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Decomposable Measure Spaces

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I discuss conditions under which a measure space will be "decomposable" or strictly localizable. In particular, I show that there can be no canonical way of enlarging σ-algebras of measurable sets so as to make measure spaces decomposable. The argument uses a discussion of subspaces (§9).

My thanks are due to M. Talagrand for helpful conversations leading to a greatly improved description of Example 8, and to L. LeCam for referring me to [4] and [7].

1. Definitions and Introduction. Following [5], I shall say that a *measure algebra* is a Boolean algebra $\mathfrak{A}$ together with a strictly positive countably additive functional $\mu: \mathfrak{A} \to [0, \infty]$. $\mathfrak{A}^f$ will then be $\{a: \mu a < \infty\}$. If (X, Σ, μ) is a measure space, I shall write $\mathfrak{A}(\Sigma, \mu)$ for the associated measure algebra; Σ^f will be $\{E: \mu E < \infty\}$. A measure algebra is *semi-finite* if it has no purely infinite elements i.e. $\sup \mathfrak{A}^f = 1$; it is *Maharam* if it is moreover Dedekind complete. A measure space is semi-finite[1] or Maharam ("localizable") if its measure algebra is. (X, Σ, μ) is *complete* if every subset of a set of zero measure is measurable; it is *locally determined* if it is semi-finite and $E \in \Sigma$ whenever $E \subseteq X$ and $E \cap F \in \Sigma$ for every $F \in \Sigma^f$. Finally, (X, Σ, μ) is *decomposable* ("strictly localizable") if there is a partition $\langle X_\iota \rangle_{\iota \in I}$ of X into sets of finite measure such that

$$\Sigma = \{E: E \subseteq X, E \cap X_\iota \in \Sigma \ \forall \iota \in I\},$$
$$\mu E = \sum_{\iota \in I} \mu(E \cap X_\iota) \qquad \forall E \in \Sigma.$$

Note that the effect of the definitions of [9] is to restrict attention to locally determined measure spaces; thus a localizable measure space in [9] corresponds to a locally determined Maharam measure space here. In [4] and [6] there is a further restriction to complete spaces.

[1] This corresponds to a "regular" upper integral in [6]

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The importance of Maharam spaces lies in the strong Radon-Nikodym theorem: (X, Σ, μ) is Maharam iff the natural duality between L^1 and L^∞ represents L^∞ as $(L^1)'$ ([5], 53B; compare [9], Theorem 5.1). All decomposable measure spaces are Maharam and locally determined ([5], 64Hb), and the most important Maharam measure spaces are decomposable (e.g. Radon measure spaces; see [5], 72B). An essential property of decomposable spaces is given in IV.3 of [6], where we see that (in the notation of this paper) a complete locally determined measure space is decomposable iff it has a lifting.

It is easy to find Maharam measure spaces which are not decomposable; see e.g. §5 below. However, the examples given there are essentially insignificant, for the following reason. Let (X, Σ, μ) be any measure space. Then it has a complete locally determined ("c.l.d.") version, constructed as follows. Set

$$\Sigma' = \{E: E \subseteq X, \forall F \in \Sigma^f \exists E_1, E_2 \in \Sigma \text{ such that } E_1 \subseteq E \cap F \subseteq E_2 \text{ and } \mu(E_2 \setminus E_1) = 0\},$$

$$\mu' E = \sup\{\mu F: F \in \Sigma^f, F \subseteq E\} \quad \forall E \in \Sigma'.$$

Then (X, Σ', μ') is a c.l.d. measure space. Moreover, the embedding $\Sigma \subseteq \Sigma'$ induces a map $\mathfrak{A}(\Sigma, \mu) \to \mathfrak{A}(\Sigma', \mu')$ which includes an isomorphism between $\mathfrak{A}(\Sigma, \mu)^f$ and $\mathfrak{A}(\Sigma', \mu')^f$; and if (X, Σ, μ) is Maharam, we actually have an isomorphism between $\mathfrak{A}(\Sigma, \mu)$ and $\mathfrak{A}(\Sigma', \mu')$. (See [5], Exercises 64 *Ja/b*. An alternative construction of μ' is from the inner measure

$$\mu_* A = \sup\{\mu F: F \in \Sigma^f, F \subseteq A\} \quad \forall A \subseteq X,$$

as in [5], 72A. Compare [1], Exercise 17.7.)

