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Canonical Representations and Convergence Criteria for Processes with Interchangeable Increments

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The elements of a finite or infinite sequence of random variables (r.v.) or more generally random elements (r.e.) are said to be *interchangeable* (ich., also exchangeable), if their joint distribution is invariant under finite permutations of elements. Furthermore, a random process (r.pr.) defined on some real interval has *interchangeable increments* if its increments (incr.) over disjoint sub-intervals of equal length are ich. r.v. Interchangeability is one of the most natural extensions of the concept of independence, being sufficiently general to provide a unifying framework for the theories of infinite divisibility, empirical distributions and sampling from finite populations, and yet sufficiently restrictive to admit an explicit treatment in terms of canonical (can.) representations.

Since the classical work of de Finetti [8], characterizations and limit theorems have been given by several authors, including Loève [17], Bühlmann [3], Blum, Chernoff, Rosenblatt, Teicher [2, 4], Billingsley [1], Davidson [5] and myself [12, 13]. Here a unified theory is presented based on representations in terms of random measures (r.m.) and point processes (p.pr.), playing the roles of can. r.e. and extending the notions of distributions, Poisson spectra and populations in the classical theories. The various parts of the theory are connected by criteria for convergence in distribution, and also by results on restriction and extension. In particular, we generalize results by Prohorov [20] and Skorohod [23] on r.pr. with independent (ind.) incr. and by Hájek [10], Rosén [22] and Hagberg [9] on sampling from finite populations.

Throughout the paper we write $\stackrel{d}{=}$ and $\stackrel{d}{\to}$ for equality and convergence in distribution of r.e. [1]. The spaces R, R_+ and $R'=R\setminus\{0\}$ are endowed with the usual topologies, $D[0,1]$ and $D[0,\infty)$ with the Skorohod J_1 topology [1] and its natural extension [16]. Product spaces are taken with their product topologies. For any locally compact second countable Hausdorff (lcscH) space S, let $\mathfrak{M}(S)$ and $\mathfrak{N}(S)$ be the spaces of R_+- and Z_+-valued Radon Borel measures on S [12], endowed with the vague or weak topologies. Writing $\mathscr{F}(S)$ for the class of bounded continuous functions $S\to R_+$ with compact support, *vague* convergence, $m_n \stackrel{v}{\to} m$, means $m_n f\to mf$, $f\in\mathscr{F}(S)$, while *weak* convergence, $m_n \stackrel{w}{\to} m$, means $m_n f\to mf$, $f\in\mathscr{F}(\bar S)$. (Here $mf=\int f(s)\,m(ds)$, and $\bar S$ is the one-point compactification of S.) Note that, for finite measures, $m_n \stackrel{w}{\to} m$ iff $m_n \stackrel{v}{\to} m$ *and* $m_n S\to mS$, but also iff $m_n \stackrel{v}{\to} m$ in $\mathfrak{M}(\bar S)$ *and* $m(\bar S\setminus S)=0$. R.e. in $\mathfrak{M}(S)$ and $\mathfrak{N}(S)$ in either topology are called r.m. and p.pr. respectively [12], and we write $\xrightarrow{vd}$ and $\xrightarrow{wd}$ for convergence in distribution in the two topologies. In product spaces, this notation refers to the measure component.

We list some further notations. Let δ_x be the Dirac measure with unit atom at x. Let 1_A be the indicator of the set A and put $1_+=1_{R_+}$. Write $I_\varepsilon=[-\varepsilon,\varepsilon]$ and define

$$h(x)\equiv x,\quad h_\varepsilon=h1_{I_\varepsilon},\quad g_1(x)\equiv x/(1+x^2),\quad g_2(x)\equiv x^2/(1+x^2).$$

For $m\in\mathfrak{M}(S)$ and measurable $f\colon S\to R$, define the (signed) measure fm by $(fm)(ds)\equiv f(s)\,m(ds)$, and put $m^1=hm$, $m^2=h^2m$. A *lattice* in R_+ is a set of the form $\{a+bz\colon z\in Z_+\}$, $a\geqq 0$, $b>0$, and $\mathcal{T}_1,\mathcal{T}_\infty$ denote the classes of sets in $[0,1]$ and R_+ respectively which are dense in some interval or contain some lattice. For $I=[0,1]$ or R_+, we define $D_0(I)=\{f\in D(I)\colon f(0)=0\}$, and similarly for $C_0(I)$. For any $f\colon R\to R$, $f(t_1,\ldots,t_k)$ means the vector $(f(t_1),\ldots,f(t_k))$.

1. Interchangeable Random Elements

Throughout this section, let S be a lcscH space with Borel algebra $\mathcal{S}$. We start with a can. representation, extending and improving a wellknown result by de Finetti [8], Loève [17], p. 365, and Bühlmann [3]. (See also [6, 11, 15, 21].)

Theorem 1.1. *Let $\xi_1,\xi_2,\ldots$ be ich.r.e. in S. Then there exists some a.s. unique r.m. μ on S which is measurable on $\{\xi_j\}$ and such that, given μ, the ξ_j are conditionally ind. with common distribution μ.*

We shall call μ the *can.r.m.* of $\{\xi_j\}$. A similar role for finite sequences $\xi_1,\ldots,\xi_n$ of ich.r.e. is played by the *can.p.pr.* $\pi=\sum_{j=1}^{n}\delta_{\xi_j}$ on S.

Partial Proof. Suppose that, given some μ, the ξ_j are conditionally ind. with distribution μ. By the strong law of large numbers,

$$\frac{1}{n}\sum_{j=1}^{n}\delta_{\xi_j}A\to\mu A\ \text{a.s.},\qquad A\in\mathcal{S}. \tag{1.1}$$

Replacing $\mathcal{S}$ by some countable DC-semi-ring $\mathcal{I}$ [12] and applying Dynkin's theorem, it follows that μ is a.s. unique and measurable on $\{\xi_j\}$.

To conclude the proof, we shall use the following result, extending L.3.1 of Rosén [22].

Theorem 1.2. *For $n\in N$, let $r_n\in N$ and let $X_n=(\xi_{n1},\xi_{n2},\ldots)$ be such that ξ_{nj}, $j=1,\ldots,r_n$, are ich.r.e. in S with can.p.pr. π_n. Suppose that $r_n\to\infty$. Then $X_n\xrightarrow{d}$ some X in S^∞ iff $\pi_n/r_n\xrightarrow{wd}$ some μ in $\mathfrak{M}(S)$. In this case, X has can.r.m. μ.*

Lemma 1.1. *Let S_1,S_2,S_3 be measurable spaces and suppose that S_2,S_3 are metric Borel. Let $\xi,\xi_1,\xi_2,\ldots$ and $\eta,\eta_1,\eta_2,\ldots$ be ind. sequences of r.e. in S_1 and S_2 respectively, and let $\varphi,\varphi_1,\varphi_2,\ldots\colon S_1\times S_2\to S_3$ be measurable. Further suppose that $\eta_n\xrightarrow{d}\eta$ and that $S_2'\subset S_2$ with $\eta\in S_2'$ a.s. is such that $\varphi_n(\xi_n,y_n)\xrightarrow{d}\varphi(\xi,y)$ whenever $y_n\to y$, $y_1,y_2,\ldots\in S_2$, $y\in S_2'$. Then $\varphi_n(\xi_n,\eta_n)\xrightarrow{d}\varphi(\xi,\eta)$.*

Proof. Let $f\colon S_3\to R$ be bounded and continuous and define $\psi(y)=\mathsf{E}f\circ\varphi(\xi,y)$, $\psi_n(y)=\mathsf{E}f\circ\varphi_n(\xi_n,y)$, $y\in S_2$, $n\in N$. By assumption, $y_n\to y\in S_2'$ implies $\psi_n(y_n)\to\psi(y)$, so Th. 5.5 in [1] yields $\psi_n(\eta_n)\xrightarrow{d}\psi(\eta)$, which in turn implies $\mathsf{E}\psi_n(\eta_n)\to\mathsf{E}\psi(\eta)$ since the ψ_n are uniformly bounded. But by Fubini's theorem, this is equivalent to $\mathsf{E}f\circ\varphi_n(\xi_n,\eta_n)\to\mathsf{E}f\circ\varphi(\xi,\eta)$, and the assertion follows.

