Z. Wahrscheinlichkeitstheorie verw. Gebiete
51, 303–322 (1980)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On the Tail σ-Field and the Minimal Parabolic Functions for One-Dimensional Quasi-Diffusions

Uwe Küchler[1] and Ulrich Lunze[2]

[1] Technische Universität Dresden, Sektion Mathematik, Mommsenstraße 13, DDR-8027 Dresden
[2] Technische Universität Dresden, Sektion Fertigungstechnik, Mommsenstraße 13, DDR-8027 Dresden

1. Introduction

The tail σ-field $\mathscr{T}$ of a Markov process has been studied by several authors, and there are many criteria in terms of the paths or of the transition probabilities which ensure that $\mathscr{T}$ is trivial or equal to the σ-field of invariant events. See e.g. [1, 3, 4, 13, 21, 31–33].

Often Markov processes are given by their infinitesimal generator. In many cases this generator turns out to be more suitable for calculations than the transition probabilities. Thus it may be useful to know the connections of the tail σ-field $\mathscr{T}$ and the infinitesimal generator. In this paper we will study one-dimensional quasi-diffusions on $[0, L)$ with reflecting boundary 0, which in particular include diffusions and birth and death processes. Their infinitesimal generator has the form of a generalized second order differential operator $D_m D_p$, where m and p are nondecreasing functions, the so-called speed measure and natural scale respectively.

We will characterize quasi-diffusions with trivial tail σ-field in terms of m and p, in terms of the spectrum σ_2 of $D_m D_p$ in $\mathbb{L}_2(m)$ and also in terms of the paths (see Theorem 1).

From the theory of Martin boundaries (see e.g. [28]) it is known that every parabolic function (i.e. every harmonic function for the space-time process), satisfying some regular conditions, can be represented by minimal parabolic functions. But it is not always easy to find the minimal parabolic functions for a given process.

Parabolic functions, which can often be given explicitly, are the so-called factorizing parabolic functions (see [22, 23, 29]). For quasi-diffusions such functions are given by $h_\mu(x,s) = \exp(-\mu s)\,\varphi(x,\mu)$ where $\mu \geqq \sup \sigma_2$ and φ is a solution of $D_m D_p \varphi = \mu \varphi$.

In Theorem 2 below we will give necessary and sufficient conditions under which such a factorizing parabolic function for a quasi-diffusion is minimal. This result is mainly interesting for processes with natural boundary L. For accessible and entrance boundaries L the minimal parabolic functions are known to be not

0044-3719/80/0051/0303/$04.00

factorizing (up to exactly one of them), in these cases they have been studied in [18, 19].

The Brownian motion on the real axis has natural boundaries. Woo [36] has proved that its minimal parabolic functions are exactly the factorizing ones. Theorem 2 and examples show that this picture is not true for all quasi-diffusions with natural boundary.

The tail σ-field for one-dimensional diffusions was also studied by Rösler [33] and Fristedt, Orey [10], see Sect. 3 below.

We would like thank H. Langer for valuable discussions around the subject of this paper, in particular for his suggestion how to prove Lemma 6. We want to express our thanks also to the referees for their careful reading the manuscript and many helpful comments.

2. Definitions and Preliminary Results

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{K}$ the set of complex numbers, $\mathscr{B}$ the σ-algebra of Borel-sets of $\mathbb{R}$ and $\mathbb{N}$ the set of natural numbers including zero. If E is a set and A a subset of E then $\mathbb{1}_A(.)$ denotes the indicator function of A on E, instead of $\mathbb{1}_E$ we write $\mathbb{1}$. By $\varnothing$ we mean the empty set.

