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Connection between the Different L_s-Predictions with Applications

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1. Introduction and Notations

Let P be a probability measure defined on a σ-algebra $\mathscr{A}$ over Ω. For each $s>1$ denote by $\mathscr{L}_s(\Omega,\mathscr{A},P)$ the space of all realvalued random variables f with $P[|f|^s]<\infty$. Let $\mathscr{B}\subset\mathscr{A}$ be a sub-σ-algebra. For each $f\in\mathscr{L}_s(\Omega,\mathscr{A},P)$ let $P_s^{\mathscr{B}}f$ be the projection of f onto the closed linear subspace $\mathscr{L}_s(\Omega,\mathscr{B},P)$ and call it the s-prediction of f. 2-prediction is the usual conditional expectation. According to [1] $P_s^{\mathscr{B}}f$ exists and is P-a.e. uniquely determined. We use the symbol $P_s^{\mathscr{B}}f$ both for a function and its corresponding equivalence class.

Let $\mathscr{P}$ be a family of probability measures on the σ-algebra $\mathscr{A}$. The σ-algebra $\mathscr{B}\subset\mathscr{A}$ is sufficient for $\mathscr{P}|\mathscr{A}$ in the usual sense iff for each $A\in\mathscr{A}$ there exists a common 2-prediction $g_2^A\in\bigcap_{P\in\mathscr{P}} P_2^{\mathscr{B}}1_A$. We call $\mathscr{B}$ s-prediction sufficient for $\mathscr{P}|\mathscr{A}$ iff for each $A\in\mathscr{A}$ there exists a common s-prediction $g_s^A\in\bigcap_{P\in\mathscr{P}} P_s^{\mathscr{B}}1_A$.

It is shown in this paper (see Theorem 1) that there exists a functional relationship between s-prediction and 2-prediction for step functions. This relation can be used to show that s-prediction-sufficiency is equivalent to sufficiency.

If $\mathscr{B}$ is sufficient for $\mathscr{P}|\mathscr{A}$, the linearity of the 2-prediction directly implies that for each $f\in\bigcap_{P\in\mathscr{P}}\mathscr{L}_2(\Omega,\mathscr{A},P)$ there exists a common 2-prediction $g_2^f\in\bigcap_{P\in\mathscr{P}}P_2^{\mathscr{B}}f$. Since s-prediction is in general not a linear operator, the corresponding consequence for s-prediction sufficiency cannot be derived in such a direct way. It, however, follows from our functional relationship between s-prediction and 2-prediction. This relationship can furthermore be applied, to obtain a generalization of Ando-Amemiya's [1] martingale theorem for s-predictions.

2. The Results

If $a\in\mathbb{R}$, $r>0$ let $a^r=|a|^r$ sign a.

To make the paper more readable we collect the following well known results (see e.g. [1], [4]).

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Remark. Let $1<s<\infty$. Then

a) $f \leqq g \Rightarrow P_s^{\mathscr{B}} f \leqq P_s^{\mathscr{B}} g \quad (f, g \in \mathscr{L}_s(\Omega, \mathscr{A}, P))$

b) $P_s^{\mathscr{B}}(\alpha f)=\alpha P_s^{\mathscr{B}} f \quad (f \in \mathscr{L}_s(\Omega, \mathscr{A}, P), \alpha \in \mathbb{R})$

c) $P_s^{\mathscr{B}}(\alpha+f)=\alpha+P_s^{\mathscr{B}} f \quad (f \in \mathscr{L}_s(\Omega, \mathscr{A}, P), \alpha \in \mathbb{R})$

d) $f_n, f \in \mathscr{L}_s(\Omega, \mathscr{A}, P)$ and $f_n \uparrow f$ or $f_n \downarrow f$ then $P_s^{\mathscr{B}} f_n \xrightarrow[n \to \infty]{} P_s^{\mathscr{B}} f$ P-a.e.