Proof of Th. 1.2 (with X and μ related as in Th. 1.1). First assume that $\pi_1, \pi_2, \ldots$ are non-random with $\pi_n/r_n \xrightarrow{w} \mu$. We then have to prove that $(\xi_{n1}, \ldots, \xi_{nk}) \xrightarrow{d} (\xi_1, \ldots, \xi_k)$, $k \in N$, where the ξ_j are ind. with distribution μ (cf. [1], p. 19), and by Th. 3.1 in [1], this is equivalent to

$$\mathsf{P} \bigcap_{j=1}^{k} \{\xi_{nj} \in A_j\} \to \prod_{j=1}^{k} \mu A_j, \quad k \in N, \quad A_j \in S, \quad \mu \partial A_j = 0, \quad j = 1, \ldots, k.$$

For $k=2$ it suffices to consider the cases $A_1 = A_2 = A$ and $A_1 \cap A_2 = \emptyset$, and we get respectively

$$\mathsf{P}\{\xi_{n1} \in A, \xi_{n2} \in A\} = \frac{\pi_n A}{r_n} \frac{\pi_n A - 1}{r_n - 1} \to (\mu A)^2,$$

$$\mathsf{P}\{\xi_{n1} \in A_1, \xi_{n2} \in A_2\} = \frac{\pi_n A_1}{r_n} \frac{\pi_n A_2}{r_n - 1} \to \mu A_1 \, \mu A_2 .$$

The proof for general k is similar. To extend the result to random $\{\pi_n\}$, use L. 1.1 with $S_2 = \mathfrak{M}(S)$, $S_2' = \{m \in \mathfrak{M}(S) \colon mS = 1\}$, $S_3 = S^\infty$, and S_1 as a suitable space supporting the randomizations leading from μ to X and from π_n to X_n, $n \in N$.

Conversely, suppose that $X_n \xrightarrow{d} X$ and note that $\{\mu_n = \pi_n/r_n\}$ is vaguely tight [12], so that any sequence $N' \subset N$ must contain a sub-sequence N'' with $\mu_n \xrightarrow{vd}$ some μ, $n \in N''$. If $\mathsf{P}\{\mu S < 1 - \varepsilon\} \geqq \varepsilon$ for some $\varepsilon > 0$, we get for any compact $F \subset S$

$$\liminf_{n \in N''} \mathsf{P}\{\mu_n F^c > \varepsilon\} = \liminf_{n \in N''} \mathsf{P}\{\mu_n F < 1 - \varepsilon\} \geqq \mathsf{P}\{\mu F < 1 - \varepsilon\} \geqq \mathsf{P}\{\mu S < 1 - \varepsilon\} \geqq \varepsilon,$$

since the set $\{m F < 1 - \varepsilon\}$ is open in $\mathfrak{M}(S)$ [12]. But since

$$\mathsf{P}\{\xi_{n1} \in F^c\} = \mathsf{E}\, \mu_n F^c \geqq \varepsilon \, \mathsf{P}\{\mu_n F^c > \varepsilon\}, \qquad n \in N,$$

this implies $\liminf_{n \in N''} \mathsf{P}\{\xi_{n1} \in F^c\} \geqq \varepsilon^2$, contrary to the tightness of $\{\xi_{n1}\}$, and hence $\mu S = 1$ a.s. The uniqueness of μ now follows from the direct assertion and the uniqueness in Th. 1.1, so we get $\mu_n \xrightarrow{wd} \mu$ by Th. 2.3 in [1].

End of the Proof of Th. 1.1. From Th. 1.2 it is seen that $\{\xi_j\}$ is *distributed* as a sequence of the asserted type. We may therefore use (1.1) (with $\mathscr{I}$ in place of $\mathscr{S}$) to *define* a r.m. μ, which will automatically possess the asserted properties.

Theorem 1.3. *Let $r \in \bar{N}$ and suppose that, for $n \in N$, X_n is an r-sequence of ich. r.e. in S with can. r.m. μ_n (p.pr. π_n). Then $X_n \xrightarrow{d}$ some X in S^r iff $\mu_n \xrightarrow{wd}$ some μ in $\mathfrak{M}(S)$ $(\pi_n \xrightarrow{wd}$ some π in $\mathfrak{N}(S))$. In this case, X has can. r.m. μ (p.pr. π).*

Proof. Proceeding as in the proof of Th. 1.2, it suffices to show that, for non-random $\{\mu_n\}$ $(\{\pi_n\})$, $\mu_n \xrightarrow{w} \mu$ $(\pi_n \xrightarrow{w} \pi)$ implies $X_n \xrightarrow{d} X$. But this follows easily from the definition of weak convergence for $r < \infty$ and from Th. 3.2 in [1] for $r = \infty$.

2. Finite Interval Processes with Interchangeable Increments

The following can. representation extends results by Davidson [5] and myself [12, 13] for r.m. and p.pr. (Compare [7, 14].)

Theorem 2.1. *If a r.pr. on* $[0, 1]$ *is separated by the binary rationals and has ich. incr., then its sample paths are a.s. continuous except for jumps and its equivalent version in* $D[0, 1]$ *has the form*

$$X(t)=X(0)+\alpha t+\sigma B(t)+\sum_{j=1}^{\infty}\beta_j[1_+(t-\tau_j)-t], \qquad t\in[0,1], \tag{2.1}$$

in the sense of a.s. uniform convergence, where

(i) $\alpha\in R$, $\sigma\in R_+$, $\beta_1\leqq\beta_3\leqq\cdots\leqq 0\leqq\cdots\leqq\beta_4\leqq\beta_2$ *are r.v. with* $\sum_j \beta_j^2<\infty$ *a.s.,*

(ii) B *is a Brownian bridge on* $[0, 1]$,

(iii) $\tau_1, \tau_2, \ldots$ *are ind. and uniformly distributed r.v. on* $[0, 1]$,

(iv) *the three groups* (i)–(iii) *of r.e. are ind.*

The sum in (2.1) *is a.s. invariant under non-random permutations of terms, and the r.e. occuring in* (2.1) *are a.s. unique and measurable (except for values of* B *as* $\sigma=0$ *and of* τ_j *as* $\beta_j=0, j\in N$, *and for the order among* τ_j *with equal* β_j*). Conversely, formula* (2.1) *subject to* (i)–(iv) *determines a r.pr. in* $D[0, 1]$ *with ich.incr.*

Corollary. *A r.pr. on* $[0, 1]$ *has ich.incr. and is continuous in probability iff it is equivalent to some r.pr.* X *as in Th.* 2.1.