The Operator $D_m D_p$ and Quasi-Diffusions

Let m and p be two nondecreasing functions on an interval $[0, L)$ $(L \leqq \infty)$ with $0 = m(0) < m(x) \leqq m(y) < m(L-0) \leqq \infty$ $(x, y \in [0, L), x < y)$ and let p be continuous and strictly increasing. For every complex valued function f on $[0, L)$ having the form

$$f(x) = a + b\,p(x) + \int_0^x (p(x) - p(s))\, g(s)\, m(ds) \qquad (x \in [0, L))$$

where a, b are complex numbers and g is a dm-locally integrable function, we define $D_m D_p f := g$. For given f (and b if $m(0+) > 0$) $D_m D_p f$ is dm-a.e. uniquely determined. We use the notation $D_p^- f(0) = b$ (for details see e.g. [7, 9, 15]). By E_m we denote the set of all points of $[0, L)$ where m increases. By definition we have $0 \in E_m$. Let $\mathbb{C}_m$ be the set of all continuous functions on the compact set $E_m \cup \{L\}$ with $f(L) = 0$ if $\int_0^L m\, dp < \infty$ or $\int_0^L p\, dm = \infty$.

Every f from $\mathbb{C}_m$ can be assumed to be a continuous function on $[0, L)$ which is linear with respect to the scale p on every subinterval of the open set $[0, L) \setminus E_m$. Define

$$\Delta = \{f \in \mathbb{C}_m \mid D_m D_p f \in \mathbb{C}_m, D_p^- f(0) = 0\}.$$

The restriction of $D_m D_p$ to Δ is the infinitesimal generator of a strongly continuous semigroup $(P_t)_{t \geqq 0}$ of linear operators on $\mathbb{C}_m$ with $P_0 = I$. (This follows from the theorem of Hille-Yosida, the proof is similar to those in [12, 26, 34], see [20].) The semigroup (P_t) generates a Hunt process $\mathfrak{X}=(X_t, \zeta, \mathcal{M}_t, \theta_t, P_x)$ with state space E_m (see [2, 8, 20, 25]) which is reflected at zero and is killed as soon as it hits L (and only there). In particular $\mathfrak{X}$ is right continuous and strongly Markovian. We will call the process $\mathfrak{X}$ the *quasi-diffusion* with speed measure m and scale p. Firstly this notation was used by Watanabe [35] as far as we know.

Let us give two basic examples.

1. Assume $\mathfrak{X}$ is a diffusion process on $[0, L)$, connected with the differential operator $a(x)\dfrac{d^2}{dx^2}+b(x)\dfrac{d}{dx}$, instantaneously reflected at zero and killed as soon as it hits L. Assume $a(x)>0$ and $\left|\int\limits_0^x \dfrac{b(s)}{a(s)}\,ds\right|<\infty$ $(x\in[0,L))$. Then $\mathfrak{X}$ is a quasi-diffusion with speed measure $dm(x)=\exp\left(-\int\limits_0^x \dfrac{b(s)}{a(s)}\,ds\right)dx$ and scale $dp(x)=a^{-1}(x)\exp\left(\int\limits_0^x \dfrac{b(s)}{a(s)}\,ds\right)dx$ (see e.g. [26]). In particular, for $L=\infty$, $a(x)\equiv\frac{1}{2}$ and $b(x)\equiv b$ the process $\mathfrak{X}$ is the Brownian motion on $[0,\infty)$ with trend b and reflecting barrier 0.

2. Let $\mathfrak{X}$ be a birth and death process on $\mathbb{N}$ with birth rates $a_k>0\,(k\geqq 0)$, death rates $b_k>0\,(k\geqq 1)$ and reflecting boundary 0. Moreover, let $\mathfrak{X}$ be killed as soon as it hits ∞. Then $\mathfrak{X}$ is the quasi-diffusion having the speed measure

$$m(x)=\sum_{k<x} m_k, \qquad m_0=1, \qquad m_k=\frac{a_0 a_1 \dots a_{k-1}}{b_1 \dots b_k} \qquad (k\geqq 1,\ x\in[0,L))$$

and the scale

$$p(x)=\sum_{j=0}^{k-1}(m_j a_j)^{-1}+\frac{(x-k)}{m_k a_k} \qquad (x\in[k,k+1),\ k\in\mathbb{N}).$$

Obviously $E_m=\mathbb{N}$ and an easy calculation shows

$$D_m D_p f(k)=a_k f(k+1)-(a_k+b_k) f(k)+b_k f(k-1) \qquad (k\geqq 1)$$
$$D_m D_p f(0)=a_0 f(1)-a_0 f(0).$$

Conversely, if $\mathfrak{X}$ is a quasi-diffusion with speed measure m, which increases exactly in the points of $\mathbb{N}$, then $\mathfrak{X}$ is a birth and death process on $\mathbb{N}$.