e) If $f \in \mathscr{L}_s(\Omega, \mathscr{A}, P)$, $g \in \mathscr{L}_s(\Omega, \mathscr{B}, P)$ then $g \in P_s^{\mathscr{B}} f$ iff $P_2^{\mathscr{B}}[(f-g)^{s-1}]=0$

f) If $f_n, f \in \bigcap_{P \in \mathscr{P}} \mathscr{L}_s(\Omega, \mathscr{A}, P)$, $f_n \uparrow f$ or $f_n \downarrow f$ and

$g_n \in \bigcap_{P \in \mathscr{P}} P_s^{\mathscr{B}} f_n$ then $\overline{\lim_{n \to \infty}} g_n \in \bigcap_{P \in \mathscr{P}} P_s^{\mathscr{B}} f$.

We remark that f) directly follows from d).

In the following theorem we state our basic relationship between s-prediction and 2-prediction.

Let

$$\Delta_m=\left\{(x_1, \ldots, x_m) \in[0,1]^m: \sum_{i=1}^m x_i=1\right\}.$$

1. Theorem. *Let $P|\mathscr{A}$ be a probability measure, $\mathscr{B} \subset \mathscr{A}$ a sub-σ-algebra and $1<s<\infty$. Then for each step function f with representations $f=\sum_{i=1}^m \alpha_i 1_{A_i}$, where $A_i \in \mathscr{A}$, $i=1, \ldots, m$, are disjoint and $\sum_{i=1}^m A_i=\Omega$, there exists a continuous function $H_{\alpha_1, \ldots, \alpha_m}^{(s)}: \Delta_m \to \mathbb{R}$ which depends on $\alpha_1, \ldots, \alpha_m$ and s but not on $P|\mathscr{A}$ and $\mathscr{B}$ such that*

$$P_s^{\mathscr{B}} f=H_{\alpha_1, \ldots, \alpha_m}^{(s)} \circ(P_2^{\mathscr{B}} A_1, \ldots, P_2^{\mathscr{B}} A_m).$$

Proof. Let g be $\mathscr{B}$-measurable and bounded: to show that $g \in P_s^{\mathscr{B}} f$ it suffices to prove according to Remark e) that

$$P_2^{\mathscr{B}}[(f-g)^{s-1}]=0. \tag{1}$$

Since $\sum_{i=1}^m 1_{A_i}=1$ and $P_2^{\mathscr{B}}$ is a linear operator, (1) is equivalent to

$$\sum_{i=1}^m P_2^{\mathscr{B}}[1_{A_i}(\alpha_i-g)^{s-1}]=0$$

and hence to

$$\sum_{i=1}^m (\alpha_i-g)^{s-1} P_2^{\mathscr{B}} A_i=0. \tag{2}$$

Now we construct a continuous function $H: \Delta_m \to \mathbb{R}$ such that $g = H \circ (P_2^{\mathscr{B}} A_1, \ldots, P_2^{\mathscr{B}} A_m)$ fulfills (2). We remark that this g is $\mathscr{B}$-measurable and bounded. W.l.g. we assume $\alpha_1 \leqq \cdots \leqq \alpha_m$. For each $(x_1, \ldots, x_m) \in \Delta_m$, $\alpha \in \mathbb{R}$, let

$$\varphi_{x_1, \ldots, x_m}(\alpha) = \sum_{i=1}^{m} (\alpha_i - \alpha)^{s-1} x_i.$$

Since $\alpha \to \varphi_{x_1, \ldots, x_m}(\alpha)$ is a strictly monotone decreasing and continuous function with $\varphi_{x_1, \ldots, x_m}(\alpha_1) \geqq 0$ and $\varphi_{x_1, \ldots, x_m}(\alpha_m) \leqq 0$ there exists a unique $H(x_1, \ldots, x_m) \in \mathbb{R}$ with

$$\varphi_{x_1, \ldots, x_m}(H(x_1, \ldots, x_m)) = 0.$$

It is easy to see that H is continuous. Since $\sum_{i=1}^{m} P_2^{\mathscr{B}} A_i = 1$ we obtain that $g = H \circ (P_2^{\mathscr{B}} A_1, \ldots, P_2^{\mathscr{B}} A_m)$ fulfills relation (2).

