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A Characterization of the Families of Finite-Dimensional Distributions Associated with Countably Additive Stochastic Processes Whose Sample Paths are in D

Lester E. Dubins* and Marjorie G. Hahn

Department of Statistics, University of California, Berkeley, California 94720 (USA)

Summary. Associated with any countably additive probability measure P on the well-known Skorohod space D is the family $\{P_\alpha\}$ of P's finite-dimensional distributions. This paper characterizes all such $\{P_\alpha\}$.

Throughout this paper, (X, ρ) is a complete separable metric space, I is the closed unit interval and, for each subset S of I, X^S is the set of all X-valued functions defined on S.

For each $\varepsilon>0$ and $f\in X^S$, $N_{\varepsilon,S}(f)$ is the supremum of all integers r such that there exist r nonoverlapping intervals $[s_i, t_i]$ with $s_i\in S$, $t_i\in S$ and with $\rho(f(s_i), f(t_i))\geqq\varepsilon$, for all i, $1\leqq i\leqq r$. Of course, $N_{\varepsilon,S}$ may assume the values 0 and $+\infty$.

Let E be the set of all $f\in X^I$ which have finite, right-hand limits at every point of $[0, 1)$ and finite, left-hand limits at every point of $(0, 1]$.

For each $S\subset I$, let $\mathscr{B}_S$ be the least σ-field of subsets of X^S such that, for each $s\in S$, the evaluation map $f\to f(s)$ is measurable. Let α be a variable which ranges over the nonempty, finite subsets of I, and let $\{P_\alpha\}$ or, more fully, $(X^\alpha, \mathscr{B}_\alpha, P_\alpha)$ be a consistent family of countably additive probability spaces.

Proposition 1. *For a consistent family of countably additive probability spaces, $(X^\alpha, \mathscr{B}_\alpha, P_\alpha)$, to be the (finite-dimensional) marginals of a countably additive stochastic process all of whose paths belong to E, it is necessary and sufficient that this condition hold.*

(C) *For all $\varepsilon>0$, $\beta>0$, there is a finite positive integer k such that, for all α,*

$$P_\alpha\{N_{\varepsilon,\alpha}>k\}<\beta.$$

Lemma 1. *For $S\subset I$ and $g\in X^S$, these three conditions are equivalent.*

(a) *For every $\varepsilon>0$, $N_{\varepsilon,S}(g)$ is finite.*

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(b) *For every monotone sequence of elements* $s \in S$, $g(s)$ *is Cauchy.*

(c) *There is an* $f \in E$ *which agrees with* g *on* S.

In particular, for $f \in X^I$ *to belong to* E *it is necessary and sufficient that, for all* $\varepsilon > 0$, $N_{\varepsilon, I}(f)$ *be finite.*

Proof. Obviously, (a) implies (b). So assume (b) and let $t \in I$. If for some $s \in S$, $s \geqq t$, let $f(t)$ be the $\lim g(s)$ as s decreases through the set of all such s. Since X is complete, and (b) holds, the limit plainly exists. Thus defined, f is clearly an extension of g. If, for all $s \in S$, $s < t$, let $f(t)$ be the $\lim g(s)$ as s increases through the set of all such s. That f agrees with g on S is obvious. Since X is complete, for f to be in E it plainly suffices to verify that, for every strictly monotone sequence of elements t_i in I for which $f(t_i) \neq f(t_{i-1})$, $f(t_i)$ is Cauchy. For such a sequence t_i there is an s_i between t_{i-1} and t_i such that $\rho(g(s_i), f(t_i)) < \frac{1}{i}$. Clearly, s_i is monotone, so $g(s_i)$, and hence $f(t_i)$, is Cauchy.

Now let f be as in (c), and let $\varepsilon > 0$. Verify that, for each $t \in I$, there exists $\delta(t) > 0$ such that

$$\rho(f(t'), f(t'')) < \varepsilon \tag{1}$$

whenever $t' \in I$, $t'' \in I$ and $t < t', t'' < t + \delta(t)$, and also whenever $t - \delta(t) < t', t'' < t$. By compactness of I, there then exists a finite set $t_1, \ldots, t_n$ of elements of I such that the n open intervals

$$t_i - \delta(t_i) < t < t_i + \delta(t_i)$$

cover I. As is now easily verified, $N_{\varepsilon, I}(f)$ is at most $4n$. Since $N_{\varepsilon, S}(g)$ is at most $N_{\varepsilon, I}(f)$, (a) obtains, which completes the proof of Lemma 1.

The following lemma is no doubt well-known but, in any event, it is an immediate consequence of the fact that any collection of real, random numbers possesses an essential supremum.

Lemma 2. *Let* $\mathcal{N}$ *be a set of nonnegative random numbers defined on a countably additive probability space and suppose that for each sequence* $N_1, N_2, \ldots$ *of elements of* $\mathcal{N}$ *there is an* $N \in \mathcal{N}$ *such that* $N_i \leqq N$ *with probability* 1. *Then there is an* $N \in \mathcal{N}$ *such that for all* $N' \in \mathcal{N}$, $N' \leqq N$ *almost surely.*

