Z. Wahrscheinlichkeitstheorie verw. Gebiete
43, 141–148 (1978)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Relative Stability and the Strong Law of Large Numbers

R.A. Maller

Australian National University and C.S.I.R.O. Division of Mathematics & Statistics
South Melbourne, Vic., Australia

Summary. Let $X_1, X_2, \dots,$ be i.i.d. random variables and $S_n = X_1 + X_2 + \dots + X_n$. In this paper we simplify Rogozin's condition for $S_n/B_n \xrightarrow{p} \pm 1$ for some $B_n \to +\infty$, which generalises Hinčin's condition for relative stability of S_n. We also consider convergence of subsequences of S_n/B_n. As an application of our methods, we extend a result of Chow and Robbins to show that $S_n/B_n \to \pm 1$ a.s. for some $B_n \to +\infty$ if and only if $0 < |EX| \leqq E|X| < +\infty$.

0. Introduction

Let $X, X_1, X_2, \dots$ be independent and identically distributed random variables with distribution F, and let $S_n = X_1 + X_2 + \dots + X_n$. We say that F is *relatively stable* if there is a sequence of norming constants $B_n \to +\infty$ for which either $\frac{S_n}{B_n} \xrightarrow{p} 1$ or $\frac{S_n}{B_n} \xrightarrow{p} -1$. This concept was introduced for positive random variables by Hinčin (1936), who showed that, when X_i are non-negative, F is relatively stable if and only if

$$\frac{x P(X > x)}{\int_0^x [1 - F(u)]\, du} \to 0, \quad \text{as } x \to +\infty;$$

see also Gnedenko (1970 p. 541) and Feller (1971 p. 236). Rogozin (1976) generalised Hinčin's condition to the case of a not necessarily positive X, using some methods of the theory of regular variation; (Rogozin (1971) having shown that, for $X \geqq 0$, relative stability is equivalent to the slow variation of $\int_0^x [1 - F(u)]\, du$).

In the present paper (Theorem 1), we state Rogozin's (1976) result in a simpler way by removing a subsidiary condition of his, and we also consider

0044-3719/78/0043/0141/$ 01.60

"compactness" of $\frac{S_n}{B_n}$ (cf. Feller (1965–66)) and convergence of a subsequence $\frac{S_{n_i}}{B_i}$ to ± 1.

Hinčin (1936) also proved that, when X is non-negative, if there are constants $B_n \to +\infty$ for which $\frac{S_n}{B_n} \xrightarrow{\text{a.s.}} 1$, then $E|X| < +\infty$. This result was proved in general by Chow and Robbins (1961), by quite different methods. We go on to consider how relative stability is related to the weak and strong laws of large numbers. Suppose $\frac{S_n}{n} \xrightarrow{p} \mu$; then if $\mu \neq 0$, $\frac{S_n}{B_n} \xrightarrow{p} \pm 1$ where $B_n = n|\mu|$. But if $\frac{S_n}{n} \xrightarrow{p} 0$, there may be a sequence B_n for which $\frac{S_n}{B_n} \xrightarrow{p} \pm 1$, as is shown by Rogozin (1976). This cannot happen if the convergence is almost sure, since if $\frac{S_n}{n} \xrightarrow{p} 0$, we show in Theorem 4 that there is no sequence $B_n \to +\infty$ for which $\frac{S_n}{B_n} \xrightarrow{\text{a.s.}} \pm 1$; combined with the result of Chow and Robbins (1961) we can summarize this almost sure behaviour by: *there is a sequence* $B_n \to +\infty$ *for which* $\frac{S_n}{B_n} \xrightarrow{\text{a.s.}} \pm 1$ *if and only if* $0 < |EX| \leqq E|X| < +\infty$.

1. Results

We say that F does not have compact support if $P(|X| > x) > 0$ for every $x > 0$.

Theorem 1. *Suppose F does not have compact support. Then F is relatively stable if and only if*

$$\frac{xP(|X| \geqq x)}{\int_{-x}^{x} u\, dF(u)} \to 0 \quad \text{as } x \to +\infty. \tag{1.1}$$

The sign of $\lim_{n \to +\infty} \frac{S_n}{B_n}$ *is determined by the ultimate sign of* $\int_{-x}^{x} u\, dF(u)$, *which is constant. The sequence* B_n *is regularly varying with index* 1, *satisfies*

$$B_n \sim n \left| \int_{-B_n}^{B_n} u\, dF(u) \right| \sim |\text{median}(S_n)| \quad \text{as } n \to +\infty,$$

and may be chosen to be non-decreasing.