Thus any Maharam measure space has a canonical c.l.d. version which has the same measure algebra and therefore the same L^p spaces, etc. The examples 5(a)–(b) below are clearly rendered decomposable by this process. The question therefore arises: can every Maharam measure space be made decomposable by taking its c.l.d. version? equivalently, is every c.l.d. Maharam measure space decomposable? or, does every c.l.d. Maharam measure space have a lifting? (See e.g. [2], p. 71.) The centrepiece of this paper is an example to show that the answer is no (Example 8).

To clear the air, I shall begin with the known positive results. The most powerful of these is the following.

2. Theorem. *Let (X, Σ, μ) be a c.l.d. measure space, and suppose that there is a disjoint family $\mathcal{A} \subseteq \Sigma^f$ such that $\sup_{E \in \mathcal{A}} E^\bullet = 1$ in $\mathfrak{A}(\Sigma, \mu)$. Then (X, Σ, μ) is decomposable.*

Proof. See [5], 64I. (The decomposition is of course $\mathcal{A} \cup \{X \setminus \cup \mathcal{A}\}$.)

Remark. This theorem amounts to a proof that the "strongly localizable" spaces of [4] are decomposable.

3. Definition. The next proposition depends on the concept of *magnitude* of a measure space, which I shall define as follows. Let $(\mathfrak{A}, \mu)$ be a semi-finite measure algebra. If $\mu 1 < \infty$, then $\mu 1 \in \mathbf{R}^+$ is the magnitude of $(\mathfrak{A}, \mu)$. Otherwise, any two maximal disjoint families $A, B \subseteq \Sigma^f$ have the same infinite cardinal (because each element of A meets only countably many elements of B, while each non-zero element of B meets at least one element of A; and vice versa). This cardinal is now the magnitude of $(\mathfrak{A}, \mu)$. If (X, Σ, μ) is a semi-finite measure space, its magnitude is the magnitude of $\mathfrak{A}(\Sigma, \mu)$ i.e. either μX or the cardinal of any maximal almost-disjoint family of sets of non-zero finite measure. Observe that for non-totally-finite spaces, the magnitude is just the "dimension" discussed in [4].

Now we have the following results. (Parts (b) and (c) are given in [4].)

4. Proposition. (a) *Any semi-finite measure space of countable magnitude is decomposable.*

(b) *Any c.l.d. measure space of magnitude $\leqq \aleph_1$ is decomposable.*

(c) *Any c.l.d. Maharam measure space of magnitude $\leqq \mathfrak{c}$ is decomposable.*

(d) *If (X, Σ, μ) is a Maharam measure space and $\#(X) < \mathfrak{c}$, then its magnitude is either finite or $\leqq \#(X)$. Consequently, if (X, Σ, μ) is also c.l.d., it must be decomposable.*

Proof. (a) The measure spaces of countable magnitude are just those which are usually called "σ-finite", and are of course decomposable ([5], 64 Ha).

(b) Let (X, Σ, μ) be a c.l.d. measure space of magnitude $\aleph_1$, and let $\mathscr{A} \subseteq \Sigma^f$ be a maximal almost-disjoint family of sets of non-zero measure. Enumerate $\mathscr{A}$ as $\langle E_\xi \rangle_{\xi < \kappa}$ where $\kappa \leqq \omega_1$, and set $F_\xi = E_\xi \setminus \bigcup_{\eta < \xi} E_\eta$ for each $\xi < \kappa$. Then $\{F_\xi : \xi < \kappa\}$ is a disjont family in Σ^f and $\sup_{\xi < \kappa} F_\xi^\bullet = \sup_{\xi < \kappa} E_\xi^\bullet = 1$ because (X, Σ, μ) is semi-finite. By Theorem 2, (X, Σ, μ) is decomposable.

