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On Multiparameter Ergodic and Martingale Theorems in Infinite Measure Spaces

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Summary. A unified proof is given of several ergodic and martingale theorems in infinite measure spaces.

Introduction

For a fixed positive integer m, define $I^m = I_1 \times \ldots \times I_m$ with each $I_k = N$ (positive integers), and partial order defined on I^m by $s=(s_1, \ldots, s_m) \leqq t=(t_1, \ldots, t_m)$ if and only if $s_k \leqq t_k$ for all $k=1, \ldots, m$. We are concerned here with almost everywhere convergence of processes indexed by I^m when all the indices s_j converge to infinity independently ("unrestricted" convergence). In a recent article [23], the second author has formulated a general principle yielding simultaneous proofs of many a.e. multiparameter convergence theorems. As a starting point, we restate this principle. Recall that a σ-complete Banach lattice E has an *order continuous norm* if $g_n \downarrow 0$ implies $\|g_n\| \to 0$. Since we only consider Banach lattices of functions over measure spaces for which convergence in order corresponds to a.e. convergence (in fact, every Banach lattice with order-continuous norm may be so represented, as shown e.g. in [14], p. 25), we only discuss a.e. convergence. The words a.e. may or may not be omitted.

Theorem 1 [23]. *Let $L(1) \supset L(2) \supset \ldots \supset L(m)$ be Orlicz spaces satisfying the Δ_2 condition, or, more generally, Banach lattices with an order-continuous norm. For $k=1, \ldots, m$, let $T(k,n)$, $n \in N$ be positive and linear operators from $L(k)$ to $L(1)$ such that*

(a) $\lim_n T(k,n)f = T(k,\infty)f$ *exists a.e. for each function f in $L(k)$ and is in $L(k)$.*

(b) $\sup_n |T(k,n)f| \in L(k-1)$ *for each f in $L(k)$, $k=2, \ldots, m$.*

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Then for each f in $L(m)$ one has

(c) $\lim T(1,s_1)\dots T(m,s_m)f=T(1,\infty)\dots T(m,\infty)f$ *a.e.*

This principle is applicable if all $L(k)$ spaces are L_p spaces for a fixed p, $1\leqq p<\infty$, since these spaces have an order-continuous norm. Thus, the principle gives multiparameter versions of purely L_p results, e.g., theorems of Akcoglu and Stein, in either finite or infinite measure spaces. The situation is different if the operators act simultaneously on the spaces L_1 and L_∞, which is the case of the martingale theorem, and theorems of Rota and Dunford-Schwartz. In *finite* measure spaces, Theorem 1 still implies the appropriate k-parameter versions, with $L(k)=L\log^{k-1}L$ for martingale and Dunford-Schwartz, $L(k)=L\log^k L$ for Rota. But in the case of *infinite* measures, the Orlicz spaces $L\log^k L$ do not have any more an order-continuous norm, and the right setting are the spaces R_k introduced by N. Fava [9], who proved the multiparameter Dunford-Schwartz theorem for R_k. A real difficulty arises because the spaces R_k, defined as intersections of Orlicz classes (see Section 1), are not known to satisfy the conditions required of the $L(k)$ classes in Theorem 1. However, we prove that R_0 is an order-continuous Banach lattice for the L_1+L_∞ norm. This is particularly gratifying because, unlike R_0, the space L_1+L_∞ has neither an order-continuous norm nor one-parameter limit theorems. The space R_0 can now play the role of $L(1)$. We require a version of the principle in which only $L(1)$ is assumed to be a Banach lattice with an order-continuous norm, and this is accomplished at the price of having the one-dimensional operators $T(k,n)$ defined on $L(1)$ rather than only on $L(k)$. As an application, we obtain a simple proof of the theorem of Fava, and also apparently new R_k versions of the martingale theorem and of Rota's theorem. No difficult properties of Banach lattices are used, and our results are easier to prove and are more widely applicable than multiparameter maximal inequalities adapted to particular cases.

Theorem 2. *Let $L(1)\supset L(2)\supset\dots\supset L(m)$ where $L(1)$ is a Banach lattice with an order-continuous norm, and $L(2),\dots,L(m)$ are (not necessarily closed) linear subspaces of $L(1)$. For $k=1,\dots,m$ let $T(k,n)$, $n\in N$ be positive and linear operators from $L(1)$ to $L(1)$ such that the assumption (b) of Theorem 1 holds. Assume also that (a′) holds:*

(a′) *For each $k=1,\dots,m$ and each f in $L(1)$, $\lim T(k,n)f=T(k,\infty)f$ exists a.e. and is in $L(1)$.*

Then for each f in $L(m)$ the condition (c) of Theorem 1 holds.

