb{P}^{(n/m)}}_{m},$$

the process $X(t)$, defined by

$$X(t)=\sqrt{Y_1^2(t)+Y_2^2(t)+\cdots+Y_m^2(t)}, \tag{3.3}$$

has the law $P_0^{(n)}$. Since

$$tX\left(\frac{1}{t}\right)=\sqrt{\left\{tY_1\left(\frac{1}{t}\right)\right\}^2+\left\{tY_2\left(\frac{1}{t}\right)\right\}^2+\cdots+\left\{tY_m\left(\frac{1}{t}\right)\right\}^2}, \tag{3.4}$$

the equivalence in law of $\{X(t)\}$ and $\left\{tX\left(\frac{1}{t}\right)\right\}$ clearly implies that of $\{Y_i(t)\}$ and $\left\{tY_i\left(\frac{1}{t}\right)\right\}$. Thus the assertion of the theorem holds when $\alpha=\frac{n}{m}$. Now, the assertion of the theorem follows at one from the continuity of $P_0^{(\alpha)}$ in α in the sense of continuity in α of $E_0^{(\alpha)}(e^{-\lambda_1 X(t_1)-\lambda_2 X(t_2)-\cdots-\lambda_n X(t_n)})$ for every $0<t_1<t_2<\cdots<t_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n>0$. q.e.d.

Of course, there are several different proofs of the theorem; e.g., one can prove it by a direct calculation using the following explicit formula of the transition probability density (cf. [1] or [3])

$$p(t,x,y)=\frac{P_x^{(\alpha)}(X_t\in dy)}{dy}=\frac{\exp[-(x^2+y^2)/2t]}{t(xy)^{\frac{\alpha}{2}-1}}\,y^{\alpha-1}\,I_{\frac{\alpha}{2}-1}\left(\frac{xy}{t}\right), \tag{3.5}$$

where I_ν is the modified Bessel function. Also, we can prove it by noting that, if $Y(t)=\left[\frac{X(t)}{2}\right]^2$, then $E_x(e^{-\lambda_1 Y(t_1)-\lambda_2 Y(t_2)-\cdots-\lambda_n Y(t_n)})$ is given in the form (1.20) where $\psi(t,\lambda)=\lambda(1+\frac{1}{2}(t\lambda))^{-1}$ and $\varphi(t,\lambda)=(1+\frac{1}{2}(t\lambda))^{-\alpha/2}$ (cf. [2] ex. 1.1).

In the following, we will apply Theorem 3.1 to study the local property of sample functions of Bessel diffusions and related diffusion processes. The following theorem is essentially due to Motoo (cf. [4] and also [1], where the proof was given when α is a positive integer but clearly applies for general $\alpha>0$).

Theorem 3.2. *Let $\mathbb{P}^{(\alpha)}=\{X_t, P_x^{(\alpha)}\}$ be a Bessel diffusion of index α $(\alpha>0)$.*

(i) *Let $\varphi(t)\uparrow\infty$ when $t\uparrow\infty$. Then,*

$$P_0^{(\alpha)}\{X(t)>\sqrt{t}\,\varphi(t)\ \ i.o.,\ t\uparrow\infty\}=1 \quad or \ \ 0 \tag{3.6}$$

according as

$$\int^{\infty} \varphi(t)^{\alpha}\, e^{-\frac{1}{2}\varphi^2(t)} \frac{dt}{t} = \infty \quad \text{or} \; < \infty. \tag{3.7}$$

(ii) *Let* $\psi(t)\downarrow 0$ *when* $t\uparrow\infty$. *Then for* $\alpha \geqq 2$,

$$P_0^{(\alpha)}\{X(t) < \sqrt{t}\,\psi(t),\ i.o.,\ t\uparrow\infty\} = 1 \quad \text{or } 0 \tag{3.8}$$

according as

$$\int^{\infty} \psi(t)^{\alpha-2} \frac{dt}{t} = \infty \quad \text{or} \; < \infty \qquad (\alpha > 2)$$

$$\int^{\infty} \frac{1}{|\log\psi(t)|}\cdot\frac{dt}{t} = \infty \quad \text{or} \; < \infty \qquad (\alpha = 2). \tag{3.9}$$

Now, combining this with Theorem 3.1, we have

Theorem 3.3. (i) *Let* $\varphi(t)\uparrow\infty$ *when* $t\downarrow 0$. *Then*

$$P_0^{(\alpha)}\{X(t) > \sqrt{t}\,\varphi(t),\ i.o.,\ t\downarrow 0\} = 1 \quad \text{or } 0, \tag{3.10}$$

according as

$$\int_{0+} \varphi(t)^{\alpha}\, e^{-\frac{1}{2}\varphi^2(t)} \frac{dt}{t} = \infty \quad \text{or} \; < \infty. \tag{3.11}$$

(ii) *Let* $\psi(t)\downarrow 0$ *when* $t\downarrow 0$. *Then, for* $\alpha \geqq 2$,

$$P_0^{(\alpha)}\{X(t) < \sqrt{t}\,\psi(t),\ i.o.\ t\downarrow 0\} = 1 \quad \text{or } 0, \tag{3.12}$$

according as

$$\int_{0+} \psi(t)^{\alpha-2} \frac{dt}{t} = \infty \quad \text{or} \; < \infty \qquad (\alpha > 2)$$

$$\int_{0+} \frac{1}{|\log\psi(t)|}\cdot\frac{dt}{t} = \infty \quad \text{or} \; < \infty \qquad (\alpha = 2). \tag{3.13}$$