In this paper we restrict ourselves to processes on intervals $[0,L)$ with reflecting boundary 0. This is done by two reasons. Firstly, in this case we can apply results of Kac and Krein from [15]. Secondly, the tail σ-field for general boundaries can be recovered from our special case. Indeed, this field depends only on the behaviour of the process near the boundaries, and this can be studied at one of them. (See also [10] for the motivation.)

The birth and death processes are Markov processes on $\mathbb{N}$ jumping only in neighbour states, i.e., in a certain sense they are continuous. Groh [12] has generalized this property to quasi-diffusions (he assumes $m(0+)=0$, but this assumption is not essential). He has proved that every path of a quasi-diffusion $\mathfrak{X}$ can be choosed $\mathscr{C}^*$-continuous. This means, with the notation

$$\rho^*(x,y)=|x-y|-\sup\{|u-v|:\ (u,v)\subseteq[x,y]\setminus E_m\}\qquad (x\leqq y;\ x,y\in[0,L))$$

that for every $\omega\in\Omega$, every $\varepsilon>0$ and all $t>0$ there exists a $\delta=\delta(t,\varepsilon)>0$ such that $\rho^*(X_s(\omega),X_t(\omega))<\varepsilon$ for every positive $s\in(t-\delta,t+\delta)$.

If $E_m=[0,L)$ then $\rho^*(x,y)=|x-y|$ and therefore $\mathscr{C}^*$-continuity coincides in this case with the usual continuity.

Define τ_l and σ_l to be the hitting times of $[l,L)$ and $[0,l]$ respectively:

$$\tau_l(\omega)=\inf\{t>0\,|\,X_t(\omega)\in[l,L)\},$$
$$\sigma_l(\omega)=\inf\{t>0\,|\,X_t(\omega)\in[0,l]\}\quad\text{with}\quad \inf\varnothing=\zeta.$$

From the $\mathscr{C}^*$-continuity and the right continuity of $\mathfrak{X}$ we have

Lemma 1. *a) Let be $l\in E_m$ and put $l^+=\inf\{y\in E_m\,|\,y>l\}$, $l^-:=\sup\{y\in E_m\,|\,y<l\}$ with $\sup\varnothing=0$. Then we have*

$$P_x(X_{\tau_l-0}=l^-,\ X_{\tau_l}=l)=1\qquad (x\in E_m,\ x<l)\tag{1}$$
$$P_x(X_{\sigma_l-0}=l^+,\ X_{\sigma_l}=l,\ \sigma_l<\zeta)=P_x(\sigma_l<\zeta)\qquad (x,l\in E_m,\ x>l).\tag{2}$$

b) If $\omega\in\Omega$ and $t>0$ such that $X_{t-0}(\omega)<X_t(\omega)$ then $E_m\cap(X_{t-0}(\omega),X_t(\omega))=\varnothing$. The analogous statement holds if $X_t(\omega)<X_{t-0}(\omega)$.

Proof. a) (1): From the right-continuity of $\mathfrak{X}$ and the definition of τ_l follows $X_{\tau_l-}\leqq l$ and $X_{\tau_l}\geqq l^+$ P_x-a.s. $(x\leqq l)$. The $\mathscr{C}^*$-continuity implies $\rho^*(X_{\tau_l-},X_{\tau_l})=0$. Thus (1) holds. (2): The proof is similar.

b) The existence of a state $z\in E_m$ in the interval considered in b) would imply $\rho^*(X_{t-0}(\omega),X_t(\omega))>0$ which contradicts the $\mathscr{C}^*$-continuity. ⌟

The Function $\varphi(x,\lambda)$

Let $\mathfrak{X}$ be a quasi-diffusion with speed measure m and scale p. Then every linear operator P_t $(t\geqq 0)$ can be uniquely extended from $\mathbb{L}_2(m)\cap\mathbb{C}_m$ to a contraction T_t on $\mathbb{L}_2(m)$, the Hilbert space of dm-quadratic integrable functions on $[0,L]$. The family $(T_t)_{t\geqq 0}$ is a strongly continuous semigroup on $\mathbb{L}_2(m)$.