(c) Now suppose that (X, Σ, μ) is a c.l.d. Maharam measure space of magnitude $\leqq \mathfrak{c}$, and let $C \subseteq \mathfrak{A}(\Sigma, \mu)^f$ be a maximal disjoint family. Index C as $\langle c_I \rangle_{I \in \mathscr{I}}$ where $\mathscr{I} \subseteq \mathscr{P}\mathbf{N}$. For $n \in \mathbf{N}$, set

$$a_n = \sup \{c_I : n \in I \in \mathscr{I}\},$$

which exists as $\mathfrak{A}$ is Dedekind complete. Choose $E_n \in \Sigma$ such that $a_n = E_n^\bullet$, and set

$$F_I = \bigcap_{n \in I} E_n \setminus \bigcup_{n \in \mathbf{N} \setminus I} E_n \quad \forall I \in \mathscr{I}.$$

Then $\langle F_I \rangle_{I \in \mathscr{I}}$ is a disjoint family in Σ^f and $\sup_{I \in \mathscr{I}} F_I^\bullet = \sup_{I \in \mathscr{I}} c_I = 1$, so that once again (X, Σ, μ) is decomposable by Theorem 2.

(d) If (X, Σ, μ) is a Maharam measure space of infinite magnitude $\aleph$, then there is a disjoint family $C \subseteq \mathfrak{A}(\Sigma, \mu) \setminus \{0\}$ of cardinal $\min(\mathfrak{c}, \aleph)$. The technique of (c) above shows that we can find a disjoint family $\mathscr{A} \subseteq \Sigma$ such that $C = \{F^\bullet : F \in \mathscr{A}\}$. As no element of $\mathscr{A}$ can be empty,

$$\#(X) \geqq \#(\mathscr{A}) = \#(C) = \min(\mathfrak{c}, \aleph).$$

Turning this round, if we know that $\#(X) < \mathfrak{c}$ then $\aleph$ must be $\leqq \#(X)$, as stated. The last part now follows from (c).

Note. See Example 11 below for the problems that can arise if $\#(X)=\mathfrak{c}$.

5. *Examples.* For the next few sections, I shall give examples to show that the hypotheses of Proposition 4 are (within their own terms) precisely what is needed.

(a) Let ω_1 be the first uncountable ordinal. Let T_0 be $\{\emptyset, \omega_1\}$ and let

$$T_1=\{E: E\subseteq\omega_1, \text{ either } E \text{ is countable or } \omega_1\setminus E \text{ is countable}\}.$$

Define ν on T_1 by

$$\nu E=0 \quad \text{if } E \text{ is countable, 1 otherwise.}$$

Then (ω_1, T_1, ν) is a measure space. Let $X=\omega_1\times\omega_1$ and for $E\subseteq X$ write

$$E_\xi=\{\eta: (\xi, \eta)\in E\} \quad \forall\, \xi<\omega_1.$$

Set

$$\Sigma=\{E: E\subseteq X, E_\xi\in T_1 \;\forall\, \xi<\omega_1, E_\xi\in T_0 \text{ for all but countably many } \xi\};$$

$$\mu E=\sum_{\xi<\omega_1}\nu E_\zeta \quad \forall\, E\in\Sigma.$$

Then it is easy to check that (X, Σ, μ) is a complete Maharam measure space, of magnitude $\aleph_1$, but is not locally determined nor decomposable. (Observe that (X, Σ, μ) has a multiplicative lifting $\theta: \Sigma\to\Sigma$ given by

$$\theta E=\{(\xi, \eta): \eta<\omega_1, \nu E_\xi=1\}.)$$

(b) Now, following the same conventions as in (a), take X to be $\omega_1\times[0, 1]$. Replace T_1 by the algebra of Lebesgue measurable sets, T_0 by the algebra of Borel measurable sets, and ν by Lebesgue measure. Proceeding as before, we obtain a locally determined Maharam measure space of magnitude $\aleph_1$ which is not complete or decomposable.