The proof is the same as the proof of Theorem 1, therefore it is omitted. However, we will explain here the role played by the order-continuity of $L(1)$: it is used to establish the *monotone continuity for order convergence* of positive operators T on $L(1)$, that is, the implication: $f_n\downarrow 0\Rightarrow Tf_n\downarrow 0$. Indeed, $f_n\downarrow 0$ implies $\|f_n\|\to 0$, hence $\|Tf_n\|\to 0$, because a positive linear operator on a Banach lattice is necessarily continuous (see e.g., [14], p. 2). Let $g=\lim Tf_n$, then $\|g\|\leqq\|Tf_n-g\|+\|Tf_n\|$ implies $g=0$. In the proof of Theorem 2, the monotone continuity for order convergence is required of the one-parameter limit operators $T(k,\infty)$.

The first section gives basic properties of the spaces R_k. Section 2 discusses conditions for the applicability of Theorem 2 to the R_k spaces. Section 3 reviews the one-parameter infinite-measure setting of Dunford-Schwartz, martingale, and Rota. To start the multiparameter induction, it is here necessary to go beyond the classical case, proving the theorems in R_0; in R_1 for Rota. The R_1 setting is an improvement over N. Starr [22]. Theorem 2 is also used to simplify the proof of the one-parameter Rota theorem. Section 4 gives the promised multiparameter theorems in R_k. Section 5 proves a new multiparameter $L\log^k L$ version of the Chacon-Ornstein theorem. Section 6 considers Banach valued processes, giving a Banach-valued version of our martingale result, and a multiparameter version of E. Mourier's ergodic theorem. This is achieved by an application of the earlier results to positive operators on spaces of real functions, which "dominate" operators acting on spaces of Banach-valued functions.

I. Definitions and Basic Notions

Let $(\Omega, \mathfrak{F}, \mu)$ be a σ-finite measure space. The relations are often considered modulo sets of measure zero. Let $L_p = L_p(\Omega, \mathfrak{F}, \mu)$, $1 \leqq p \leqq \infty$ be the usual Banach spaces of real valued functions defined on Ω. By $L_1 + L_\infty = Y$ we understand the Banach space of functions f which can be written as $g+h$ where $g \in L_1$ and $h \in L_\infty$, endowed with the norm

$$\|f\| = \inf\{\|g\|_1 + \|h\|_\infty : f = g+h\}.$$

The completeness of the norm $\|\cdot\|$ follows from the completeness of the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$. For various properties of Y, see e.g., [14], p. 119.

For each $k=1,2,\ldots$, let R_k be the class of functions f such that

$$\int_{\{|f|>1/n\}} |nf|(\log|nf|)^k \, d\mu = \int |nf|(\log^+|nf|)^k \, d\mu < \infty$$

for all $n \in N$. R_0 is the class of functions f such that for each $n \in N$,

$$\int_{\{|f|>1/n\}} |f| \, d\mu < \infty.$$

An equivalent definition of R_k is as follows:

Let for $s>0$, $\Phi_k^s(x) = sx(\log^+ sx)^k$: $\mathbb{R}^+ \to \mathbb{R}^+$, then Φ_k^s is an Orlicz function. Let

$$L_k^s = \{f : \int \Phi_k^s(|f|) \, d\mu < \infty\}.$$

Each L_k^s is an Orlicz class (see e.g., [12], p. 60). Now $R_k = \bigcap_N L_k^n$. Since the classes L_k^n are convex, it follows that the spaces R_k are linear.

By $L\log^k L$ we understand the Orlicz spaces corresponding to Orlicz functions $\Phi_k(x) = x(\log^+ x)^k$. The following proposition collects simple properties of the R_k spaces ([9]).

Proposition 1.1. i) *Each* R_k *is a linear space.*

ii) $L_1 \subset R_0 \subset L_1 + L_\infty$.

iii) $R_k \subset L\log^k L$, $k=1,2,\ldots$; $R_k = L\log^k L$ *if and only if* $\mu(\Omega)<\infty$.

iv) $R_0 \supset R_1 \supset R_2 \supset \ldots$.

v) *For each* k, R_k *contains the linear space spanned by* $\bigcup_{p>1} L_p$.

The containment in (v) is proper, as shown by the example of $f(x)=1/\log x$ for $x \geqq 2$, $f(x)=0$ for $x<2$. It is also known that the spaces R_k, $k\geqq 1$, are rearrangement invariant Banach lattices for the norm of $L\log^k L$, closure in this norm of the class of simple functions (see remarks about spaces H_M, [14], p. 120). Here we do not use these properties. We only require the following:

Theorem 1.2. *The space* R_0 *is an order continuous Banach lattice for the norm* $\|\cdot\|$ *(the* L_1+L_∞ *norm).*

Proof. We first prove that $(R_0, \|\cdot\|)$ is a closed subspace of the Banach lattice $Y = L_1 + L_\infty$.