Theorem 1 differs essentially from the theorem of Rogozin (1976) only in that Rogozin assumes $\int_{-x}^{x} u\, dF(u)$ is slowly varying. We show this is implied by (1.1).

We remark that no symmetric F can be relatively stable, but we do not need to exclude symmetric F if we interpret the expression in (1.1) as being infinite in this case.

As an application of Theorem 1, we show that there is no "compactness" version of the convergence $\frac{S_n}{B_n} \xrightarrow{p} \pm 1$:

Theorem 2. *Suppose F does not have compact support and* $B_n \to +\infty$ *are constants. If for every sequence* $n'' \to +\infty$ *there is a subsequence* $n' \to +\infty$ *for which* $\frac{S_{n'}}{B_{n'}} \xrightarrow{p} c'$, *where* c' *is a constant, possibly depending on the choice of* n', *with* $0 < |c'| < +\infty$, *then F is relatively stable.*

We now give a subsequential version of Theorem 1:

Theorem 3. *Suppose F does not have compact support. If there are sequences* n_i, $B_i \to +\infty$ *for which* $\frac{S_{n_i}}{B_i} \xrightarrow{p} \pm 1$, *then*

$$\liminf_{x \to +\infty} \frac{x P(|X| \geqq x)}{\left| \int_{-x}^{x} u \, dF(u) \right|} = 0. \tag{1.2}$$

Conversely, if (1.2) *holds and either X is non-negative or X is not in the domain of partial attraction of the normal distribution* (Lévy (1937, p. 113)) *then there are sequences* $n_i, m_i, B_i \to +\infty$ *for which either* $\frac{S_{n_i}}{B_i} \xrightarrow{p} +1$, $\frac{S_{m_i}}{B_i} \xrightarrow{p} -1$, *or both.*

We remark that we can equivalently take $\frac{S_{n_i}}{B_i} \xrightarrow{\text{a.s.}} \pm 1$ in Theorem 3, so that we are considering "strong limit points" of $\frac{S_n}{B_n}$ (cf. Erickson and Kesten (1974)). Let's mention what happens when F has compact support; in fact, suppose $EX^2 < +\infty$. Then if $EX \neq 0$, we have $\frac{S_n}{B_n} \xrightarrow{p} \pm 1$, where $B_n = n|EX|$, while if $EX = 0$, there are no sequences $n_i, B_i \to +\infty$ for which $\frac{S_{n_i}}{B_i} \xrightarrow{p} \pm 1$, as is easy to see.

Now we look at the almost sure convergence of $\frac{S_n}{B_n}$:

Theorem 4. *If* $\liminf_{x \to +\infty} \left| \int_{-x}^{x} u \, dF(u) \right| = 0$ *there is no* $B_n \to +\infty$ *for which* $\frac{S_n}{B_n} \xrightarrow{\text{a.s.}} \pm 1$.

The result of Chow and Robbins (1961) states that if $\frac{S_n}{B_n} \xrightarrow{\text{a.s.}} \pm 1$, then $E|X| < +\infty$; we give a short proof of this following the proof of Theorem 4. Combining this result with that of Theorem 4 gives: *there is a sequence* $B_n \to +\infty$ *for which* $\frac{S_n}{B_n} \xrightarrow{\text{a.s.}} \pm 1$ *if and only if* $0 < |EX| \leqq E|X| < +\infty$; *we then have* $B_n \sim n|EX|$.