The construction of H shows that H depends only on $\alpha_1, \ldots, \alpha_m$ and s.

Although the s-prediction is in general not a linear operator it turns out to be possible to describe the s-prediction of a step function $\sum_{i=1}^{m} \alpha_i 1_{A_i}$ $\left(\sum_{i=1}^{m} A_i = \Omega\right)$ in terms of the s-predictions $P_s^{\mathscr{B}} A_i$, $i = 1, \ldots, m$. Moreover the functional connection is independent of $P|\mathscr{A}$ and $\mathscr{B}$.

2. Corollary. *Let $P|\mathscr{A}$ be a probability measure, $\mathscr{B} \subset \mathscr{A}$ be a sub-σ-field and $1 < s < \infty$. If $f = \sum_{i=1}^{m} \alpha_i 1_{A_i}$ where $A_i \in \mathscr{A}$, $i = 1, \ldots, m$, are disjoint and $\sum_{i=1}^{m} A_i = \Omega$, there exists a continuous function $\hat{H}: \Delta_m \to \mathbb{R}$ which depends on $\alpha_1, \ldots, \alpha_m$ and s but not on $P|\mathscr{A}$ and $\mathscr{B}$ such that*

$$P_s^{\mathscr{B}} f = \hat{H} \circ (P_s^{\mathscr{B}} A_1, \ldots, P_s^{\mathscr{B}} A_m).$$

Proof. Let $H_{0,1}^{(s)}: \Delta_2 \to \mathbb{R}$ be the function appearing in Theorem 1 for $\alpha_1 = 0$, $\alpha_2 = 1$. It is easy to see that $\varphi(x) = H_{0,1}^{(s)}(x, 1-x) = \dfrac{x^r}{x^r + (1-x)^r}$ $\left(\text{where } r = \dfrac{1}{s-1}\right)$ is a continuous strictly increasing function from $[0, 1]$ onto $[0, 1]$. Let

$$\hat{H} = H_{\alpha_1, \ldots, \alpha_m}^{(s)} \circ (\varphi^{-1}, \ldots, \varphi^{-1}).$$

Since $P_2^{\mathscr{B}} A = \varphi^{-1} \circ P_s^{\mathscr{B}} A$, Theorem 1 yields the assertion.

As $P_s^{\mathscr{B}} A = \varphi \circ P_2^{\mathscr{B}} A$ and $P_2^{\mathscr{B}} A = \varphi^{-1} \circ P_s^{\mathscr{B}} A$ with a strictly increasing and continuous function φ which does not depend on $P|\mathscr{A}$ and $\mathscr{B}$ we obtain:

3. Corollary. *Let $\mathscr{P}|\mathscr{A}$ be a family of probability measures. If a σ-algebra $\mathscr{B} \subset \mathscr{A}$ is s-prediction sufficient for $\mathscr{P}|\mathscr{A}$ for some $s \in (1, \infty)$ then it is s-prediction sufficient for all $s \in (1, \infty)$.*

Hence s-prediction sufficiency is equivalent to sufficiency.

4. Corollary. *Let $\mathscr{P}|\mathscr{A}$ be a family of probability measures, and $\mathscr{B} \subset \mathscr{A}$ be a s-prediction sufficient σ-algebra for $\mathscr{P}|\mathscr{A}$ with $s \in (1, \infty)$. Then for each $f \in \bigcap_{P \in \mathscr{P}} \mathscr{L}_s(\Omega, \mathscr{A}, P)$ there exists a common s-predictor $g \in \bigcap_{P \in \mathscr{P}} P_s^{\mathscr{B}} f$.*

Proof. According to Corollary 2 there exists for each $\mathscr{A}$-measurable step function a common s-predictor. Hence the assertion follows from Remark f).