We introduce the p.pr. $\beta=\sum_j \delta_{\beta_j}$ on R' and call α, σ, β the *can.r.e.* of $X-X(0)$.

Partial Proof[1] *of Th. 2.1.* Consider non-random α, σ, β satisfying (i)–(iv). By Th. 3 in [9], the right side of (2.1) converges in distribution to some r.e. Y in $D[0, 1]$, and Y has clearly ich. incr. Since the set of fixed discontinuities of Y is at most countable ([1], p. 124), it must be empty by interchangeability, and so (2.1) holds for some X in the sense of a.s. uniform convergence by [14]. Since $\beta^2 R<\infty$, the L_2-limit exists for each t and is invariant under permutations, so the a.s. limit has the same properties, and the a.s. invariance in (2.1) follows by right continuity. These results on interchangeability, a.s. uniform convergence and invariance under permutations extend to random α, σ, β by Fubini's theorem.

To prove the assertion on uniqueness and measurability, let X be defined by (2.1), and note that $\alpha=X(1)-X(0)$ a.s. and that, by (iii), the pairs (β_j, τ_j) a.s. represent the sizes and positions of the jumps, which are clearly measurable (cf. [13]). Next reduce by subtraction to the case $X=\sigma B$, and note that X has a.s. square variation σ^2 since any Brownian motion has a.s. square variation 1 ([17], p. 559). We finally get $B=X/\sigma$ whenever $\sigma\neq 0$. Now suppose that X' is another r.pr. in $D[0, 1]$ such that $X'\overset{d}{=}X$. Starting from X' and proceeding as above, we may then construct r.e. X', α', β', $\{\tau_j'\}$, σ', B' which are jointly distributed as X, α, β, $\{\tau_j\}$, σ, B, and since (2.1) is a property of this joint distribution, it must hold for X' as well.

To complete the proof, consider r.pr. in $D[0, 1]$ of the form

$$X_n(t)=\sum_{j\leqq r_n t}\xi_{nj}, \qquad t\in[0,1]. \tag{2.2}$$

[1] My first proof of Th. 2.1 extended the one given in [13]. P. Jagers called my attention to the work of Hagberg [9].

The following theorem extends results by Prohorov ([20], p. 197), Skorohod ([23], Th. 2.7). Billingsley ([1], Th. 24.2) and Hagberg ([9], Th. 5). (See also [18].) To avoid repetitions we introduce the conditions

$$\text{(C)}\ X_n(t) \xrightarrow{d} \text{some } \zeta_t,\ t \in T^k,\ k \in N, \qquad \text{(D)}\ X(t) \overset{d}{=} \zeta_t,\ t \in T^k,\ k \in N.$$

Theorem 2.2. *For $n \in N$, let $r_n \in N$, let ξ_{nj}, $j=1, \ldots, r_n$, be ich.r.v. with can.p.pr. π_n and define X_n by (2.2). Suppose that $r_n \to \infty$ and that $T \in \mathscr{T}_1$. Then $X_n \xrightarrow{d}$ some X iff*

$$(\pi_n^1 R, \pi_n^2) \xrightarrow{wd} \textit{some } (\alpha, \sigma^2 \delta_0 + \beta^2) \textit{ in } R \times \mathfrak{M}(R), \tag{2.3}$$

and also iff (C) *holds. In this case, X has can.r.e. α, σ, β and* (D) *holds.*

Note that (2.3) is equivalent to

$$(\pi_n^1 R, \pi_n^2 R, \pi_n) \xrightarrow{vd} \text{some } (\alpha, \sigma^2 + \beta^2 R, \beta) \text{ in } R \times R_+ \times \mathfrak{N}(R'),$$

or more explicitly

$$\Big(\sum_j \xi_{nj}, \sum_j \xi_{nj}^2, \tilde{\xi}_{n1}, \tilde{\xi}_{n2}, \ldots\Big) \xrightarrow{d} \text{some } \Big(\alpha, \sigma^2 + \sum_j \beta_j^2, \beta_1, \beta_2, \ldots\Big) \text{ in } R^\infty,$$

where the $\tilde{\xi}_{nj}$ are obtained by ordering $\{\xi_{nj}\}$ just as $\{\beta_j\}$ was ordered in Th. 2.1. A similar reformulation in terms of vague convergence is possible in all subsequent limit theorems. To appreciate (C), compare Th. 15.1 in [1].

Proof. We first prove that (2.3) implies $X_n \xrightarrow{d} X$. For non-random $\{\pi_n\}$ with $\pi_n^1 R \equiv 0$, this is Th. 5 in [9]. If $\pi_n^1 R \equiv \alpha_n \to \alpha$, define X_n' as X_n but with ξ_{nj} replaced by $\xi_{nj}' = \xi_{nj} - \alpha_n / r_n$, and verify that the corresponding p.pr. π_n' satisfy $(\pi_n')^1 R \equiv 0$, $(\pi_n')^2 R \to \sigma^2 + \beta^2 R$, $\pi_n' \xrightarrow{v} \beta$, so that $X_n' \xrightarrow{d} X - \alpha h$, and hence $X_n' + \alpha h \xrightarrow{d} X$ by Th. 5.1 in [1]. Now

$$|X_n'(t) + \alpha t - X_n(t)| = |\alpha t + \alpha_n [r_n t]/r_n| \leqq |\alpha - \alpha_n| + |\alpha_n|/r_n \to 0$$

uniformly in t, so we get $X_n \xrightarrow{d} X$ by Th. 4.1 in [1]. To extend to random π_n, use L. 1.1 with $S_2 = R \times \mathfrak{M}(R)$, $S_3 = D[0,1]$ and S_1 as a suitable space supporting B, $\{\tau_j\}$ and the randomizations leading from $\{\pi_n\}$ to $\{\xi_{nj}\}$.

Next suppose that $X_n \xrightarrow{d} X$. Then (C) follows since X is continuous in probability, and in particular we get $\pi_n^1 R = X_n(1) \xrightarrow{d} X(1) = \alpha$. Moreover, $\pi_n \xrightarrow{vd}$ some β. To prove tightness of $\eta_n^2 = \pi_n^2 R$, $n \in N$, suppose on the contrary that the distribution of η_n converges improperly [6] as $n \to \infty$ through some $N' \subset N$. Without loss of generality [24], we may assume that the π_n are all defined on the same probability space and satisfy $\pi_n^1 R \to \alpha$, $\pi_n \xrightarrow{v} \beta$ and $\eta_n \to \eta$ a.s., $n \in N'$, where $\mathsf{P}\{\eta = \infty\} > 0$, and that $\{X_n\}$ is defined from $\{\pi_n\}$ by ind. randomizations. Since conditioning on the set $\{\eta = \infty\}$ preserves the interchangeability and tightness of X_n, we may assume that $\eta_n \to \infty$ a.s. By the direct assertion we get $X_n/\eta_n \xrightarrow{d}$ some Y as $n \in$ some $N'' \subset N'$, where $\mathsf{P}\{Y \neq 0\} > 0$. By the tightness we get for any $t \in [0,1]$

$$\begin{aligned}\mathsf{P}\{Y(t) \neq 0\} &= \lim_{\varepsilon \to 0} \mathsf{P}\{|Y(t)| > \varepsilon\} \leqq \lim_{\varepsilon \to 0} \liminf_{n \in N''} \mathsf{P}\{|X_n(t)/\eta_n| > \varepsilon\} \\ &\leqq \lim_{\varepsilon \to 0} \limsup_{n \in N} \mathsf{P}\{|X_n(t)| > \varepsilon^{-1}\} + \lim_{\varepsilon \to 0} \limsup_{n \in N'} \mathsf{P}\{\eta_n < \varepsilon^{-2}\} = 0,\end{aligned}$$

which yields the contradiction $Y = 0$ a.s., proving tightness of $\{\eta_n\}$. Now (2.3) follows as in the proof of Th. 1.2.