(c) Finally, try $X=\omega_2\times\omega_2$, where ω_2 is the first ordinal of greater cardinal than $\aleph_1$. Let T be

$$\{F: F\subseteq\omega_2, \textit{either } F \text{ is countable } \textit{or } \omega_2\setminus F \text{ is countable}\},$$

and define ν on T by saying that $\nu F=0$ if F is countable, 1 otherwise. For $E\subseteq X$, write

$$E_\xi=\{\eta: (\xi, \eta)\in E\}, \qquad E^\eta=\{\xi: (\xi, \eta)\in E\}.$$

Let Σ be

$$\{E: E\subseteq X, E_\xi \text{ and } E^\eta\in T \;\forall\, \xi, \eta<\omega_2\}$$

and define μ on Σ by

$$\mu E = \sum_{\xi < \omega_2} \nu E_\xi + \sum_{\eta < \omega_2} \nu E_\eta .$$

Then (X, Σ, μ) is complete and locally determined, of magnitude ω_2, but is not Maharam.

6. For the fourth and principal example of this paper, we need the following (attributed in [3] to Z. Hedrlin).

Proposition. *Let I be any index set, and let $\mathscr{A}$ be a disjoint collection of subsets of $\{0, 1\}^I$ such that each $A \in \mathscr{A}$ is countably-determined (i.e. there is a countable $J \subseteq I$ such that if $t \in A$, $u \in \{0, 1\}^I$ and $t(\iota) = u(\iota)$ for every $\iota \in J$, then $u \in A$). Then $\#(\mathscr{A}) \leqq \mathfrak{c}$.*

Proof. This is a special case of [3], Theorem 3.13.

7. Construction. Let $(\mathfrak{B}, \lambda)$ be any Dedekind σ-complete measure algebra. Then it can be represented as the measure algebra of a measure space $(Z, T, \dot{\nu})$ where Z is the Stone space of $\mathfrak{B}$ ([5], 61I). Now Z, the space of maximal ideals in $\mathfrak{B}$, can be regarded as a subset of $X = \{0, 1\}^{\mathfrak{B}}$, identifying $\mathscr{I} \in Z$ with the function t given by $t(a) = 0$ if $a \in \mathscr{I}$, 1 otherwise. Since, for any $a \in \mathfrak{B}$,

$$E_a = \{t : t(a) = 1\}$$

corresponds to

$$\{\mathscr{I} : a \notin \mathscr{I}\} = F_a \in T,$$

the embedding $Z \subseteq X$ is measurable for T and the Baire σ-algebra $\Sigma_0 \subseteq \mathscr{P}X$ generated by the sets E_a. So we have a measure μ_0 on Σ_0 given by

$$\mu_0(E) = \nu(E \cap Z) \qquad \forall\, E \in \Sigma_0 .$$

Since each $F \in T$ is ν-equivalent to some F_a, we see that the map $E \mapsto E \cap Z : \Sigma_0 \to T$ induces an isomorphism between $\mathfrak{A}(\Sigma_0, \mu_0)$ and $\mathfrak{A}(T, \nu) \cong (\mathfrak{B}, \lambda)$.

Accordingly, if the original algebra $(\mathfrak{B}, \lambda)$ is Maharam, we can take the c.l.d. version of (X, Σ_0, μ_0) to obtain a c.l.d. Maharam measure space (X, Σ, μ) with measure algebra isomorphic to $(\mathfrak{B}, \lambda)$.

The essential property of (X, Σ, μ) is the following: if $\mathscr{A} \subseteq \Sigma$ is disjoint and $\mu E > 0$ for every $E \in \mathscr{A}$, then $\#(\mathscr{A}) \leqq \mathfrak{c}$. This is because each $E \in \mathscr{A}$ has $\mu E = \sup\{\mu_0 H : H \in \Sigma_0^f, H \subseteq E\}$, so there must be a non-empty Baire set $H_E \subseteq E$. Now each H_E is countably determined, so by Theorem 6

$$\#(\mathscr{A}) = \#\{H_E : E \in \mathscr{A}\} \leqq \mathfrak{c}.$$

8. Example. If, in the construction of § 7, $(\mathfrak{B}, \lambda)$ is a Maharam measure algebra of magnitude $> \mathfrak{c}$ (e.g. $\mathfrak{B} = \mathscr{P}Y$, where $\#(Y) > \mathfrak{c}$, and $\lambda\{u\} = 1$ for every $u \in Y$), then the measure space (X, Σ, μ) obtained is a c.l.d. Maharam measure space which is not decomposable.