Let $(f_n) \subset R_0$, $f \in Y$ be such that $\|f_n - f\| \to 0$. We will show that $f \in R_0$. Let $t>0$. Then there exists $p \in N$ such that $\|f-f_p\| \leqq t$. Since $f-f_n \in Y$, there exist $g_p \in L_1$, $h_p \in L_\infty$ such that

$$f-f_p = g_p + h_p \quad \text{and} \quad \|g_p\|_1 \leqq t, \quad \|h_p\|_\infty \leqq t.$$

Now $|f| \leqq |f_p| + |g_p| + |h_p|$, and

$$\begin{aligned}\{|f|>3t\} &\subseteq \{|f_p|+|g_p|+|h_p|>3t\} \subseteq \{|f_p|+|g_p|>2t\} \\ &\subseteq \{|f_p|>t\} \cup \{|g_p|>t\}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{\{|f|>3t\}} |f|\,d\mu \leqq \int_{\{|f_p|>t\}} |f_p|\,d\mu + \int_{\{|g_p|>t\}} |f_p|\,d\mu \\ +2\|g_p\|_1 + t\mu(\{|f_p|>t\}) + t\mu(\{|g_p|>t\}).\end{aligned}$$

The third and fifth terms on the right are finite because $g_p \in L_1$. The remaining terms on the right are finite because $f_p \in R_0$. Thus, $\int_{\{|f|>3t\}} |f|\,d\mu < \infty$, and since t is arbitrary, it follows that $f \in R_0$.

We next show that $\|\cdot\|$ restricted to R_0 is an order-continuous norm. Let $(f_n) \subset R_0$, $f_n \downarrow 0$ a.e., then $f_n 1_{\{f_n>t\}}$ is in L_1 and hence $\|f_n 1_{\{f_n>t\}}\|_1 \downarrow 0$. Also

$$\|f_n\| \leqq \|f_n 1_{\{f_n>t\}}\|_1 + t,$$

therefore $\lim \|f_n\| \leqq t$ for all $t>0$. Thus $\lim \|f_n\| = 0$. ///

II. A Maximal Inequality

To apply Theorem 2 of the Introduction to the R_k spaces, it is necessary to show that the condition (b) holds with $L(k)=R_{k-1}$. In fact, this will follow from a discussion of the L_k^n classes.

Given an Orlicz function Φ with $\Phi' = \phi$, set

$$\xi(x) = x\,\phi(x) - \Phi(x).$$

If $\Phi_k^n(x) = n x (\log^+ n x)^k$ then $\xi_k^n(x) = k\,\Phi_{k-1}^n(x)$.

Proposition 2.1. *Let f, g be positive functions with $f \in L_k^n$ for some $k \geqq 1$. Suppose that for some $c \geqq 1$ and every $\lambda > 0$*

$$\mu(\{g > c\lambda\}) \leqq \frac{1}{\lambda} \int_{\{f > \lambda\}} f\,d\mu. \tag{1}$$

Then $g \in L_{k-1}^{n/c}$. Hence if $f \in R_k$, then $g \in R_{k-1}$.

Proof. Let $t > 0$ be fixed. From (1) we have

$$\mu\left(\left\{g > c\frac{\lambda}{t}\right\}\right) \leqq \frac{t}{\lambda} \int_{\{f > \lambda/t\}} f\,d\mu.$$

Hence applying Fubini, we have:

$$\begin{aligned}
\int \xi_k\left(\frac{t}{c} g\right) d\mu &= \int \int_0^{\frac{t}{c} g} d\xi_k(\lambda)\,d\mu = \int_0^\infty \mu\left(g > \frac{c}{t}\lambda\right) d\xi_k(\lambda) \\
&\leqq \int_0^\infty \frac{t}{\lambda} \int_{\{f > \lambda/t\}} f\,d\mu\,d\xi_k(\lambda) \\
&= \int t f \int_0^{tf} \frac{d\xi_k(\lambda)}{\lambda}\,d\mu \\
&= \int t f\,\phi_k(tf)\,d\mu \\
&= \int t f (\log^+ t f)^k\,d\mu + k \int t f (\log^+ t f)^{k-1}\,d\mu < \infty.
\end{aligned}$$

Since t is arbitrary, it follows that $g \in L_{k-1}^{n/c}$. ///

III. One Parameter Theory

In this section, we discuss the behavior of one parameter sequences in σ-finite measure spaces.

Lemma 3.1. *Let $(T_n, n \in N)$ be a sequence of positive linear operators on $Y = L_1 + L_\infty$ such that $\sup_n \|T_n\|_\infty = c < \infty$. ($\|T\|_\infty$ denotes the L_∞ norm of T.) Assume that $\lim T_n f$ exists a.e. for all $f \in L_1$. Then*

i) $\limsup T_n f < \infty$ *a.e. for all $f \in Y$.*

ii) $\lim T_n f$ *exists a.e. for all $f \in R_0$.*

Proof. i) Let $f \in Y$. Then $f = g + h$, $g \in L_1$, $h \in L_\infty$, $\lim T_n g$ exists a.e. and $|T_n h| \leqq c\,\|h\|_\infty$. Consequently, $\limsup T_n f$ is finite a.e. ii) Let $T_\infty f = \limsup T_n f$, $f \in Y$. The operator T_∞ is sublinear and $\|T_\infty\|_\infty \leqq c$. Now let $f \in R_0$, then

$$f^t = f\,1_{\{|f| > t\}} \in L_1$$

for each $t>0$, hence

$$\begin{aligned} T_\infty f &= \limsup (T_n f^t + T_n f 1_{\{|f| \leqq t\}}) \\ &= \lim T_n f^t + \limsup T_n f 1_{\{|f| \leqq t\}} \\ &= T_\infty f^t + T_\infty f 1_{\{|f| \leqq t\}}. \end{aligned}$$