Let us now assume that $\{X_n(t)\}$ is tight for some fixed $t \in (0,1)$. If $\{X_n\}$ is not, then neither is $\eta_n^2 = (\pi_n^1 R)^2 + \pi_n^2 R$, $n \in N$. As above it then suffices to assume that $\eta_n \to \infty$ a.s. and $X_n/\eta_n \xrightarrow{d} Y$, $n \in N'' \subset N$, where Y has a representation of form (2.1) and satisfies $\mathsf{P}\{Y \neq 0\} > 0$. By Fubini's theorem, this implies $\mathsf{P}\{Y(t) \neq 0\} > 0$, which again leads to a contradiction, now proving tightness of $\{X_n\}$. In particular, (C) implies tightness and therefore convergence in distribution of $\{X_n\}$ by Th. 5.3 below, since any limiting process is continuous in probability.

End of the Proof of Th. 2.1. Let X be a r. pr. on $[0,1]$ which is separated by the set T of binary rationals and has ich. incr. Put

$$\xi_{nj} = X(j\,2^{-n}) - X((j-1)\,2^{-n}), \qquad j = 1, \dots, 2^n, \ n \in N,$$

and let X_n be defined by (2.2) with $r_n = 2^n$, $n \in N$. Then (C) is trivially satisfied, so by Th. 2.2, $X_n \xrightarrow{d}$ some Y of form (2.1) satisfying $X(t) \stackrel{d}{=} Y(t)$, $t \in T^k$, $k \in N$. By the proof of Th. 15.8 in [1] it follows that the sample paths of X are a.s. continuous except for jumps. Since X is continuous in probability, its right continuous version is distributed as Y.

Proof of the Corollary to Th. 2.1. If X has ich. incr. and is continuous in probability, then there exists an equivalent separable process X' ([17], p. 507) which has clearly the same properties, and we may take the binary rationals as the separating set ([17], p. 510). By Th. 2.2 there exists an equivalent version X'' in $D[0,1]$ of form (2.1). The converse part is obvious.

Our next theorem extends results by Skorohod ([23], Th. 2.7) and Hagberg ([9], Th. 4).

Theorem 2.3. *For $n \in N$, let X_n be a r. pr. in $D_0[0,1]$ with ich. incr. and can. r. e. $\alpha_n, \sigma_n, \beta_n$. Suppose that $T \in \mathcal{T}_1$. Then $X_n \xrightarrow{d}$ some X iff*

$$(\alpha_n, \sigma_n^2\,\delta_0 + \beta_n^2) \xrightarrow{wd} \text{some } (\alpha, \sigma^2\,\delta_0 + \beta^2) \text{ in } R \times \mathfrak{M}(R), \tag{2.4}$$

and also iff (C) *holds. In this case, X has can. r. e. α, σ, β and* (D) *holds.*

Proof. Suppose that (2.4) holds for some non-random $\alpha_n, \sigma_n, \beta_n$. For any $N' \subset N$ there exist some $N'' \subset N'$ and σ', σ'' with $\sigma'^2 + \sigma''^2 = \sigma^2$ such that $\sigma_n \to \sigma'$, $\beta_n^2 R \to \sigma''^2 + \beta^2 R$, $n \in N''$. Let X' and X'' be ind. with can. r. e. $(\alpha, \sigma', 0)$, $(0, \sigma'', \beta)$, and for $n \in N''$, let X_n' and X_n'' be ind. with can. r. e. $(\alpha_n, \sigma_n, 0)$, $(0, 0, \beta_n)$. Then $X_n' \xrightarrow{d} X'$ is obvious while $X_n'' \xrightarrow{d} X''$ holds by Th. 4 in [9], so we get $(X_n', X_n'') \xrightarrow{d} (X', X'')$ by independence ([1], Th. 3.2), and hence $X_n \stackrel{d}{=} X_n' + X_n'' \xrightarrow{d} X' + X'' \stackrel{d}{=} X$, $n \in N''$, since addition is continuous $C[0,1] \times D[0,1] \to D[0,1]$. By Th. 2.3 in [1] we obtain $X_n \xrightarrow{d} X$, $n \in N$. The remainder of the proof is similar to that of Th. 2.2.

The following tightness criteria which are implicit in Th. 2.2–3 and their proofs will be needed below. Similar results hold in all subsequent limit theorems and also in Th. 1.2–3.

Lemma 2.1. *The sequence $\{X_n\}$ is tight in Th. 2.2 iff $\{\pi_n^1 R\}$ and $\{\pi_n^2 R\}$ are tight, in Th. 2.3 iff $\{\alpha_n\}$, $\{\sigma_n\}$ and $\{\beta_n^2 R\}$ are tight, and in both iff $\{X_n(t)\}$ is tight for some (any) $t \in (0,1)$.*

3. Infinite Interval Processes with Interchangeable Increments

We recall [6, 17] that a r.v. ξ is infinitely divisible iff

$$\log \mathsf{E} e^{iu\xi} = iu\Gamma + \int_R (e^{iux} - 1 - iu g_1(x))(g_2(x))^{-1} \Lambda(dx), \qquad u \in R,$$

for some $\Gamma \in R$ and $\Lambda \in \mathfrak{M}(R)$ with $\Lambda R < \infty$, i.e.

$$\log \mathsf{E} e^{iu\xi} = iu\gamma_\varepsilon - u^2\sigma^2/2 + \int_{R'} (e^{iux} - 1 - iu h_\varepsilon(x)) \lambda(dx), \qquad u \in R,$$

for some (any) $\varepsilon > 0$. Here and below, (Γ, Λ) and $(\gamma_\varepsilon, \sigma, \lambda)$ are related by

$$\Gamma = \gamma_\varepsilon + \lambda(g_1 - h_\varepsilon), \qquad \Lambda = \sigma^2 \delta_0 + g_2 \lambda.$$

For r.pr. with stationary ind.incr. they refer to intervals of unit length.

The following can. representation improves results by Bühlmann [3]. (See also Davidson [5] and myself [12] for the particular case of p.pr.)

Theorem 3.1. *If a r.pr. on R_+ is separated by the binary rationals and has ich. incr., then its sample paths are a.s. continuous except for jumps and its equivalent version X in $D[0, \infty)$ determines a.s. uniquely and measurably a r.v. Γ and a r.m. Λ on R such that, given (Γ, Λ), X has stationary ind. incr. with distribution determined by (Γ, Λ).*

This time we take Γ, Λ or $\gamma_\varepsilon, \sigma, \lambda$ as our can.r.e. The Corollary to Th. 2.1 carries over with obvious changes.

Partial Proof. To prove the uniqueness and measurability of Γ and Λ, note that they are continuously determined from the conditional distribution of $X(1) - X(0)$ ([6], p. 561), which in turn is unique and measurable by Th. 1.1. If X' is distributed as X, we may define can.r.e. Γ', Λ' through these mappings to obtain the desired representation.