Its infinitesimal generator is the restriction $\mathbb{D}_m\mathbb{D}_p$ of D_mD_p to $\Delta_2:=H_1\cap H_2$ with

$$H_1=\{f\in\mathbb{L}_2(m)\,|\,D_mD_pf\in\mathbb{L}_2(m),\ D_p^-f(0)=0\}\quad\text{and}$$

$$H_2=\begin{cases} H_1 & \text{if } \int_0^L p^2\,dm=\infty,\\ \{f\in H_1\,|\,f(L)=0\} & \text{if } \int_0^L p^2\,dm<\infty \text{ and } \int_0^L m\,dp<\infty,\\ \{f\in H_1\,|\,D_p^-f(L)=0\} & \text{if } \int_0^L p^2\,dm<\infty \text{ and } \int_0^L m\,dp=\infty,\end{cases}$$

(see [9] for strongly increasing m, [17] for birth and death processes and [19, 20, 25]). The operator $\mathbb{D}_m\mathbb{D}_p$ is self-adjoint and nonpositive, in particular its spectrum σ_2 is included in $(-\infty,0]$. Put $\lambda_0=\sup\sigma_2$.

A detailed study of $\mathbb{D}_m \mathbb{D}_p$ and its spectrum σ_2 was given by Kac and Krein. A summary can be found in [15], see also [7]. The following lemma is an immediately consequence of [15], § 11, formula 11.5.

Lemma 2. *If $p(L)=\infty$ then $\lambda_0=0$, if $p(L)<\infty$ and if moreover λ_0 is an isolated point of σ_2 then $\lambda_0<0$.*

For every complex μ the solution of

$$D_m D_p \varphi(\,.\,, \mu)=\mu\varphi(\,.\,, \mu), \qquad D_p^- \varphi(0, \mu)=0, \qquad \varphi(0, \mu)=1 \tag{3}$$

exists and is uniquely determined. Indeed (3) is equivalent to

$$\varphi(x, \mu)=1+\mu \int_0^x (p(x)-p(s))\, \varphi(s, \mu)\, m(ds) \qquad (x\in[0, L)) \tag{4}$$

and this equation is solved by the function

$$\varphi(x, \mu) = \sum_{n=0}^{\infty} \mu^n \varphi_n(x) \quad \text{with} \quad \varphi_0(x)\equiv 1 \quad \text{and}$$

$$\varphi_{n+1}(x)=\int_0^x \varphi_n(s)\, m(ds) \qquad (x\in[0, L)).$$

Obviously $\varphi(l, \bar{\mu})=\overline{\varphi(l, \mu)}$ ($\mu\in\mathbb{K}$) holds. The function $\varphi(l, .)$ is entire and its zeros $(\lambda_j(l))_{j\in\mathbb{N}}$ are simple and strictly negative. Thus the representation formula

$$\varphi(l, \mu)=\prod_{j=0}^{\infty}\left(1-\frac{\mu}{\lambda_j(l)}\right) \qquad (\mu\in\mathbb{K}) \tag{5}$$

holds (see [15] for details).

Obviously, if $\mu>0$ then $\varphi(\,.\,, \mu)$ is strictly increasing with

$$\varphi(l, \mu)\geqq 1+\mu \int_0^x m(s)\, dp(s) \qquad (l\in[0, L)) \tag{6}$$

Hitting Times and Some Path Properties

The resolvent operator $\mathbb{R}_\mu=(\mu\mathfrak{I}-\mathbb{D}_m \mathbb{D}_p)^{-1}$ ($\mu>0$) is given by

$$\mathbb{R}_\mu f(x)=\int_0^L r_\mu(x, y)\, f(y)\, m(dy) \qquad (f\in\mathbb{L}_2(m))$$

with

$$r_\mu(x, y)=\varphi(x, \mu)\, \chi(y, \mu) \qquad (x\leqq y;\ x, y\in E_m)$$

and

$$\chi(y, \mu)=\varphi(y, \mu)\int_y^L \varphi^{-2}(z, \mu)\, p(dz) \qquad \text{(see e.g. [7, 15, 19, 26]).}$$