Next, let $X(t)$ be a Bessel diffusion of index α ($\alpha > 0$) such that $X(0) = 0$ and set

$$Y(t) = \left(\tfrac{1}{2}X(t)\right)^2. \tag{3.14}$$

$Y(t)$ is a sample path, starting at 0, of the CBI-diffusion on $[0, \infty)$ determined by the generator

$$L = \frac{1}{2}\left(x\frac{d^2}{dx^2} + \beta\frac{d}{dx}\right), \qquad \beta = \frac{\alpha}{2}. \tag{3.15}$$

Thus we have:

Corollary 1. (i) *Let* $\varphi(t)\uparrow\infty$ *when* $t\downarrow 0$. *Then*

$$P\{Y(t) > t\,\varphi(t),\ i.o.\ t\downarrow 0\} = 1 \quad \text{or } 0$$

according as

$$\int_{0+} \varphi(t)^{\beta}\, e^{-2\varphi(t)} \frac{dt}{t} = \infty \quad \text{or} < \infty. \tag{3.16}$$

(ii) *Let* $\psi(t)\downarrow 0$ *when* $t\downarrow 0$. *Then for* $\beta \geqq 1$,

$$P\{Y(t) < t\,\psi(t),\ i.o.,\ t\downarrow 0\} = 1 \quad \text{or } 0 \tag{3.17}$$

according as

$$\int_{0+}\psi(t)^{\beta-1}\frac{dt}{t}=\infty \quad or <\infty \qquad (\beta>1)$$
$$\int_{0+}\frac{1}{|\log\psi(t)|}\cdot\frac{dt}{t}=\infty \quad or <\infty \qquad (\beta=1). \tag{3.18}$$

Finally, let $X(t)$ be a Bessel diffusion of index α $(0<\alpha<2)$ such that $X(0)=0$ and set

$$Z(t)=2^{\frac{1}{2}(2-\alpha)}(2-\alpha)^{-(2-\alpha)}X(t)^{2-\alpha}. \tag{3.19}$$

Then $Z(t)$ is a sample path, starting at 0, of the diffusion on $[0,\infty)$ determined by the local generator

$$L=x^{1-\gamma}\frac{d^2}{dx^2}, \qquad \gamma=\frac{\alpha}{2-\alpha} \tag{3.20}$$

(γ ranges over $(0,\infty)$) with the point 0 as a reflecting boundary. Thus we have,

Corollary 2. *Let $\varphi(t)\downarrow 0$ and $\varphi(t)^{1+\gamma}t^{-1}\uparrow\infty$ when $t\downarrow 0$. Then*

$$P\{Z(t)>\varphi(t),\ i.o.,\ t\downarrow 0\}=1 \quad or\ 0 \tag{3.21}$$

according as

$$\int_{0+}t^{-\frac{1+2\gamma}{1+\gamma}}\varphi(t)^{\gamma}\exp\left[-\frac{\varphi(t)^{1+\gamma}}{t(1+\gamma^2)}\right]dt=\infty \quad or <\infty. \tag{3.22}$$

Or, setting

$$\varphi(t)=\left[(1+\gamma)^2\, t\log\frac{1}{\rho(t)}\right]^{\frac{1}{1+\gamma}} \tag{3.23}$$

where $\rho(t)\downarrow 0$ when $t\downarrow 0$, (3.21) *holds according as*

$$\int_{0+}\frac{1}{t}\left(\log\frac{1}{\rho(t)}\right)^{\frac{\gamma}{1+\gamma}}\rho(t)\,dt=\infty \quad or <\infty. \tag{3.24}$$

Corollary 2 completes the result of Vencel' [5].

References

1. Itô, K., McKean, H.P., Jr.: Diffusion processes and their sample paths. Berlin-Heidelberg-New York: Springer 1965
2. Kawazu, K., Watanabe, S.: Branching processes with immigration and related limit theorems. Teor. Verojatnost. i Primenen. **16**, 34–51 (1971)
3. Molchanov, S.A., Ostrovskii, E.: Symmetric stable processes as traces of degenerate diffusion processes. Theor. Propability Appl., **14**, 128–131 (English translation) (1969)
4. Motoo, M.: Proof of the law of iterated logarithm through diffusion equation. Ann. Inst. Statist. Math. **10**, 21–28 (1959)
5. Vencel', A.D.: On the local behavior of the trajectories of diffusion processes. (In Russian.) All Union Congress on Probability Theory and Mathematical Statistics. Erevan 1960, 236–238
6. Yamada, T., Watanabe, S.: On the uniqueness of solutions of stochastic differential equations. J. Math. Kyoto Univ. **11**, 155–167 (1971)

Tokuzo Shiga
Department of Mathematics
Faculty of Science
Nara Women's University
630 Nara, Japan

Shinzo Watanabe
Department of Mathematics
Faculty of Science
Kyoto University
606 Kyoto, Japan

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