The proof is completed as for Th. 2.1 by means of the following result, where we consider r.pr. in $D[0, \infty)$ of the form

$$X_n(t) = \sum_{j \leq r_n t} \xi_{nj}, \qquad t \in R_+. \tag{3.1}$$

Theorem 3.2. *For $n \in N$, let $r_n \in R_+$, let $\xi_{nj}, j \in N$, be ich.r.v. with can.r.m. μ_n and define X_n by* (3.1). *Suppose that $r_n \to \infty$ and that $T \in \mathscr{T}_\infty$. Then $X_n \xrightarrow{d}$ some X iff*

$$r_n(\mu_n g_1, g_2 \mu_n) \xrightarrow{wd} \text{some } (\Gamma, \Lambda) \text{ in } R \times \mathfrak{M}(R), \tag{3.2}$$

and also iff (C) *holds. In this case, X has can.r.e. Γ, Λ and* (D) *holds.*

Proof. By [16] and Th. 5.5 in [1] we may assume that $r_n \in N$, $n \in N$. Then $X_n \xrightarrow{d} X$ iff (C) holds, according to L. 2.1, Th. 5.3 and [16]. For non-random $\{\mu_n\}$, (3.2) implies (C) and therefore $X_n \xrightarrow{d} X$ by [6], p. 564, and this result extends to random $\{\mu_n\}$ by L. 1.1. Finally suppose that (C) holds with $T = N$. Applying Th. 1.3 to the increments $X_n(j) - X_n(j-1)$, $j \in N$, yields $\mu_n^{*r_n} \xrightarrow{wd}$ some μ in $\mathfrak{M}(R)$ ($*$ for convolution power). For non-random $\{\mu_n\}$ this implies $r_n \mu_n g_1 \to \Gamma$, $r_n g_2 \mu_n \xrightarrow{w} \Lambda$ ([6], p. 564), which by [24] extends to random $\{\mu_n\}$.

Theorem 3.3. *For $n \in N$, let X_n be a r.pr. in $D_0[0,\infty)$ with ich. incr. and can.r.e. Γ_n, Λ_n. Suppose that $T \in \mathscr{T}_\infty$. Then $X_n \xrightarrow{d}$ some X iff*

$$(\Gamma_n, \Lambda_n) \xrightarrow{wd} \text{some } (\Gamma, \Lambda) \text{ in } R \times \mathfrak{M}(R),$$

and also iff (C) *holds. In this case, X has can.r.e. Γ, Λ and* (D) *holds.*

Proof. Proceed as in the last proof except that $\mu_n^{*r_n}$, $r_n \mu_n g_1$ and $r_n g_2 \mu_n$ are replaced by μ_n, Γ_n and Λ_n respectively, where μ_n is the r.m. corresponding to Γ_n, Λ_n.

4. Limit Theorems for Processes on Increasing Intervals

The following result extends Th. 5.1 of Hájek [10].

Theorem 4.1. *For $n \in N$, let $r_n \in R_+$ and $m_n \in N$, let ξ_{nj}, $j \in N$, be r.v. such that ξ_{nj}, $j=1,\dots,m_n$, are ich. with can.p.pr. π_n, and let X_n be defined by* (3.1). *Suppose that $r_n \to \infty$ and $c_n = r_n/m_n \to 0$, and that $T \in \mathscr{T}_\infty$. Then $X_n \xrightarrow{d}$ some X iff*

$$c_n(\pi_n g_1, g_2 \pi_n) \xrightarrow{wd} \text{some } (\Gamma, \Lambda) \text{ in } R \times \mathfrak{M}(R), \tag{4.1}$$

and also iff (C) *holds. In this case, X has can.r.e. Γ, Λ, and* (D) *holds.*

Proof. The equivalence of $X_n \xrightarrow{d} X$ and (C) is proved as in Th. 3.2. Next suppose that (4.1) holds for some non-random $\{\pi_n\}$ with uniformly bounded atom positions. For $n \in N$, extending ideas of Hájek [10] and Hagberg [9], let x_{nj}, $j=1,\dots,m_n$, be the atom positions of π_n, let the r.v. τ_{nj}, $j=1,\dots,m_n$, be ind. and uniformly distributed over the set $\{j/r_n : j=1,\dots,m_r\}$, and put

$$Y_n(t) = \sum_{j=1}^{m_n} x_{nj} 1_+(t-\tau_{nj}), \qquad t \in R_+.$$

Furthermore, suppose that the r.v. v_{nj}, $j \in N$, are such that v_{nj}, $j=1,\dots,m_n$, have a multinomial distribution corresponding to m_n trials and equal probabilities m_n^{-1}, and put

$$R_n(t) = \sum_{j \leq r_n t} v_{nj}, \qquad \tilde{Y}_n(t) = \sum_{j \leq R_n(t)} \xi_{nj}, \qquad t \in R_+.$$

Since $\{c_n \pi_n^2 R\}$ is bounded, it follows from Hájek's L. 2.1 [10] that the finite-dimensional distributions of $\{X_n\}$ and $\{\tilde{Y}_n\}$ converge simultaneously, and that their limits agree in case of convergence. Since the restrictions of Y_n and $\tilde{Y}_n$ to $[0, c_n^{-1}]$ are equally distributed, it thus suffices to prove (C) with Y_n in place of X_n.

For arbitrary $m \in N$ and $0 = t_0 < t_1 < \cdots < t_m$, define

$$\xi_{njk} = x_{nj}[1_+(t_k - \tau_{nj}) - 1_+(t_{k-1} - \tau_{nj})], \quad k=1,\dots,m, \; j=1,\dots,m_n, \; n \in N.$$

Proceeding as in § 4 of [10] we get

$$\lim_{n\to\infty} \sum_{j=1}^{m_n} \mathsf{P}\{\xi_{njk} - \mathsf{E}\,\xi_{njk} \in I\} = (t_k - t_{k-1})\,\lambda I, \qquad k=1,\dots,m, \tag{4.2}$$

for any λ-continuity interval $I \subset R$ which is bounded away from the origin. Further, by the uniform boundedness of x_{nj},

$$\lim_{n\to\infty} \sum_{j=1}^{m_n} \operatorname{Var} \xi_{njk} = (t_k - t_{k-1})(\sigma^2 + \lambda^2 R), \qquad k=1,\dots,m, \tag{4.3}$$

and also

$$\mathsf{E}\,\xi_{njk} \sim (t_k - t_{k-1})\, c_n x_{nj} \to 0, \qquad n \to \infty,\ k = 1, \ldots, m,$$

uniformly in j, yielding for $\varepsilon > 0$ and $k, l \in \{1, \ldots, m\}$, $k \neq l$,

$$\begin{aligned} &\limsup_{n\to\infty} \sum_{j=1}^{m_n} \mathsf{P}\{|\xi_{njk} - \mathsf{E}\,\xi_{njk}| \wedge |\xi_{njl} - \mathsf{E}\,\xi_{njl}| > \varepsilon\} \\ &\qquad \leqq \limsup_{n\to\infty} \sum_{j=1}^{m_n} \mathsf{P}\{|\xi_{njk}| \wedge |\xi_{njl}| > \varepsilon/2\} = 0. \end{aligned} \tag{4.4}$$

Finally, by (4.1),

$$\begin{aligned} \sum_{j=1}^{m_n} \mathrm{Cov}(\xi_{njk}, \xi_{njl}) &\sim -\sum_{j=1}^{m_n} \mathsf{E}\,\xi_{njk}\, \mathsf{E}\,\xi_{njl} \\ &\sim -(t_k - t_{k-1})(t_l - t_{l-1})\, c_n^2 \pi_n^2 R \to 0, \qquad k \neq l. \end{aligned} \tag{4.5}$$

By [6], p. 585, it follows easily from (4.2-5) that

$$\sum_{k=1}^{m} a_k [Y_n(t_k) - Y_n(t_{k-1})] \xrightarrow{\mathrm{d}} \sum_{k=1}^{m} a_k [X(t_k) - X(t_{k-1})], \qquad a_1, \ldots, a_m \in R,$$

and so (C) holds by Th. 7.7 in [1].