Using $\mathbb{R}_\mu f(x) = E_x \int_0^\infty e^{-\mu t} f(X_t)\,dt$ and the strong Markov property of $\mathfrak{X}$ we get for $f = \mathbb{1}_{[l,L)}$ and every $\lambda > 0$:

$$E_x \exp(-\lambda \tau_l) = \frac{\mathbb{R}_\lambda f(x)}{\mathbb{R}_\lambda f(l)} = \frac{\varphi(x,\lambda)}{\varphi(l,\lambda)} \qquad (x, l \in E_m,\ x \leqq l) \tag{7}$$

(see e.g. [14], if $m(L-0) = \infty$ then $\mathbb{R}_\lambda f = \lim_{c \uparrow L} \mathbb{R}_\lambda f_c$ with $f_c = \mathbb{1}_{[l,c)}$).

From the theory of Laplace transforms it follows (see [6]) that

$$E_x \tau_l^k = (-1)^k \lim_{\lambda \downarrow 0} \frac{\partial^k}{\partial \lambda^k} \frac{\varphi(x,\lambda)}{\varphi(l,\lambda)}. \tag{8}$$

Now from (8) and (4) we have (with $\tau_L := \lim_{l \uparrow L} \tau_l$)

$$E_x \tau_l = \int_x^l m\,dp \qquad (x, l \in E_m \cup \{L\},\ x < l). \tag{9}$$

Assume $\int_0^L m\,dp = \infty$. From (6) and (7) it follows that $P_x(\tau_L = \infty) = 1$ $(x \in E_m)$, i.e., the boundary L is inaccessible for $\mathfrak{X}$. Thus for the life time ζ holds $P_x(\zeta < \infty) = 0$ $(x \in E_m)$. Depending on $\int_0^L p\,dm < \infty$ or $= \infty$ an inaccessible boundary L is called an *entrance* or *natural boundary* (Feller's classification, see e.g. [9, 14, 26]).

Throughout this paper we make the

Assumption.

$$\int_0^L m\,dp = \infty, \qquad \text{i.e. } L \text{ is inaccessible.} \tag{10}$$

Now let us give some properties of the paths of a quasi-diffusion, necessary for the proofs in Sect. 3.

Let be $x, l \in E_m$ with $x \leqq l < l^+$, i.e. m is constant on (l, l^+). From (9) it follows that $E_x \tau_l < E_x \tau_{l^+}$, and by Blumenthals zero-one-law (see e.g. [2]) and the strong Markov property we have

$$P_x(\tau_l < \tau_{l^+}) = P_l(\tau_{l^+} > 0) = 1. \tag{11}$$

Analogously it follows for $x, l \in E_m$ with $x \geqq l > l^-$ that

$$P_x(\sigma_l < \sigma_{l^-}) = P_l(\sigma_{l^-} > 0) = 1. \tag{12}$$

By virtue of (11) and (12) and the strong Markov property the process $\mathfrak{X}$ jumps over every interval $I = (l_1, l_2)$ where m is constant only finite often in finite times. Denote by $\tau_I^{(k)}(\omega)$ $(\sigma_I^{(k)}(\omega))$ the k-th jump of $X_\cdot(\omega)$ over I from below (from above) if it exists and define it to be ∞ if it does not exist.

Lemma 3. *Assume $k \geqq 1$, $I = (l_1, l_2)$ is an interval from $[0, L) \setminus E_m$ with $l_1, l_2 \in E_m$ and let $x \in E_m$. Then for P_x-a.e. ω with $\tau_I^{(k)}(\omega) < \infty$ $(\sigma_I^{(k)}(\omega) < \infty)$ there exists a strongly increasing sequence $t_n \uparrow \tau_I^{(k)}(\omega)$ $(t_n \uparrow \sigma_I^{(k)}(\omega))$ with $X_{t_n}(\omega) = l_1$ $(X_{t_n}(\omega) = l_2)$.*