Remark. Prof. LeCam has pointed out that this is in effect a counterexample to (1)$\Rightarrow$(5) of [7], Theorem 5.

9. Under a suitable set-theoretic assumption, we can find an even more striking version of this example. I begin with a lemma on subspaces.

Subspaces of Measure Spaces. Let (X, Σ, μ) be a measure space, and Y a subset of X. Write

$$\Sigma_Y = \{E \cap Y : E \in \Sigma\},$$

$$\mu_Y(F) = \mu^*(F) = \inf\{\mu E : E \in \Sigma, E \supseteq F\} \qquad \forall F \in \Sigma_Y.$$

Then

(a) (Y, Σ_Y, μ_Y) is a measure space;
(b) if (X, Σ, μ) is complete, so is (Y, Σ_Y, μ_Y);
(c) if (X, Σ, μ) is c.l.d., (Y, Σ_Y, μ_Y) is semi-finite;
(d) if (X, Σ, μ) is c.l.d. Maharam, so is (Y, Σ_Y, μ_Y);
(e) if (X, Σ, μ) is decomposable, so is (Y, Σ_Y, μ_Y).

Proof. (a) is elementary; cf. [1], Ex. 17.10.

(b) is trivial.

(c) Suppose that $H \in \Sigma_Y$ is such that $\mu_Y(H \cap F) = 0$ for every $F \in \Sigma_Y^f$. Then $\mu^*(H \cap E) = 0$ for every $E \in \Sigma^f$; as (X, Σ, μ) is complete, $\mu(H \cap E) = 0$ for every $E \in \Sigma^f$; as (X, Σ, μ) is locally determined, $H \in \Sigma$ and $\mu H = 0$, so that $\mu_Y(H) = 0$.

(d) We know already that (Y, Σ_Y, μ_Y) is complete and semi-finite.

(i) Suppose that $H \subseteq Y$ and that $H \cap F \in \Sigma_Y$ for every $F \in \Sigma_Y^f$. Consider

$$\{E^\bullet : E \in \Sigma^f, E \cap Y \subseteq H\} \subseteq \mathfrak{A}(\Sigma, \mu).$$

As (X, Σ, μ) is Maharam, this has a supremum of the form $E_0^\bullet$ where $E_0 \in \Sigma$. It is easy to show that (because (X, Σ, μ) is complete)

$$\mu(F \cap ((E_0 \cap Y) \triangle H)) = 0 \qquad \forall F \in \Sigma^f,$$

from which it follows (because (X, Σ, μ) is locally determined) that $\mu((E_0 \cap Y) \triangle H) = 0$ and $H \in \Sigma_Y$. Because (X, Σ_Y, μ_Y) is semifinite, this is enough to show that it is locally determined.

(ii) To see that (Y, Σ_Y, μ_Y) is Maharam, observe that the map $E \mapsto E \cap Y : \Sigma \to \Sigma_Y$ induces an isomorphism between $\mathfrak{A}(\Sigma_Y, \mu_Y)$ and the quotient $\mathfrak{A}(\Sigma, \mu)/\mathscr{I}$, where

$$\mathscr{I} = \{E^\bullet : E \in \Sigma, \mu(E \cap Y) = 0\}.$$

Because $\mathfrak{A}(\Sigma, \mu)$ is Dedekind complete, $\mathscr{I}$ is a principal ideal, and the quotient is isomorphic to the complementary principal ideal, therefore also Dedekind complete.

(e) is easy.