In the case of unbounded non-random atoms, let $\varepsilon > 0$ be arbitrary and choose a $u > 0$ with $\lambda\, \partial I_u = 0$, $\lambda I_u^c < \varepsilon$. Let $X', X_1', X_2', \ldots$ be the r.pr. obtained from $X, X_1, X_2, \ldots$ by omitting jumps of modulus $> u$. Let $v, v_1, v_2, \ldots$ be the number of such jumps in the interval $T = [0, 1]$. Then

$$\mathsf{P}\{X(t) = X'(t),\ t \in T\}^c = \mathsf{P}\{v > 0\} \leqq \mathsf{E}\, v = \lambda I_u^c < \varepsilon \tag{4.6}$$

by [17], p. 550, while

$$\mathsf{E}\, v_n = [r_n]\, \pi_n I_u^c / m_n \leqq c_n \pi_n I_u^c \to \lambda I_u^c < \varepsilon \tag{4.7}$$

implies

$$\mathsf{P}\{X_n(t) = X_n'(t),\ t \in T\}^c = \mathsf{P}\{v_n > 0\} \leqq \mathsf{E}\, v_n \leqq \varepsilon \tag{4.8}$$

for large n. Furthermore, (4.1) clearly remains true for $X', X_1', X_2', \ldots$, so we get $X_n' \xrightarrow{\mathrm{d}} X'$ by uniform boundedness. Now (C) follows from (4.6, 8) by Th. 4.2 in [1]. The result extends to random $\{\pi_n\}$ by L. 1.1.

Next assume that $X_n \xrightarrow{\mathrm{d}} X$. To prove (4.1) it clearly suffices to show that the sequences $\{c_n \pi_n^k I_u\}$, $k = 1, 2$, are tight for arbitrarily large $u > 0$ and that

$$\lim_{u\to\infty} \limsup_{n\to\infty} \mathsf{P}\{c_n \pi_n I_u^c > \varepsilon\} = 0, \qquad \varepsilon > 0. \tag{4.9}$$

For the first assertion, proceed as in the proof of Th. 2.2, putting

$$\eta_n^2 = (c_n \pi_n^1 I_u)^2 + c_n \pi_n^2 I_u, \qquad n \in N, \tag{4.10}$$

where X has a.s. no jump of modulus u, and observing that $\{X_n'/\eta_n\}$ is tight and $X_n' \xrightarrow{\mathrm{d}} X'$. To prove (4.9), let A_u be the set of functions in $D[0, \infty)$ with no jump of modulus $>u$ in [0, 1]. Then

$$\mathsf{P}\{X_n \in A_u | \pi_n\} = \binom{\pi_n I_u}{r_n} \Big/ \binom{m_n}{r_n} \leqq \left(\frac{\pi_n I_u}{m_n}\right)^{r_n} = \left(1 - \frac{\pi_n I_n^c}{m_n}\right)^{r_n} \leqq \exp(-c_n \pi_n I_u^c) \quad (4.11)$$

provided $r_n \in N$ (which involves no real restriction), so we get

$$\liminf_{n\to\infty} \mathsf{E} \exp(-c_n \pi_n I_u^c) \geqq \liminf_{n\to\infty} \mathsf{P}\{X_n \in A_u\} \geqq \mathsf{P}\{X \in A_{2u}\},$$

where the right side tends to zero as $u \to \infty$.

For the next result, let the can. r.e. of a r. pr. X on $[0, s]$ with ich. incr. be defined as those of the process $X(st)$, $t \in [0, 1]$.

Theorem 4.2. *For $n \in N$, let $s_n > 0$ and let X_n be a r. pr. in $D_0[0, \infty)$ whose restriction to $[0, s_n]$ has ich. incr. and can. r.e. $\alpha_n, \sigma_n, \beta_n$. Suppose that $c_n = s_n^{-1} \to 0$ and that $T \in \mathscr{T}_\infty$. Then $X_n \xrightarrow{d}$ some X iff*

$$c_n(\alpha_n - \beta_n(h - g_1), \sigma_n^2 \delta_0 + g_2 \beta_n) \xrightarrow{wd} \text{some } (\Gamma, \Lambda) \text{ in } R \times \mathfrak{M}(R), \quad (4.12)$$

and also iff (C) *holds. In this case, X has can. r.e. Γ, Λ, and* (D) *holds.*

Proof. Suppose that (4.12) holds for some non-random $\alpha_n, \sigma_n, \beta_n$ such that the β_n have uniformly bounded atom positions. Then clearly

$$c_n \alpha_n \to \gamma, \qquad c_n(\sigma_n^2 \delta_0 + \beta_n^2) \xrightarrow{w} \sigma^2 \delta_0 + \lambda^2,$$

where $\gamma = \lim_{u\to\infty} \gamma_u$. As in the proof of Th. 2.3, we may assume that $(\alpha_n, \sigma_n, \beta_n)$ takes either of the three forms $(\alpha_n, 0, 0)$, $(0, \sigma_n, 0)$, $(0, 0, \beta_n)$. The first case is trivial. In the second case, $X_n(t) = \sigma_n B_n(c_n t)$ on $[0, s_n]$ for some Brownian bridge B_n, so for $s \leqq t$ we get by [1], p. 65,

$$\operatorname{Cov}(X_n(s), X_n(t)) = \sigma_n^2 c_n s(1 - c_n t) \to \sigma^2 s,$$

and hence $X_n \xrightarrow{d} \sigma M$ where M is Brownian motion. In the third case we have

$$X_n(t) = \sum_{j=1}^{\infty} \beta_{nj} [1_+(t - \tau_{nj}) - c_n t], \qquad t \in [0, s_n],\ n \in N, \quad (4.13)$$

where, for $n \in N$, $\beta_n = \sum_j \delta_{\beta_{nj}}$ and the τ_{nj} are ind. and uniformly distributed on $[0, s_n]$. By Th. 4.2 in [1] and Čebyšev's inequality, we may assume the number of non-zero terms in (4.13) to be finite. For fixed $m \in N$ and $0 = t_0 < t_1 < \cdots < t_m$ we put

$$\xi_{njk} = \beta_{nj} [1_+(t_k - \tau_{nj}) - 1_+(t_{k-1} - \tau_{nj}) - c_n(t_k - t_{k-1})], \qquad k = 1, \ldots, m,\ j, n \in N.$$

Then (4.3) (with ∞ in place of m_n) follows from

$$\operatorname{Var} \xi_{njk} = \beta_{nj}^2 c_n(t_k - t_{k-1}) [1 - c_n(t_k - t_{k-1})], \qquad k = 1, \ldots, m,\ j, n \in N.$$