2. Proofs

Proof of Theorem 1. A minor modification of Rogozin's (1976) theorem shows that (1.1) and the regular variation of B_n are necessary for $\frac{S_n}{B_n} \xrightarrow{p} \pm 1$. (A treatment of the regular variation of sequences, rather than that of functions, is given in Galambos and Seneta (1973)). The fact that $B_n \sim |\text{median}(S_n)|$ follows by symmetrisation, since clearly $\frac{S_n}{B_n} \xrightarrow{p} \pm 1$ implies $[S_n\text{-median}(S_n)]/B_n \xrightarrow{p} 0$. Conversely, suppose (1.1) holds; to apply Rogozin's theorem we need only show that $\int_{-x}^{x} u\, dF(u)$ is slowly varying. Let $A(x) = \int_0^x G(u)\, du$, where $G(x) = 1 - F(x) - F(-x)$ for $x > 0$, and let (here and elsewhere) $H(x) = 1 - F(x) + F(-x) \leqq P(|X| \geqq x)$. Then (1.1) implies $xH(x)/A(x) \to 0$ as $x \to +\infty$, and since $|G| \leqq H$, we also have $xG(x)/A(x) \to 0$. Given $\varepsilon > 0$ choose $x_0 = x_0(\varepsilon)$ so that $x \geqq x_0$ implies $|A(x)| \geqq \varepsilon^{-1} x H(x)$. Since F doesn't have compact support, $H(x) > 0$ for every $x > 0$ and hence $|A(x)| > 0$ for $x \geqq x_0$. Since A is continuous this means either $A(x) > 0$ or $A(x) < 0$ for $x \geqq x_0$. Suppose $A(x) > 0$ for $x \geqq x_0$ (the other case leads to $\frac{S_n}{B_n} \xrightarrow{p} -1$). Then for $x \geqq x_0$, and $\lambda > 0$,

$$\left|\log \frac{A(\lambda x)}{A(x)}\right| = \left|\int_x^{\lambda x} \frac{uG(u)}{\int_0^u G(y)\, dy} \frac{du}{u}\right| \leqq |\log \lambda^{\varepsilon}|.$$

Hence $A(\lambda x) \sim A(x)$ and A is slowly varying. But since

$$\int_{-x}^{x} u\, dF(u) = -\int_0^x u\, dG(u) = -xG(x) + A(x) = [1 + o(1)]\, A(x),$$

$\int_{-x}^{x} u\, dF(u)$ is also slowly varying. Hence the result.

Proof of Theorem 2. We must have

$$0 < \liminf_{n \to +\infty} B_{n+1}/B_n \leqq \limsup_{n \to +\infty} B_{n+1}/B_n < +\infty,$$

since, if not, we could take $n'' \to +\infty$ such that $B_{n''+1}/B_{n''} \to 0$ or $+\infty$. By taking subsequences, we can make $S_{n''}/B_{n''} \xrightarrow{p} c''$ and $S_{n''+1}/B_{n''+1} \xrightarrow{p} c'''$. Thus

$$\begin{aligned} c''' &= \lim S_{n''+1}/B_{n''+1} = \lim S_{n''}/B_{n''+1} = \lim (S_{n''}/B_{n''})(B_{n''}/B_{n''+1}) \\ &= c'' \cdot (0 \text{ or } +\infty), \end{aligned}$$

which is impossible since neither c'' nor c''' is 0 or $+\infty$. Also, as in Rogozin (1976), for each n'' there is a subsequence n' for which

$$B_{n'} H(B_{n'}) / \int_{-B_{n'}}^{B_{n'}} u\, dF(u) \to 0, \quad \text{so} \quad B_n H(B_n) / \int_{-B_n}^{B_n} u\, dF(u) \to 0.$$

Using the boundedness of B_{n+1}/B_n, a proof just like that of Rogozin shows that (1.1) holds, so F is relatively stable.

Proof of Theorem 3. Our proof is closely related to the method of Lévy's (1937, p. 113) characterization of the domain of partial attraction of the normal distribution. We have $\frac{S_{n_i}}{B_i} \xrightarrow{p} \pm 1$ if and only if (Gnedenko and Kolmogorov (1968 p. 124)) (letting $V(x)=\int_{-x}^{x} u^2\,dF(u)$),

$$nH(xB_n)\to 0 \quad \text{as} \quad n\to+\infty \qquad \text{for } x>0, \tag{2.1}$$

$$nB_n^{-2}\left\{V(xB_n)-\left[\int_{-xB_n}^{xB_n} u\,dF(u)\right]^2\right\}\to 0 \quad \text{as} \quad n\to+\infty \qquad \text{for } x>0, \tag{2.2}$$

and

$$nB_n^{-1}\int_{-xB_n}^{xB_n} u\,dF(u)\to\pm 1 \quad \text{as} \quad n\to+\infty \qquad \text{for } x>0, \tag{2.3}$$