Also

$$\begin{aligned} |T_n f - T_\infty f| &= |T_n f^t + T_n f 1_{\{|f| \leqq t\}} - T_\infty f^t - T_\infty f 1_{\{|f| \leqq t\}}| \\ &\leqq |T_n f^t - T_\infty f^t| + 2ct. \end{aligned}$$

Thus, $\limsup |T_n f - T_\infty f| \leqq 2ct$. Since t is arbitrary, $|T_n f - T_\infty f| \to 0$ a.e. ///

Lemma 3.2. *Let $(T_n, n \in N)$ be a sequence of positive linear operators on Y such that $\|T_n\|_\infty \leqq c < \infty$. For each $f \in Y$, let $f^* = \sup_n T_n |f|$. Assume that for each $f \in L_1$ and every $\lambda > 0$,*

$$\mu(\{f^* > c\lambda\}) \leqq \frac{1}{\lambda} \|f\|_1. \tag{1}$$

Then for every $f \in R_0$ and every $\lambda > 0$,

$$\mu(\{f^* > 2c\lambda\}) \leqq \frac{1}{\lambda} \int_{\{|f| > \lambda\}} |f| \, d\mu. \tag{2}$$

Proof. Let $f \in R_0$, then $f^\lambda \in L_1$, and since $|f| \leqq |f^\lambda| + \lambda$, we have $f^* \leqq \sup_n T_n |f^\lambda| + c\lambda$. Hence

$$\mu(\{f^* > 2c\lambda\}) \leqq \mu(\{\sup_n T_n |f^\lambda| > c\lambda\}) \leqq \frac{1}{\lambda} \|f^\lambda\|_1 = \frac{1}{\lambda} \int_{\{|f| > \lambda\}} |f| \, d\mu.$$

Let $(\mathfrak{F}_n, n \in N)$ be either an increasing (or decreasing) sequence of sub-σ-fields of $\mathfrak{F}$, and let $\mathfrak{F}_\infty = \bigvee_n \mathfrak{F}_n \, (\bigcap_n \mathfrak{F}_n)$. We assume that μ is σ-finite on each σ-field $\mathfrak{F}_n$ (but not necessarily on $\bigcap_n \mathfrak{F}_n$).

Remark. Observe that the weak maximal inequality (1) in Lemma 3.2 holds for $f \in L_1$ if $f^* = \sup_n E(|f| \,|\, \mathfrak{F}_n|)$ and $c=1$. It is the usual weak martingale inequality which extends to the σ-finite case by considering $\Omega = \bigcup_{i=1}^\infty \Omega_i$, $\mu(\Omega_i) < \infty$, $\Omega_i \in \mathfrak{F}_n$ (see C. Dellacherie and P.A. Meyer [5], p. 33).

Theorem 3.3. *If $f \in R_0$, then $E(f \,|\, \mathfrak{F}_n)$ converges a.e.*

Proof. The main case, where $f \in L_1$, is known (see [5], pp. 34, 35). Since $\|E(\cdot \,|\, \mathfrak{F}_n)\|_\infty \leqq 1$, we can apply Lemma 3.1 to obtain convergence for all $f \in R_0$. ///

Let T_i be positive operators on Y such that $T_i 1 = 1$, $T_i^* 1 = 1$ (bistochastic operators). Let $U_n f = T_1 \ldots T_n T_n^* \ldots T_1^* f$. The following result is Rota's theorem in σ-finite measure spaces.

Theorem 3.4. *If $f \in R_1$, then $U_n f$ converges a.e.*

Proof. In the finite measure case, the proof of this result (see [21, 5, 7]) shows that U_n can be represented as $E(\cdot|\mathfrak{F}_n|\mathfrak{B})$ where $\mathfrak{F}_n\downarrow$ and $\mathfrak{B}$ is a fixed σ-field. The argument extends to the σ-finite case. The previous discussion (Remark, Lemma 3.2, Proposition 2.1) shows that if $f \in R_1$ then $\sup_n E(|f|\,|\mathfrak{F}_n) \in R_0$. The a.e. convergence now follows from Theorem 3.3 and Theorem 2 of the Introduction, applied with $L(1)=R_0$, $L(m)=L(2)=R_1$, $T(1,n)=E(\cdot|\mathfrak{F}_n)$ and $T(2,n)=E(\cdot|\mathfrak{B})$, $n \in N$. ///

Theorem 3.4 was obtained by N. Starr [22] for $f \in (\bigcup_{1 \leqq p < \infty} L_p) \cap L\log L$. One has that $(\bigcup_{1 \leqq p < \infty} L_p) \cap L\log L \subset R_1$ and the inclusion is proper: Let again $f(x) = 1/\log x$ for $x \geqq 2$, $f(x)=0$ otherwise.