Further, for any λ-continuity interval $I \subset R$ which is bounded away from the origin, we get for sufficiently large n

$$\mathsf{P}\{\xi_{njk} \in I\} = c_n(t_k - t_{k-1})\, 1_I(\beta_{nj}[1 - c_n(t_k - t_{k-1})]), \qquad k = 1, \ldots, m,$$

uniformly in $j \in N$, so by approximation we easily obtain

$$(t_k - t_{k-1})\,\lambda I \leqq \liminf_{n\to\infty} \sum_j \mathsf{P}\{\xi_{njk} \in I\} \leqq \limsup_{n\to\infty} \sum_j \mathsf{P}\{\xi_{njk} \in I\} \leqq (t_k - t_{k-1})\,\lambda I$$

for all k, and (4.2) follows. Finally we get for $k \neq l$ and $\varepsilon > 0$

$$\sum_j \mathsf{P}\{|\xi_{njk}| \wedge |\xi_{njl}| > \varepsilon\} \to 0, \qquad \sum_j \operatorname{Cov}(\xi_{njk}, \xi_{njl}) \to 0 \tag{4.14}$$

as in (4.4–5). From (4.2–3, 14) we obtain (C) as in the preceding proof.

The remainder of the proof follows that of Th. 4.1, except that formulas (4.7, 10–11) are now replaced by

$$\mathsf{E}\,v_n = c_n\,\beta_n\,I_u^c \to \lambda\,I_u^c < \varepsilon,$$

$$\eta_n^2 = (c_n\,\alpha_n)^2 + c_n(\sigma_n^2 + \beta_n^2\,R),$$

$$\mathsf{P}\{X_n \in A_u \mid \beta_n\} = (1 - c_n)^{\beta_n I_u^c} \leqq \exp(-c_n\,\beta_n\,I_u^c).$$

5. Restriction and Extension

For any r.m. μ and p.pr. π, π', we say that π is a *subordinated* Poisson pr. *directed* by μ if, given μ, π is conditionally distributed as a Poisson pr. with intensity μ. Further, π' is a *p-thinning* of π, if π' is obtained from π by deleting the atoms independently with probability $1-p$ (cf. [19]). For any r.e. $(\gamma, \sigma, \lambda)$ in $R \times R_+ \times \mathfrak{M}(R')$ we define the *FL-transform* (FL = Fourier-Laplace) H by $H(u, v, f) = \mathsf{E}\exp(i\,u\,\gamma - v\,\sigma^2 - \lambda f)$ for $u \in R$, $v \in C_+ = \{x + i\,y \colon x \in R_+,\, y \in R\}$, $f \in \mathscr{F}_+ = \{f \colon f$ measurable $R' \to C_+$ and $f(x) = O(x^2)$ as $x \to 0$. Write $N(0, 1)$ for the standard normal distribution.

Theorem 5.1. *Let X be a r.pr. in $D_0[0, \infty)$ with ich. incr. and can.r.e. $\gamma_\varepsilon, \sigma, \lambda$, and let α, σ', β be the can.r.e. of its restriction to $[0, 1]$. Then*

(i) *β is a subordinated Poisson pr. on R' directed by λ,*

(ii) *$\sigma' = \sigma$, $\alpha = \sigma\,\vartheta + \lim_{\varepsilon\to 0}(\gamma_\varepsilon + \beta^1\,I_\varepsilon^c)$ a.s., where ϑ is $N(0, 1)$,*

(iii) *ϑ and the randomization involved in* (i) *are mutually ind. and ind. of $(\gamma_\varepsilon, \sigma, \lambda)$.*

The FL-transforms H_1 of (α, σ, β) and H_∞ of $(\gamma_\varepsilon, \sigma, \lambda)$ are related by

$$H_1(u, v, f) = H_\infty(u, v + u^2/2, i\,u\,h_\varepsilon + 1 - e^{iuh - f}), \qquad u \in R,\ v \in C_+,\ f \in \mathscr{F}_+. \tag{5.1}$$

Proof. By conditioning, the first assertion is reduced to the case of non-random $\gamma_\varepsilon, \sigma, \lambda$, for which (i) follows by [17], p. 550. To prove (ii)–(iii), let us first assume that $\lambda I_\varepsilon = 0$ for some $\varepsilon > 0$. Then $X = X_1 + X_2$ where X_1 is a pure jump pr. while $X_2 = \gamma_0\,h + \sigma M$ for some Brownian motion M ind. of X_1. Now $\vartheta = M(1)$ is $N(0, 1)$ and ind. of the Brownian bridge $B(t) = M(t) - t\,M(1)$, $t \in [0, 1]$, since

$$\operatorname{Cov}[\vartheta, B(t)] = \operatorname{Cov}[M(1), M(t)] - t\operatorname{Var} M(1) = t - t = 0, \qquad t \in [0, 1],$$

while $\alpha = \gamma_0 + \sigma\,\vartheta + \beta^1\,R$, and this proves (ii)–(iii). For general λ, let λ_ε be the restriction of λ to I_ε^c and write $X = X_\varepsilon + X'_\varepsilon$, where X_ε is ind. of X'_ε with can.r.e. $\gamma_\varepsilon, \sigma, \lambda_\varepsilon$, constructed as in [14]. By martingale convergence, we easily obtain $X_\varepsilon(1) \to X(1) = \alpha$ a.s. as $\varepsilon \to 0$, which proves (ii). Furthermore,

$$\alpha = \sigma\,\vartheta + \gamma_\varepsilon + \lim_{c\to 0}\,[\lambda(h_c - h_\varepsilon) + \beta\,h'_c],$$

where $h'_c = h - h_c$, so for $u \in R$, $v \in C_+$, $f \in \mathscr{F}_+$

$$\begin{aligned}
&\mathsf{E}\exp\left[iu(\sigma\vartheta+\gamma_\varepsilon+\lambda(h_c-h_\varepsilon)+\beta h'_c)-v\sigma^2-\beta f\right]\\
&\quad=\mathsf{E}\left\{\exp\left[iu(\gamma_\varepsilon+\lambda(h_c-h_\varepsilon))-v\sigma^2\right]\mathsf{E}\left[e^{iu\sigma\vartheta}|\sigma\right]\mathsf{E}\left[e^{\beta(iuh'_c-f)}|\lambda\right]\right\}\\
&\quad=\mathsf{E}\exp\left[iu(\gamma_\varepsilon+\lambda(h_c-h_\varepsilon))-v\sigma^2-u^2\sigma^2/2-\lambda(1-e^{iuh'_c-f})\right]\\
&\quad=H_\infty\left(u,v+u^2/2,iu(h_\varepsilon-h_c)+1-e^{iuh'_c-f}\right),
\end{aligned}$$

and (5.1) follows as $c \to 0$ by dominated convergence.