Proof of Necessity. Assume that there is such a stochastic process. Then, as is routine, there is a countably additive probability measure P defined on $\mathscr{B}' = E \cap \mathscr{B}_I$ whose α-marginals are the given $(X^\alpha, \mathscr{B}_\alpha, P_\alpha)$. Fix $\varepsilon > 0$. Though $N_{\varepsilon, I}$ need not be measurable with respect to $\mathscr{B}'$, $N_{\varepsilon, I}(f)$ is finite for all $f \in E$, as Lemma 1 asserts. So, for each countable subset S of I, $N_{\varepsilon, S}$ when deemed to be defined on, and only on, E is certainly finite. Moreover, for each sequence $S_1, S_2, \ldots$ of countable subsets of I, N_{ε, S_i} is at most $N_{\varepsilon, S}$ where S is the set-theoretic union of the S_i. As is trivially verified, each $N_{\varepsilon, S}$ is $\mathscr{B}'$-measurable, so by Lemma 2 there exists a countable set S such that, for all finite subsets α of I,

$$N_{\varepsilon, \alpha} \leqq N_{\varepsilon, S} \quad P\text{-almost surely.} \tag{2}$$

Let $\beta > 0$ be fixed, too. Since P_S is countably additive, there plainly exists an integer k such that

$$P_S(N_{\varepsilon,S} \leqq k) > 1 - \beta. \tag{3}$$

Now calculate, thus.

$$\begin{aligned} P_\alpha(N_{\varepsilon,\alpha} \leqq k) &= P(N_{\varepsilon,\alpha} \leqq k) \\ &\geqq P(N_{\varepsilon,S} \leqq k) \\ &= P_S(N_{\varepsilon,S} \leqq k) \\ &> 1 - \beta, \end{aligned} \tag{4}$$

where the two inequalities hold because of (2) and (3) respectively, and the two equalities are evident. This completes the proof of necessity.

Proof of Sufficiency. Suppose that the consistent family $\{P_\alpha\}$ satisfies Condition (C) and let P be the unique countably additive probability measure defined on $\mathscr{B}_I$ whose marginals are the given P_α. Plainly, it suffices to show that $P^*(E)$, the outer P-measure of E, is 1. With this end in view, let $E \subset B \in \mathscr{B}_I$. Of course, for some countable subset S of I, fixed for the remainder of this proof,

$$E \subset B \in \mathscr{B}_S. \tag{5}$$

The proof will be complete once it is shown that $P(B) = 1$.

Let $\varepsilon(i) = i^{-1}$. As Condition (C) implies, for each $\beta > 0$ and $i \geqq 1$, there is a least $k = k(i, \beta)$ such that

$$P_\alpha[N_{\varepsilon(i),\alpha} > k(i, \beta)] < \beta 2^{-i} \quad \text{for all } \alpha. \tag{6}$$

Defined the event

$$A = [\forall\, i \geqq 1,\ \bar{N}_{\varepsilon(i),S} \leqq k(i, \beta)], \tag{7}$$

where Π is the projection of X^I onto X^S and $\bar{N}_{\varepsilon,S} = N_{\varepsilon,S} \circ \Pi$. It is routine to verify that $P(A) > 1 - \beta$.

Let $f \in A$ and let g be the restriction of f to S. In view of Lemma 1, there exists $f^* \in E$ which agrees with f on S. By (5), $f^* \in B$. So, since $B \in \mathscr{B}_S$, $f \in B$, too. In sum, $A \subset B$, so $P(B) > 1 - \beta$. This completes the proof.

For the special case of Proposition 1 in which X is also assumed to be locally compact, some may prefer an alternative proof of sufficiency based on [Gihman and Skorohod, Chapter III, § 2, Theorem 2; and § 4, Lemma 1, and Definition 2 (of O_ε, the maximum number of consecutive ε-oscillations)], together with this easily verified inequality:

$$N_{2\varepsilon} \leqq O_\varepsilon \leqq N_\varepsilon. \tag{8}$$

Say that a stochastic process $\{Y(t),\ t \in I\}$ is *strongly right-continuous in distribution* (s.r.c.i.d.) if, for every n elements of I, $t(1), \ldots, t(n)$, whenever $\tau(i) \downarrow t(i)$ for each i, $Y(\tau(1)), \ldots, Y(\tau(n))$ converges in distribution to $Y(t(1)), \ldots, Y(t(n))$.

As is obvious, the property of being s.r.c.i.d. depends only on the "finite-dimensional" distributions of the process, which justifies this definition: A consistent family $\{P_\alpha\}$ is *right-continuous in distribution* if any stochastic process Y whose marginals are the $\{P_\alpha\}$ is s.r.c.i.d.

For each countably additive stochastic process Y with paths in E, define $Y_+(t)$ to be the lim $Y(t+\varepsilon)$ as $\varepsilon>0$ converges to 0, where it is understood that $Y_+(1)$ is $Y(1)$. As is easily verified, Y_+ has paths in D.

Lemma 3. *Let Y be a countably additive stochastic process with paths in E. For Y_+ to have the same "finite-dimensional" distributions as does Y it is necessary, and sufficient, that Y be strongly right-continuous in distribution.*

In view of Proposition 1 and Lemma 3 the following theorem is evident.

Theorem 1. *For the consistent family $\{P_\alpha\}$ of countably additive probability measures to be the family of "finite-dimensional" marginals of a countably additive stochastic process with paths in D it is necessary and sufficient that the family be right-continuous in distribution and that Condition* (C) *hold.*

It would be good to give a direct argument that the sufficient conditions for a process to have sample paths in E which are in [Chentsov, Theorem 1, p. 140] and in [Gihman and Skorohod, Chapter III, §4, Theorem 1, p. 179] imply Condition (C).

References

Chentsov, N.N.: Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the "heuristic" approach to the Kolmogorov-Smirnov tests. Theor. Probability Appl. **1**, 140–149 (1956)

Dubins, L.E.: Which families of finite-dimensional joint distributions are associated with continuous-path, countably additive, stochastic processes? Annali de Matematica Pura ed Applicata (IV), **CXIII**, 237–243 (1977)

Gihman, I.I., Skorohod, A.V.: The Theory of Stochastic Processes I. Berlin-Heidelberg-New York: Springer 1974

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