hold with n_i in place of n and B_i in place of B_n. Hence the necessity of (1.2) is obvious. For the sufficiency, using the notation of the proof of Theorem 1, by (1.2) we can find a sequence $x_i\to+\infty$ for which $x_i H(x_i)/A(x_i)\to 0$. Since F does not have compact support, $|A(x_i)|>0$ for i large. Define $n_i\to+\infty$ by $n_i^2\sim x_i/[H(x_i)\,|A(x_i)|]$, so that $[n_i H(x_i)]^2\sim x_i H(x_i)/|A(x_i)|\to 0$ as $i\to+\infty$. Define $B_i>0$ by $B_i=n_i\,|A(x_i)|$; then

$$x_i^{-2}B_i^2=x_i^{-2}\,n_i^2A^2(x_i)\sim|A(x_i)|/[x_i H(x_i)]\to+\infty,$$

so $B_i\to+\infty$ and $x_i=o(B_i)$. Given $x>0$ we can make $xB_i\geqq x_i$, so $n_i H(xB_i)\leqq n_i H(x_i)\to 0$. This is (2.1). Also, for $x>0$,

$$n_iB_i^{-1}A(xB_i)=n_iB_i^{-1}A(x_i)+n_iB_i^{-1}\int_{x_i}^{xB_i} G(u)\,du=\pm 1+o(1).$$

because $\left|n_iB_i^{-1}\int_{x_i}^{xB_i} G(u)\,du\right|\leqq x\,n_i H(x_i)\to 0$. By taking a further subsequence, if necessary, we can make either $n_iB_i^{-1}A(xB_i)\to+1$ or $\to-1$, or possibly both. This proves (2.3). To prove (2.2), we have for $x>0$

$$\begin{aligned} n_iB_i^{-2}V(xB_i)&=n_iB_i^{-2}V(x_i)+n_iB_i^{-2}\int_{x_i<|u|\leqq xB_i} u^2\,dF(u)\\ &\leqq\frac{V(x_i)}{B_i\,|A(x_i)|}+x^2\,n_i H(x_i)\\ &=\frac{x_i}{B_i}\,\frac{V(x_i)}{x_i\,|A(x_i)|}+o(1)\\ &=o(1)\,\frac{V(x_i)}{x_i\,|A(x_i)|}+o(1). \end{aligned}$$

At this stage we need the further information that X is non-negative, in which case

$$\frac{V(x_i)}{x_i|A(x_i)|}=\frac{-\int_0^{x_i} u^2\,dH(u)}{x_i\int_0^{x_i} H(u)\,du}\leqq\frac{2\int_0^{x_i} uH(u)\,du}{x_i\int_0^{x_i} H(u)\,du}\leqq 2,$$

or that X is not in the domain of partial attraction of the normal distribution, equivalently, by Lévy (1937, p. 113), $\liminf_{x\to+\infty} x^2 H(x)/V(x)>0$, and then

$$\frac{V(x_i)}{x_i|A(x_i)|}\leqq o(1)\frac{V(x_i)}{x_i^2 H(x_i)}=o(1).$$

In either case, we see that (2.2) holds, and this proves the sufficiency of (1.2).

Proof of Theorem 4. We require the following Lemma (cf. Chow and Robbins (1961 Lemma 1), Feller (1968 Lemma 3.2)):

Lemma 1. *If* B_n *is non-decreasing and* $\liminf_{n\to+\infty} B_{n\lambda_0}/B_n>1$ *for some integer* $\lambda_0>1$, *then either* $X_n/B_n \xrightarrow{\text{a.s.}} 0$ *or* $\limsup_{n\to+\infty}|X_n|/B_n=+\infty$ a.s., *according as* $\sum_{n=1}^{\infty} H(B_n)$ *converges or diverges.*

Proof of Lemma 1. First, note that, given any integer $\lambda>\lambda_0>1$, and defining $k=k(\lambda)$ by $\lambda_0^k\leqq\lambda<\lambda_0^{k+1}$, we have

$$\frac{B_{n\lambda}}{B_n}\geqq\frac{B_{n\lambda_0^k}}{B_n}=\frac{B_{n\lambda_0^k}}{B_{n\lambda_0^{k-1}}}\,\frac{B_{n\lambda_0^{k-1}}}{B_{n\lambda_0^{k-2}}}\cdots\frac{B_{n\lambda_0}}{B_n}\geqq(c-\varepsilon)^k,$$