10. Remarks. Part (d) above is a surprising property of c.l.d. Maharam spaces. It is not difficult, using the techniques of §5, to find (i) a subspace of a complete Maharam measure space which is not semi-finite (take $Y = \{(\xi, 0) : \xi < \omega_1\}$ in

5(a)); (ii) a subspace of a locally determined Maharam measure space which is not semi-finite; (iii) a subspace of a c.l.d. space which is not locally determined (take $Y=\omega_1\times\omega_1$ in 5(c)).

11. Example. Observe first that if I is any set of cardinal $2^{\mathfrak{c}}$, and $X=\{0,1\}^I$, there is a set $Y\subseteq X$, of cardinal $\mathfrak{c}$, which meets every non-empty Baire set in X. For there is certainly a dense set $Y_0\subseteq X$ of cardinal $\mathfrak{c}$. Now for each sequence $\langle t_n\rangle_{n\in\mathbf{N}}$ in Y_0, choose one $t\in X$ such that $t(\iota)=\lim_{n\to\infty} t_n(\iota)$ for all those $\iota\in I$ for which the limit exists. Let Y be the set of all the t's obtained. Then $\#(Y)=\mathfrak{c}$. If $E\subseteq X$ is a non-empty Baire set, let $u\in E$ and let $J\subseteq I$ be a countable set such that any $t\in X$ which agrees with u on J must belong to E. Then there is a sequence $\langle t_n\rangle_{n\in\mathbf{N}}$ in Y_0 such that $u(\iota)=\lim_{n\to\infty} t_n(\iota)$ for every $\iota\in J$. The corresponding t belongs to $Y\cap E$, as required.

(b) Let us now suppose that $2^{\aleph}=2^{\mathfrak{c}}$ for some $\aleph>\mathfrak{c}$; this is consistent with the usual axioms of set theory (for instance, we may take $\mathfrak{c}=\aleph_1$, $\aleph=\aleph_2$, $2^{\mathfrak{c}}=\aleph_3$; see [8], Chapter 8). In § 7, take $\mathfrak{B}$ to be $\mathscr{P}W$, where $\#(W)=\aleph$, and $\lambda\{v\}=1$ for each $v\in W$. Then $\#(\mathfrak{B})=2^{\mathfrak{c}}$, so, by part (a), there is a set $Y\subseteq X=\{0,1\}^{\mathfrak{B}}$ such that $\#(Y)=\mathfrak{c}$ and Y meets every non-empty element of Σ_0. Consequently Y meets every non-negligible set in Σ. If we construct Σ_Y and μ_Y from Σ and μ as in § 10, we see that (X,Σ_Y,μ_Y) is a c.l.d. Maharam measure space of magnitude $\aleph$, because $\mathfrak{A}(\Sigma_Y,\mu_Y)$ is isomorphic to $\mathfrak{A}(\Sigma,\mu)$.

Thus we have a c.l.d. Maharam space with magnitude actually greater than the cardinal of the underlying space. This means that not merely is (Y,Σ_Y,μ_Y) not decomposable, but that it cannot be made decomposable without either shrinking the measure algebra or enlarging Y.

12. Remark. Of course this example is eliminated by assuming the generalized continuum hypothesis. If (X,Σ,μ) is a Maharam measure space of infinite magnitude $\aleph$, we certainly have

$$2^{\aleph}\leqq\#(\mathfrak{A}(\Sigma,\mu))\leqq\#(\Sigma)\leqq 2^{\#(X)},$$

so (subject to *GCH*) $\aleph\leqq\#(X)$.

From 4(d) above we see also that the problem cannot arise if the underlying space has cardinal $<\mathfrak{c}$.

13. Application to Product Measures. There is no generally accepted formulation of "the" product of two measure spaces of uncountable magnitude, but the strongest candidate is in my view the following. Given measure spaces (X,Σ,μ) and (Y,T,ν), let ρ be the outer measure on $X\times Y$ given by

$$\rho A=\inf\{\sum_{i\in\mathbf{N}}\mu E_i\cdot\nu F_i\colon E_i\in\Sigma,\ F_i\in T\ \forall\, i\in\mathbf{N},\ A\subseteq\bigcup_{i\in\mathbf{N}} E_i\times F_i\}$$

(with the usual convention that $0.\infty=0$); let λ_ρ be the measure defined by ρ; and let λ be the c.l.d. version of λ_ρ. (This is the completion of the product measure defined in [1], §39. Note that λ_ρ need not be semi-finite, so that λ and λ_ρ may fail to have isomorphic measure algebras; cf. [1], Ex. 39.17.) It is easy to see that this construction – the "c.l.d. product measure" – is associative, commutative, and distributive over direct sums; so that the c.l.d. product of two decomposable spaces is always decomposable. But the c.l.d. product of two Maharam spaces need not be Maharam, because of the following.