In Theorem 3.4, R_1 cannot be replaced by L_1 even if the measure space is finite. This follows from a counterexample of D. Ornstein [19].

Assume that T is an L_1-contraction and also L_∞-bounded, i.e., $\sup_n \|T^n\|_\infty \leqq c < \infty$. We can assume $c \geqq 1$.

Lemma 3.5. *If $f \in L_1^+$ and $g = \sup_n \frac{1}{n}\sum_{k=0}^{n-1} T^i f$, then*

$$\mu(\{g > c\lambda\}) \leqq \frac{1}{\lambda}\|f\|_1.$$

The proof is similar to the case $\|T\|_1 \leqq 1$, $\|T\|_\infty \leqq 1$.

Theorem 3.6. *Let $\|T\|_1 \leqq 1$, $\sup_n \|T^n\|_\infty < \infty$. If $f \in R_0$, then $\lim \frac{1}{n}\sum_{i=0}^{n-1} T^i f$ exists a.e.*

Proof. For $f \in L_1$, and $\|T\|_1 \leqq 1$, $\|T\|_\infty \leqq 1$, the results is the Dunford-Schwartz theorem (see e.g., [11], p. 27). This result remains valid assuming $\sup_n \|T^n\|_\infty < \infty$ instead of $\|T\|_\infty \leqq 1$, as observed by D. Ornstein and L. Sucheston [20]. If $U_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i$, then $\sup_n \|U_n\|_\infty \leqq c$, hence Lemma 3.1 applies and we obtain convergence for $f \in R_0$. ///

The following simple examples show that the convergence theorems do not hold in $L_1 + L_\infty$.

Let (a_n) be a sequence of real numbers such that $|a_n|=1$ and $(a_0 + a_1 + \ldots + a_{2^n})/2^n + 1$ diverges. Consider the σ-finite measure space $([0,\infty), \mathfrak{F}, \mu)$, $\mathfrak{F}$ being the Borel σ-field and μ the Lebesgue measure. Let $g(x) = a_n$ if $n \leqq x < n+1$, $n = 0, 1, 2, \ldots$. Clearly, $g \in Y = L_1 + L_\infty$, but $g \notin R_0$.

Let

$$A_k^n = \{x : k2^n \leqq x < (k+1)2^n\}, \qquad k = 0, 1, \ldots.$$

Define $\mathfrak{F}_n = \sigma(A_k^n, k = 0, 1, \ldots)$, then $\mathfrak{F}_{n+1} \subset \mathfrak{F}_n$ and

$$E(g|\mathfrak{F}_n) = \sum_{i=1}^{2^n} a_i/2^n \quad \text{on } [0,2)$$

diverges. This is a counter-example to the reversed martingale theorem on $L_1 + L_\infty$.

Next consider the shift operator $Tf(x)=f(x+1)$ on L_1+L_∞. Then

$$\frac{1}{n}\sum_{i=0}^{n-1} T^i g(x)=\frac{a_0+\ldots+a_{n-1}}{n} \quad \text{on } [0,1),$$

again diverges. This is a counter-example to Dunford-Schwartz on L_1+L_∞.

IV. Multiparameter Results

For each fixed k, let $\mathfrak{F}_n^k$, $n\in N$ be either increasing (or decreasing) sub-σ-fields of $\mathfrak{F}$, and let $\mathfrak{F}_\infty^k=\bigvee_n \mathfrak{F}_n^k(\bigcap_n \mathfrak{F}_n^k)$. We assume that μ is σ-finite on each σ-field $\mathfrak{F}_n^k$ (but not necessarily on $\bigcap_n \mathfrak{F}_n^k$). Let $E_n^k=E(\cdot|\mathfrak{F}_n^k)$, $n\in N$, $U_s=E_{s_1}^1 E_{s_2}^2 \ldots E_{s_m}^m$ for $s=(s_1,s_2,\ldots,s_m)\in I^m$. By Theorem 3.3, Lemma 3.2, Proposition 2.1, the system (E_n^k) satisfies the conditions (a′) and (b) of the Introduction. Let $T(k,n)=E_n^k$ and for $f\in R_0$, $T(k,\infty)f=\lim_n E_{s_m}^m f$. Applying Theorem 2 with $L(k)=R_{k-1}$ for $k=1,2,\ldots,m$, we obtain:

Theorem 4.1. *If* $f\in R_{m-1}$, *then*

$$E_{s_1}^1 E_{s_2}^2 \ldots E_{s_m}^m f$$

converges to

$$T(1,\infty)\ldots T(m,\infty)f$$

a.e. as the indices $s_i\to\infty$ *independently.*

Let $(\mathfrak{F}_s, s\in I^m)$ be an increasing (decreasing) net of sub-σ-fields of $\mathfrak{F}$. Let $\mathfrak{F}_\infty=\bigvee \mathfrak{F}_s(\bigcap_s \mathfrak{F}_s)$. Now for $s=(s_1,s_2,\ldots,s_m)\in I^m$, $\mathfrak{F}_s^k$ is defined as the σ-field obtained by lumping together the σ-fields on all axes except for the k-th one. That is

$$\mathfrak{F}_s^k=\bigvee \mathfrak{F}_{(s_1,s_2,\ldots,s_m)},$$

where $\bigvee$ is taken over all $s_1\in I_1,\ldots,s_{k-1}\in I_{k-1}$, $s_{k+1}\in I_{k+1},\ldots,s_m\in I_m$. Let for $k\leq m$, $s\in I^m$

$$E_s^k=E(\cdot|\mathfrak{F}_s^k).$$

The commutation assumption is the assumption that the operators $E_s^k=E_{s_k}^k$ commute; then

$$E(\cdot|\mathfrak{F}_s)=E_{s_1}^1 E_{s_2}^2 \ldots E_{s_m}^m.$$

Applying this representation, we obtain:

Theorem 4.2. *Let* $(f_s, s\in I^m)$ *be a martingale or a reversed martingale of the form*

$$f_s=E(f|\mathfrak{F}_s)$$

for $f\in R_{m-1}$. *Suppose that* $(\mathfrak{F}_s)$ *satisfies the commutation assumption. Then* $\lim_{I^m} f_s$ *exists a.e.*

The next theorem is the multiparameter version of Rota's theorem in σ-finite measure spaces.

Theorem 4.3. *Let for $i=1,2,\dots,m$ and for $n\in N$, T_n^i be a bistochastic operator on $Y=L_1+L_\infty$. Set*

$$U_n^i=T_1^i\dots T_n^i(T_n^i)^*\dots(T_1^i)^*.$$

If $f\in R_m$ then

$$\lim U_{s_1}^1\dots U_{s_m}^m f$$

exists a.e. as the indices $s_i\to\infty$ independently.

Proof. For $i=1,2,\dots,m$, and $n\in N$ there exist σ-fields $\mathfrak{B}^i$ and $\mathfrak{F}_n^i$ such that $U_n^i=E(\cdot|\mathfrak{F}_n^i|\mathfrak{B}^i)$ (see Theorem 3.4). By Jensen's inequality, $E(\cdot|\mathfrak{B}^i)$ maps each class L_k^n to itself; it follows that if $f\in R_k$ then $\sup_n U_n^i f=\sup_n E(f|\mathfrak{F}_n^i|\mathfrak{B}^i)\in R_{k-1}$. Hence Theorem 2 of the Introduction is applicable with $L(k)=R_{k-1}$ for $k=1,2,\dots,m+1$. ///

Let $T_1,\dots,T_m$ be linear positive operators on Y. Assume that each T_i is an L_1-contraction and L_∞-power bounded.

Theorem 4.4. *If $f\in R_{m-1}$, then*

$$\lim\frac{1}{s_1 s_2\dots s_m}\sum_{k_1=0}^{s_1-1}\dots\sum_{k_m=0}^{s_m-1}T_1^{k_1}\dots T_m^{k_m}f$$

exists a.e. as the indices $s_i\to\infty$ independently.

Proof. Let $T(k,n)=\frac{1}{n}\sum_{i=0}^{n-1}T_k^i$. By Theorem 3.6, Lemmas 3.5, 3.2 and Proposition 2.1, the system $(T(k,n))$ satisfies conditions (a′) and (b) of Theorem 2 with $L(k)=R_{k-1}$ for $k=1,2,\dots,m$.

V. Multiparameter Chacon-Ornstein Theorem

Let $T_1, T_2,\dots,T_m$ be positive linear contraction operators on L_1. We first recall the following weak maximal inequality.

Lemma 5.1. *Let f, $g\in L_1^+$, $g>0$ and set*

$$h=\sup_n\frac{\sum_{i=0}^n T^i f}{\sum_{i=0}^n T^i g}.$$

Then for every $\lambda>0$

$$\lambda\int_{\{h>\lambda\}}g\,d\mu\leqq\int_{\{h>\lambda\}}f\,d\mu.$$

Proof. Let $B(\lambda)=\bigcup_{n\geqq 0}\left\{\sum_{i=0}^n T^i f>\sum_{i=0}^n T^i\lambda g\right\}$. Since $T^i\lambda g=\lambda T^i g$, we have $B(\lambda)=\{h>\lambda\}$. The maximal inequality now follows from Hopf's inequality (see e.g., [17], p. 114 or [11], p. 23). ///

Let $dv=g\,d\mu$, then v is a finite measure and v and μ have exactly the same null sets. Then Lemma 5.1 implies that for each $\lambda>0$ and each $f\in L_1(\mu)$

$$\begin{aligned}2\lambda v(\{h>2\lambda\}) &\leqq \int_{\{h>2\lambda\}} f/g\,dv\\ &\leqq \int_{\{f/g>\lambda\}} f/g\,dv+\int_{\{f/g<\lambda,h>2\lambda\}} f/g\,dv\\ &\leqq \int_{\{f/g>\lambda\}} f/g\,dv+\lambda v(\{h>2\lambda\})\end{aligned}$$

from which we obtain

$$v(\{h>2\lambda\})\leqq\frac{1}{\lambda}\int_{\{f/g>\lambda\}} f/g\,dv.$$

Now Proposition 2.1 (with $n=1$) implies:

Lemma 5.2. *If $f/g\in L\log^k L(v)$ then $h\in L\log^{k-1}L(v)$.*

Theorem 5.3. *Let $(\Omega,\mathfrak{F},\mu)$ be a finite measure space and let g be a fixed element of $L_1^+(\mu)$ bounded away from zero* (essinf $g=c>0$). *Let v be the measure $g\cdot\mu$. Then for each f such that $f/g\in L\log^{m-1}(v)$ one has that*

$$\lim\frac{\sum_{k_1=0}^{t_1}T_1^{k_1}}{\sum_{k_1=0}^{t_1}T_1^{k_1}g}\cdots\frac{\sum_{k_m=0}^{t_m}T_m^{k_m}f}{\sum_{k_m=0}^{t_m}T_m^{k_m}g}$$

exists a.e. as the indices $t_i\to\infty$ independently.

Proof. Let for $k\geqq 0$, $S_k=\{f: f/g\in L\log^k L(v)\}$. The elements of S_k are defined modulo sets of v, or μ, measure zero. Each S_k is a linear space, and $L_1(\mu)=S_0\supset S_1\supset S_2\supset\ldots\supset S_m$. Let

$$T(k,n)=\frac{\sum_{i=0}^{n}T_k^i}{\sum_{i=0}^{n}T_k^i g}.$$

Clearly the operators $T(k,n)$ are from $L_1(\mu)$ to $L_1(v)$; since $g\geqq c$, $L_1(v)\subset L_1(\mu)$. Furthermore, Lemma 5.2 implies that if $f\in S_k$ then $h\in L\log^{k-1}L(v)$, hence again from $g\geqq c$ it follows that $h\in S_{k-1}$. Let $L(k)=S_{k-1}$ for $k=1,2,\ldots,m$; then the condition (b) in Theorems 1 and 2 is satisfied. The condition (a′) in Theorem 2 follows from the one-parameter Chacon-Ornstein theorem. The convergence now follows from Theorem 2. ///

VI. Banach Spaces

Let $(E,|\,|)$ be a Banach space. All functions considered are from Ω to E, strongly measurable and Bochner integrable. As in the real case, we define $(Y(E),\|\cdot\|)$ to be the Banach space of all E-valued functions that can be written as a sum of an L_1 and L_∞-function. $R_k(E)$, $L\log^k L(E)$ are spaces of functions f such that $|f|$ are in R_k, respectively in $L\log^k L$.

A linear operator T on $Y(E)$ is said to be *positively dominated* (by $\hat{T}$) if there exists a positive linear operator $\hat{T}$ on Y such that

$$\|Tf\| \leqq \hat{T}\|f\| \quad \text{a.e.}$$

for all $f \in Y(E) = L_1(E) + L_\infty(E)$. The operator $\hat{T}$ will be called a *positive domimant* of T.

We now give some examples of positively dominated operators.

1. Any linear operator T on $L_1(\mathbb{R}) + L_\infty(\mathbb{R})$ is positively dominated by its modulus $|T|$, defined by

$$|T| f = \sup_{|g| \leqq f} |Tg| \qquad \text{for all } f \geqq 0.$$

This follows from Dunford-Schwartz [8], Lemma 4, p. 672; see also Chacon-Krengel [3].

2. Assume that the Banach space E has the Radon-Nikodym property (RNP). Let $\mathfrak{B}$ be any sub-σ-field of $\mathfrak{F}$ such that μ restricted to $\mathfrak{B}$ is σ-finite. The conditional expectation $E(\cdot|\mathfrak{B})$ is defined on $Y(E)$. Indeed, for $f \in Y(E)$, $E(f|\mathfrak{B})$ is the Radon-Nikodym density of the measure $f \cdot \mu$ with respect to σ-finite measure μ on $\mathfrak{B}$. Since $\|E(f|\mathfrak{B})\| \leqq E(\|f\||\mathfrak{B})$ a.e. for all $f \in Y(E)$, the operator $E(\cdot|\mathfrak{B})$ is positively dominated by $E(\cdot|\mathfrak{B})$ restricted to $Y = L_1(\mathbb{R}) + L_\infty(\mathbb{R})$.

3. Let θ be a measure preserving point-transformation on $(\Omega, \mathfrak{F}, \mu)$. The linear operator $Tf = f \circ \theta$, $f \in Y(E)$ is positively dominated by $\hat{T}$ where $\hat{T}\|f\| = \|f \circ \theta\|$.