Theorem 5.2. *Let X be a r.pr. in $D_0[0,1]$ with ich. incr. and can.r.e. α, σ, β, and let α', σ', β' be the can.r.e. of its restriction to $[0,p]$, $p \in (0,1)$. Then*

(i) *β' is a p-thinning of β,*

(ii) *$\sigma' = \sigma\sqrt{p}$, $\alpha' = \alpha p + \sigma\vartheta\sqrt{p(1-p)} + \lim_{\varepsilon\to 0}(\beta' - p\beta)^1 I^c_\varepsilon$ a.s. where ϑ is $N(0,1)$,*

(iii) *ϑ and the randomization involved in* (i) *are mutually ind. and ind. of (α, σ, β).*

The FL-transforms H_p of $(\alpha', \sigma', \beta')$ and H_1 of (α, σ, β) are related for $u \in R$, $v \in C_+$, $f \in \mathscr{F}_+$ by

$$H_p(u,v,f)=H_1\left(up,\ vp+u^2p(1-p)/2,\ iuph-\log[1-p(1-e^{iuh-f})]\right). \quad (5.2)$$

It is interesting to observe that Th. 5.1–2 essentially contain the main result of Mecke [19]. In fact, if X is a r.pr. on $[0,1]$ with ich. incr. and can.r.e. α, σ, β, then by Th. 5.1 its distribution may be extended to R_+ for some α provided β is a subordinated Poisson pr., and by Th. 5.2 provided β is a p-thinning for each $p \in (0,1)$, so these two conditions must be equivalent. Th. 5.1–2 have simple analogues for sequences of ich.r.e., but there the notions of sample pr. [12] and sampling take over the roles of Poisson pr. and thinning here.

Proof. As for Th. 5.1, it suffices to prove (i)–(iii) in the case of non-random α, σ, β with $\beta I_\varepsilon = 0$ for some $\varepsilon > 0$. Then (i) is obvious, while the continuous component of X takes the form $X_2 = (\alpha - \beta^1 R)h + \sigma B$ for some Brownian bridge B. Defining $\vartheta = B(p)/\sqrt{p(1-p)}$; $B'(s) = [B(sp) - sB(p)]/\sqrt{p}$, $s \in [0,1]$, and verifying that ϑ is $N(0,1)$, that B' is a Brownian bridge ind. of ϑ and that

$$X_2(sp) = (\alpha - \beta^1 R)sp + \sigma s\vartheta\sqrt{p(1-p)} + \sigma\sqrt{p}B'(s), \qquad s \in [0,1],$$

we obtain (ii) and (iii). Finally, (5.2) follows by proceeding as in Th. 5.1 and using Hilfssatz 4.1 of Mecke [19].

Theorem 5.3. *Let X and Y be r.pr. in $D_0[0,1]$ or $D_0[0,\infty)$ with ich. incr. and let $T \in \mathscr{T}_1$ or $\mathscr{T}_\infty$ respectively. Then $X \stackrel{d}{=} Y$ iff $X(t) \stackrel{d}{=} Y(t)$, $t \in T^k$, $k \in N$.*

Note that the corresponding statement for sequences of ich.r.e. is false. For a strengthening in the case of simple p.pr., see [12], Th. 5.2.

Proof. For $T \in \mathscr{T}_1$, the extension to the closure of T is unique by continuity. (Only this fact was needed in the proof of Th. 2.1.) By interchangeability and induction, we may assume that $T = [0,p]$ for some fixed $p \in (0,1)$, so it suffices to prove that H_p determines H_1 in Th. 5.2. Substitution in (5.2) yields with $q = 1/p$

$$H_1(u,v,f)=H_p\left(uq,vq+u^2q(1-q)/2,iuqh-\log[1-q(1-e^{iuh-f})]\right) \quad (5.3)$$

for $u\in R$, $\mathrm{Re}\,[v\,q+u^2 q(1-q)/2]\geqq 0$, and for $f\in\mathscr{F}_+$ with

$$i\,u q h-\log[1-q(1-e^{iuh-f})]\in\mathscr{F}_+. \tag{5.4}$$

Now $H_1(u,v,f)$ is analytic in $v\in C_+$ for fixed $u\in R, f\in\mathscr{F}_+$, so H_1 is determined by (5.3) for all $v\in C_+$. Put $p=1/2$ and note that the principal branch of the function

$$w=-\log[1-p(1-e^{-z})]=-\log[(1+e^{-z})/2]$$

maps $\{z\colon \mathrm{Re}\,z>0\}$ onto $D=\{w\colon |\mathrm{Im}\,w|<\arccos e^{-\mathrm{Re}\,w}\}$, except for $w=\log 2$. By (5.4), H_1 is therefore determined for all $f\in\mathscr{F}_+$ such that $i\,u\,x-f(x)\in D\setminus\{\log 2\}$, $x\in R'$. The exception for $\log 2$ is removed by continuity.

For fixed $u\in R$, $v\in R_+$, let f_0 be some fixed function of this type with $f_0(x)=O(x^2)$, $x\to 0$, which exists since $\mathrm{Re}\,w\sim(\mathrm{Im}\,w)^2$, $w\to 0$, on the boundary of D. Let I be any compact sub-set of R', and choose some disjoint partitioning $I_1,\dots,I_k$ of I and some $z_1,\dots,z_k\in C_+$, such that

$$f(x)=\begin{cases} f_0(x), & x\notin I,\\ z_j, & x\in I_j,\ j=1,\dots,k,\end{cases} \tag{5.5}$$

has the same properties. Since the values of $z_1,\dots,z_k$ may be varied around the initially chosen numbers, it follows by analyticity that H_1 is determined for any f of the form (5.5) with $z_1,\dots,z_k\in C_+$. By successive sub-divisioning of $I_1,\dots,I_k$, it is seen that f may be chosen arbitrarily on I, and this result extends to $I=R'$ by dominated convergence.

Finally, let T contain some lattice, and assume without real loss that $T=Z_+$. By Th. 1.1, the can. r. m. μ of $X(j)-X(j-1), j\in N$, is then uniquely determined, and μ determines Γ and Λ by [6], p. 564.

Our last result improves and extends Th. 4.9.6 of Bühlmann [3].

Theorem 5.4. *For $I=[0,1]$ or R_+, let X and Y be r. pr. in $C_0(I)$ with ich. incr., and suppose that $T\subset I$ has a limit point in the interior of I. Then $X\overset{d}{=}Y$ iff $X(t)\overset{d}{=}Y(t)$, $t\in T$.*

Proof. It suffices to take $I=[0,1]$. Let $t_0\in(0,1)$ be a limit point of T, let $X=\alpha h+\sigma B$, and define the FL-transform H by $H(s,v)=\mathsf{E}\exp(i s\alpha-v\sigma^2)$, $s\in R$, $v\in C_+$. For $u\in R$ and $t\in I$ we get

$$\begin{aligned} C(u,t)=\mathsf{E}\,e^{iuX(t)}&=\mathsf{E}\,\mathsf{E}[e^{iu\alpha t+iu\sigma B(t)}|\alpha,\sigma]=\mathsf{E}[e^{iu\alpha t}\ \mathsf{E}(e^{iu\sigma B(t)}|\sigma)]\\ &=\mathsf{E}\big[e^{iu\alpha t}\exp\big(-u^2\sigma^2 t(1-t)/2\big)\big]=H\big(u\,t, u^2 t(1-t)/2\big),\end{aligned}$$

so

$$C(s/t,t)=H\big(s,(t^{-1}-1)\,s^2/2\big)=H(s,v),\qquad s\in R,\ t\in I,\ v=(t^{-1}-1)\,s^2/2.$$

For fixed $s\in R'$, the left side is assumed to be known for all $t\in T$, so $H(s,v)$ is determined for some v-set with limit point $v_0=(t_0^{-1}-1)\,s^2/2>0$, and hence by analyticity for all $v>0$.

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