14. Proposition. *A c.l.d. measure space (X, Σ, μ) is decomposable iff its c.l.d. product with every probability space is Maharam.*

Proof. (a) If (X, Σ, μ) is decomposable, so is its c.l.d. product with any decomposable measure space; so such products have to be Maharam.

(b) Suppose that (X, Σ, μ) has Maharam c.l.d. products. Let $\langle a_\iota\rangle_{\iota\in I}$ be a maximal disjoint family in $\mathfrak{A}(\Sigma, \mu)^f$. Let (Y, T, ν) be a probability space such that $\#(\mathfrak{A}(T, \nu)) \geqq \#(I)$ (e.g. $Y=\{0,1\}^I$ with the usual product measure), and let $\langle b_\iota\rangle_{\iota\in I}$ be a family of distinct elements of $\mathfrak{A}(T, \nu)$. Choose $A_\iota\in\Sigma$, $B_\iota\in T$ such that $A_\iota^\bullet = a_\iota$, $B_\iota^\bullet = b_\iota$ for each $\iota\in I$. Let $(X\times Y, \Lambda, \lambda)$ be the c.l.d. product of (X, Σ, μ) and (Y, T, ν).

In $\mathfrak{A}(\Lambda, \lambda)$, consider the family $\{(A_\iota\times B_\iota)^\bullet : \iota\in I\}$. By hypothesis, this has a supremum, which is of the form $E^\bullet$ where $E\in\Lambda$ and

$$(A_\iota\times B_\iota)^\bullet \subseteq E^\bullet, \quad (A_\iota\times(Y\setminus B_\iota))^\bullet \cap E^\bullet = 0 \qquad \forall\, \iota\in I.$$

For $t\in X$ write $E_t=\{u: (t,u)\in E\}$. On $A_\iota\times Y$, λ is the ordinary (completed) product measure, so we can use Fubini's theorem to see that

$$A'_\iota = \{t: t\in A_\iota,\ \nu(E_t \triangle B_\iota)=0\}$$

is μ-measurable and μ-almost the whole of A_ι. Now (because the b_ι are all distinct) $\{A'_\iota : \iota\in I\}$ is a disjoint family and $\sup_{\iota\in I}(A'_\iota)^\bullet = \sup_{\iota\in I} a_\iota = 1$ in $\mathfrak{A}(\Sigma, \mu)$. By Theorem 2, (X, Σ, μ) is decomposable.

15. Corollary. *There exist Maharam measure spaces which have a non-Maharam c.l.d. product.*

16. Remark. Observe that the c.l.d. version of any measure space (X, Σ, μ) is isomorphic to the c.l.d. product of (X, Σ, μ) with the elementary probability space (S, Ω, p) with one point. Consequently, if (X, Σ, μ) and (Y, T, ν) are two Maharam measure spaces of magnitude $\leqq \mathfrak{c}$,

$$(X, \Sigma, \mu)\times(Y, T, \nu) \cong ((X, \Sigma, \mu)\times(Y, T, \nu))\times((S, \Omega, p)\times(S, \Omega, \rho))$$

(because the product is certainly c.l.d.)

$$\cong ((X, \Sigma, \mu)\times(S, \Omega, p))\times((Y, T, \nu)\times(S, \Omega, p))$$

(because the c.l.d. product is associative and commutative). Thus $(X, \Sigma, \mu)\times(Y, T, \nu)$ is isomorphic to the product of decomposable spaces, therefore decomposable, therefore Maharam.

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