provided $n\geqq n_0(\lambda_0,\varepsilon)$ where $c=\liminf_{n\to+\infty} B_{n\lambda_0}/B_n>1$ and $c-\varepsilon>1$ for some $\varepsilon>0$. Since $k+1>\log\lambda/\log\lambda_0$, we then have

$$\frac{B_{n\lambda}}{B_n}\geqq(c-\varepsilon)^{-1}\lambda^{\log(c-\varepsilon)/\log\lambda_0}=(c-\varepsilon)^{-1}\lambda^{2\delta}\geqq\lambda^{\delta}$$

for $\lambda\geqq\lambda_1(c,\varepsilon)\geqq\lambda_0$, where $2\delta=\log(c-\varepsilon)/\log\lambda_0>0$.

Now suppose $\sum H(x_0B_n)=+\infty$ for some $x_0>0$, so $\sum H(xB_n)=+\infty$ for $x\leqq x_0$. Consider $x>x_0$; define an integer $i=i(x)$ so that $i^\delta\geqq x/x_0$. Then $x_0B_{in}\geqq x_0 i^\delta B_n\geqq xB_n$ for $i\geqq\lambda_1$, $n\geqq n_0$, so

$$\sum_{n\geqq n_0} H(xB_n)\geqq\sum_{n\geqq n_0} H(x_0B_{in})\geqq\sum_{n\geqq n_0} H(x_0B_{in+j})$$

for every $j>0$, since B_n is non-decreasing. But

$$\sum_{j=0}^{i-1}\sum_{n=1}^{\infty} H(x_0B_{ni+j})=\sum_{n=i}^{\infty} H(x_0B_n)=+\infty,$$

so $\sum H(xB_n)=+\infty$ for this case also. The remainder of Lemma 1 follows from the Borel-Cantelli lemma.

The proof of Theorem 4, and the result of Chow and Robbins (1961) follow from:

Lemma 2. *If either*

$$\liminf_{x\to+\infty}\left|\int_{-x}^{x} u\,dF(u)\right|=0 \quad or \quad \limsup_{x\to+\infty}\left|\int_{-x}^{x} u\,dF(u)\right|=+\infty,$$

then there is no sequence $B_n\to+\infty$ *for which* $\frac{S_n}{B_n}\xrightarrow{\text{a.s.}}\pm 1$.

Proof of Lemma 2. We can assume F does not have compact support (see Section 2). Suppose $\frac{S_n}{B_n}\xrightarrow{\text{a.s.}}+1$; the other case may be treated similarly. Then $\frac{S_n}{B_n}\xrightarrow{p}1$, and also $\sum_{n=1}^{\infty} H(B_n)<+\infty$; because, from Theorem 1 we may choose B_n to be non-decreasing and regularly varying with index 1, so that B_n satisfies the conditions of Lemma 1, and then $\limsup_{n\to+\infty}|S_n|/B_n<+\infty$ a.s. implies $\limsup_{n\to+\infty}|X_n|/B_n<+\infty$ a.s. and hence $\sum_{n=1}^{\infty} H(B_n)<+\infty$. Now, instead of B_n we can use any asymptotically equivalent sequence, and we define $B(x)=\sup\{y>0\,|\,y^{-1}A(y)\geqq x^{-1}\}$; by the continuity of A, this means $B(x)=xA[B(x)]$. If $B_n^*=B(n)$, then by a proof the same as Rogozin's, $\frac{S_n}{B_n^*}\xrightarrow{\text{a.s.}}1$, so $\sum_{n=1}^{\infty} H(B_n^*)<+\infty$. Now $B(x)$ is a differentiable function, because

$$B'(x)=A[B(x)]+xG[B(x)]\,B'(x),$$

or

$$B'(x)=A[B(x)]/[1-xG[B(x)]]=[1+o(1)]\,A[B(x)],$$

since $x|G[B(x)]|\leqq xH[B(x)]\to 0$ as $x\to+\infty$, when $\frac{S_n}{B_n^*}\xrightarrow{p}1$, by (2.1). Also, as we saw in the proof of Theorem 1, $A(x)>0$ for $x\geqq x_0$, so $B(x)$ is ultimately increasing. Hence

$$\begin{aligned}\infty>\sum_{n=1}^{\infty} H(B_n^*)\geqq \sum_{n=1}^{\infty}\int_{n}^{n+1} H[B(x)]\,dx=\int_{1}^{\infty} H[B(x)]\,dx\\ \geqq\int_{x_0}^{\infty}\frac{H(x)\,dx}{B'[B^{-1}(x)]},\end{aligned}$$