Now we apply once more our relation between s-predictions and 2-predictions to obtain a generalized martingale theorem. For the special case $s=2$ and $\mathscr{B}_n=\mathscr{B}_\infty$ this is the result of Gänßler-Pfanzagl [2]. For the special case $P_n=P$ it yields immediately the result of Ando-Amemiya [1] (use Remark d)).

5. Corollary. *Let $P_n|\mathscr{A}$, $n\in\mathbb{N}\cup\{\infty\}$, be probability measures, dominated by a measure $\mu|\mathscr{A}$. Let $\mathscr{B}_n\subset\mathscr{A}$, $n\in\mathbb{N}$, σ-algebras decreasing or increasing to the σ-algebra $\mathscr{B}_\infty$ and assume that μ is σ-finite on $\bigcap_{n\in\mathbb{N}}\mathscr{B}_n$. Let h_n be a density of $P_n|\mathscr{A}$ with respect to $\mu|\mathscr{A}$ and h_n^* be a density of $P_n|\mathscr{B}_n$ with respect to $\mu|\mathscr{B}_n$, $n\in\mathbb{N}\cup\{\infty\}$. Assume that $h_n\to h$ μ-a.e. and $h_n^*\to h^*$ μ-a.e.*

Then for each bounded $\mathscr{A}$-measurable function f and each $s\in(1,\infty)$

$$(P_n)_s^{\mathscr{B}_n}f\to(P_\infty)_s^{\mathscr{B}_\infty}f\qquad P_\infty\text{-a.e.}$$

Proof. We show at first that the assertion holds for all step functions f. Represent f by $f=\sum_{i=1}^{n}\alpha_i 1_{A_i}$ with $A_i\in\mathscr{A}$, $i=1,\dots,n$, disjoint and $\sum_{i=1}^{m}A_i=\Omega$. According to Theorem 1 of [3] we have for each $i=1,\dots,m$

$$(P_n)_2^{\mathscr{B}_n}A_i\to(P_\infty)_2^{\mathscr{B}_\infty}A_i\qquad P_\infty\text{-a.e.}\tag{1}$$

According to Theorem 1 there exists a continuous function $H:\Delta_m\to\mathbb{R}$, not depending on $P_n|\mathscr{A}$ and $\mathscr{B}_n$ such that

$$(P_n)_s^{\mathscr{B}_n}f=H\circ((P_n)_2^{\mathscr{B}_n}A_1,\dots,(P_n)_2^{\mathscr{B}_n}A_m)$$

for all $n\in\mathbb{N}\cup\{\infty\}$. Since H is continuous we obtain from (1) the assertion for the step function f.

Now let f be bounded and $\mathscr{A}$-measurable. Then there exists $\mathscr{A}$-measurable simple functions f_k such that $f_k\leqq f\leqq f_k+\frac{1}{k}$, $k\in\mathbb{N}$. Hence

$$(P_n)_s^{\mathscr{B}_n}f_k\leqq(P_n)_s^{\mathscr{B}_n}f\leqq(P_n)_s^{\mathscr{B}_n}f_k+\frac{1}{k},\qquad k\in\mathbb{N}.$$

Since the assertion holds for each f_k we obtain

$$\begin{aligned}(P_\infty)_s^{\mathscr{B}_\infty}f_k&\leqq\varliminf_{n\to\infty}(P_n)_s^{\mathscr{B}_n}f\leqq\varlimsup_{n\to\infty}(P_n)_s^{\mathscr{B}_n}f\\&\leqq(P_\infty)_s^{\mathscr{B}_\infty}f_k+\frac{1}{k}\end{aligned}\tag{2}$$

for all $k\in\mathbb{N}$. According to Remark a) and c) we have $(P_\infty)_s^{\mathscr{B}_\infty}f_k\xrightarrow[k\to\infty]{}(P_\infty)_s^{\mathscr{B}_\infty}f$ P-a.e. and (2) implies the assertion.

Examples, given in [2] and [3] show that the assumptions of the preceding result cannot be weakened.

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