4. Assume that E has RNP, and $\mu(\Omega) = 1$. Let $P_1(\omega, F)$, $F \in \mathfrak{F}$, be a transition probability defined on $(\Omega, \mathfrak{F}, \mu)$ that is, $P_1(\omega, \cdot)$ is a probability measure for each $\omega \in \Omega$, and $P_1(\cdot, F)$ is a measurable function for each $F \in \mathfrak{F}$. Assume furthermore that P_1 preserves μ-null sets, i.e., $\mu(A) = 0 \Rightarrow P(\cdot, A) = 0$, μ-a.e. Let $f \in L_1(E)$ and define the following measure

$$(P_1 f \mu)(F) = \int f P_1(\cdot, F)\, d\mu, \qquad F \in \mathfrak{F}.$$

Clearly $(P_1 f \mu) \ll \mu$. Hence there exists an E-valued function Tf such that

$$Tf = \frac{d(P_1 f \mu)}{d\mu}.$$

Thus T maps $L_1(E)$ to itself. For $f \in L_1(\mathbb{R})$ define

$$\hat{T} f = \frac{d(P_1 f \mu)}{d\mu}.$$

For $f \in L_1(E)$ we have

$$(P_1 f \mu)(F) = \int f P_1(\cdot, F)\, d\mu = \int_F Tf\, d\mu$$

$$(P_1 \|f\| \mu)(F) = \int \|f\| P_1(\cdot, F)\, d\mu = \int_F \hat{T}\|f\|\, d\mu.$$

Consequently,

$$\|(P_1 f\mu)(F)\| \leqq \int_F \hat{T}\|f\|\,d\mu$$

for all $F\in\mathfrak{F}$.

Let $|v|$ denote the variation of an E-valued measure v (see, e.g., [6]). Then the above inequality implies

$$|P_1 f\mu|(F) \leqq \int_F \hat{T}\|f\|\,d\mu$$

for all $F\in\mathfrak{F}$. But

$$|P_1 f\mu|(F) = \int_F \|Tf\|\,d\mu$$

([6], p. 46). Hence

$$\int_F \|Tf\|\,d\mu \leqq \int_F \hat{T}\|f\|\,d\mu$$

for all $F\in\mathfrak{F}$. This implies

$$\|Tf\| \leqq \hat{T}\|f\| \quad \text{a.e.} \quad ///$$

We now state a version of Theorem 2 of the Introduction for positively dominated operators on $R_0(E)$. The argument in [23] proves also this version.

Theorem 6.1. *For $k=1,2,\ldots,m$ let $T(k,n)$, $n\in N$ be positively dominated operators on $R_0(E)$, such that*

a) $\lim_n T(k,n)f$ and $\lim_n \hat{T}(k,n)\|f\|$ exist a.e. for each function f in $R_0(E)$, $k=1,2,\ldots,m$.

b) $\sup_n \hat{T}(k,n)\|f\|\in R_{k-1}$ for each $f\in R_k(E)$, $k=1,2,\ldots$.

Then for each $f\in R_m(E)$

$$\lim T(1,s_1)\,T(2,s_2)\ldots T(m,s_m)f$$

exists strongly a.e. as $s_i\to\infty$ independently.

The following Theorems 6.2 and 6.3 follow from Theorem 6.1 and the one-parameter results.

Theorem 6.2. *Let E be a Banach space with the Radon-Nikodym Property. If $f\in R_{m-1}(E)$, then*

$$E^1_{s_1} E^2_{s_2} \ldots E^m_{s_m}(f)$$

converges strongly a.e. as the indices $s_i\to\infty$ independently.

Let θ_i, $i=1,2,\ldots,m$ be measure preserving point transformations. The next result is the multiparameter version of E. Mourier's theorem [16].

Theorem 6.3. *If $f\in R_{m-1}(E)$, then*

$$\lim \frac{1}{s_1 s_2 \ldots s_m} \sum_{k_1=0}^{s_1-1} \ldots \sum_{k_m=0}^{s_m-1} f\circ\theta_1^{k_1}\ldots\theta_m^{k_m}$$

exists strongly a.e. as the indices $s_i\to\infty$ independently.

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Notes Added in Proof

1. In fact, Theorem 1 stated in the Introduction is applicable throughout the paper and provides the simplest proofs of all the results. The reason is that not only R_0, but also the spaces R_k, $k \geqq 1$, and the spaces S_k appearing in the proof of Theorem 5.2, are order-continuous Banach lattices. As observed following Proposition 1.1, the spaces R_k are Banach lattices for the (Luxemburg) norm defined by the Orlicz function $\Phi(x)=x(\log^+ x)^k$, closure in this norm of simple integrable functions. The order continuity is now easy: Let f_n be in R_k, $f_n \downarrow 0$. Let $\delta>0$ be arbitrary and observe that $\Phi(f_n/\delta)$ is integrable, hence by the continuity of Φ and the Lebesgue theorem, $\int \Phi(f_n/\delta) \leqq 1$ for large n, which implies that $\|f_n\| \leqq \delta$. *2.* Theorem 2 of the Introduction is also applicable, but its proof requires a slight variant of the argument in [23], letting the index n in $T_s X_n$ run over a directed set. This variant is also useful to obtain a version of Theorem 1 which reduces multiparameter local ergodic theorems to one parameter local ergodic theorems. *3.* The first counterexample to Rota's theorem in L_1 is due to D.L. Burkholder, "Semi-Gaussian subspaces", Trans. Am. Math. Soc. **104**, 123–131 (1962). Ornstein's example (19) is concerned with the important particular case involving alternating applications of two conditional expectations.