where B^{-1} is the inverse function to B, existing ultimately by the monotonicity. From the above, $B'[B^{-1}(x)]=[1+o(1)]\,A(x)$ as $x\to+\infty$, so we have the integral $\int_{x_0}^{\infty}\frac{H(x)}{A(x)}dx$ converging. But this is impossible if either $A(x_n)\to 0$ or $A(x_n)\to+\infty$ for some x_n, because then

$$\infty = \lim_n \left| \log \frac{A(x_n)}{A(x_0)} \right| = \lim_n \left| \int_{x_0}^{x_n} \frac{G(x)\,dx}{\int_0^x G(u)\,du} \right|$$

$$\leqq \liminf_n \int_{x_0}^{x_n} \frac{|G(x)|\,dx}{A(x)}$$

$$\leqq \int_{x_0}^{\infty} \frac{H(x)}{A(x)}\,dx.$$

This contradiction completes the proof of Lemma 2.

Theorem 4 is immediate from Lemma 2, and we now deduce the result of Chow and Robbins (1961). If $\frac{S_n}{B_n} \xrightarrow{\text{a.s.}} 1$ then $\frac{S_n}{B_n} \xrightarrow{p} 1$, and from Lemma 2, $\limsup_{x\to+\infty} \left| \int_{-x}^{x} u\,dF(u) \right| < +\infty$. Also $B_n \sim n \int_{-B_n}^{B_n} u\,dF(u)$, so $B_n = O(n)$ as $n \to +\infty$, and since, as we saw, $\sum_{n=1}^{\infty} H(B_n) < +\infty$, we have $\sum_{n=1}^{\infty} H(n) < +\infty$ and $E|X| < +\infty$.

Acknowledgement. This paper forms part of the author's Ph. D. studies in the Department of Statistics, School of General Studies, Australian National University. I am grateful for the support of a C.S.I.R.O. Division of Mathematics and Statistics Postgraduate Studentship, and for the valuable comments of the referees.

References

1. Chow, Y.S., Robbins, H.: On sums of independent random variables with infinite moments and "fair" games. Proc. Nat. Acad. Sci. **47**, 330–335 (1961)
2. Erickson, K.B., Kesten, H.: Strong and weak limit points of a normalised random walk. Ann. Probability **2**, 553–580 (1974)
3. Feller, W.: On regular variation and local limit theorems. Proc. 5th Berkeley Sympos. Math. Statist. Probab., Univ. Calif. Vol. i. pt 1, 373–388 (1965–66)
4. Feller, W.: An extension of the law of the iterated logarithm to variables without variance. J. Math. Mech. **18**, 343–355 (1968)
5. Feller, W.: An Introduction to Probability Theory and its Applications 2. 2nd ed. New York: Wiley 1971
6. Galambos, J., Seneta, E.: Regularly varying sequences. Proc. Amer. Math. Soc. **41**, 110–116 (1973)
7. Gnedenko, B.V., Kolmogorov, A.N.: Limit distributions for sums of independent random variables. 2nd ed. Reading Mass.: Addison-Wesley 1968
8. Gnedenko, B.V.: Limit theorems for sums of a random number of positive independent random variables. Proc. 6th Berkeley Sympos. Math. Statist. Probab. Univ. Calif. **2**, 537–549 (1970)
9. Hinčin, A.: Su una legge dei grandi numeri generalizzata. Giorn. Ist. Ital. Attuari, Anno VII, Vol. **14**, 365–377 (1936)
10. Lévy, P.: Théorie de l'Addition des Variables Aléatoires. Paris: Gautiers-Villars 1937
11. Rogozin, B.A.: Ladder moments and fluctuations. Theor. Probability Appl. **16**, 575–594 (1971)
12. Rogozin, B.A.: Relatively stable walks. Theor. Probability Appl. **21**, 375–379 (1976)

Received July 15, 1976; in revised form September 10, 1977

Added in Proof. A recent paper by M. Klass and H. Teicher (Ann. Probability **5**, 861–874 (1977)) contains results related to those given herein. Also, a statement of our Lemma 1 is in Klass, M.J. (Z. Wahrscheinlichkeitstheorie verw. Gebiete, **36**, 165–178 (1976)).