Proof. By virtue of the strong Markov property and Lemma 1 it suffices to prove the proposition for $\tau_I^{(1)}$, $\sigma_I^{(1)}$ only. We restrict ourselves to $\tau_I^{(1)}$, for $\sigma_I^{(1)}$ the proof is completely analogous. We put $l=l_1$, $l^+=l_2$. Obviously we can assume $x\leqq l$. Then the equation $\tau_I^{(1)}=\tau_{l^+}$ holds P_x-a.s. If $l^-<l$ then l is an isolated point of E_m. By virtue of $P_x(X_{\tau_{l^+}-0}=l)=1$ (Lemma 1) it follows that $X_{\tau_{l^+}-s}=l$ for every sufficient small positive s P_x-a.s.

Let be $l^-=l$. Then l cannot be a holding point. (Indeed, define $\xi_l=\inf\{t>0|X_t\neq l\}$ and assume $P_l(\xi_l>0)\neq 0$. By Dynkin [8] Theorems 5.4 and 5.5 it follows for every $f\in\Delta$ that

$$D_m D_p f(l)=-af(l)+a\lim_{h\downarrow 0} E_l f(X_{\xi_l+h}) \quad \text{for some } a>0. \tag{13}$$

We have $P_l(\sigma_x<\tau_{l^+})=\dfrac{p(l^+)-p(l)}{p(l^+)-p(x)} \xrightarrow[x\uparrow l]{} 1$.

Thus $E_l f(X_{\xi_l+h})\to f(l)$, i.e. $D_m D_p f(l)=0$.

Therefore l is absorbing ([8], Lemma 5.3). This contradicts $E_l\tau_{l^+}=\int_l^{l^+} m\,dp<\infty$. Therefore we have

$$P_l(\xi_l>0)=0.) \tag{14}$$

From (11) and (14) it follows that

$$P_l(\sigma_{y_n}<\tau_{l^+} \text{ for some } n\in\mathbb{N})=1 \tag{15}$$

where (y_n) is any strongly increasing sequence from E_m with $y_n\uparrow l$. For every $t\geqq 0$ we define

$$A_t=\{\omega|\tau_{l^+}(\omega)>t,\ \nexists\, s\in(t,\tau_{l^+}(\omega)):\ X_s(\omega)=l\}.$$

Then we have

$$\{\omega|\nexists\,(t_n),\ t_n\uparrow\tau_{l^+}(\omega):\ X_{t_n}(\omega)=l\}=\bigcup\{A_t|t\geqq 0,\ t \text{ rational}\}$$

and from the Markov property it follows that

$$P_x(A_t)=\int_{\{\tau_{l^+}>t\}} P_{X_t}(A_0)\,dP_x.$$

Lemma 1 implies

$$P_y(A_0)=0 \quad \text{for } y<l. \tag{16}$$

Thus $P_x(A_t)=P_l(A_0)P_x(\tau_{l^+}>t)$. From (15), (16) and Lemma 1 we get

$$\begin{aligned} P_l(A_0)&=\lim_{n\to\infty} P_l(A_0,\sigma_{y_n}<\tau_{l^+})\leqq\limsup_{n\to\infty} P_l(A_{\sigma_{y_n}},\sigma_{y_n}<\tau_{l^+})\\ &=\limsup_{n\to\infty} P_l(\sigma_{y_n}<\tau_{l^+})P_{y_n}(A_0)=0.\end{aligned}$$

Therefore $P_x(A_t)=0$, and the lemma follows immediately.

The Tail σ-Field and Parabolic Functions

Let $\mathfrak{X}=(X_t, \mathscr{M}_t, \theta_t, P_x)$ be a quasi-diffusion with state space E_m and infinite life time, i.e. with (10). We denote by $\mathscr{F}_t^\infty$ the σ-field generated by $\{X_s, s\geqq t\}$, by $\mathscr{T}^*$ the intersection $\bigcap_{t>0} \mathscr{F}_t^\infty$, and we identify all elements of $\mathscr{T}^*$ being equal P_x-a.s. for every $x\in E_m$. The σ-field $\mathscr{T}$ of all equivalence classes thus defined is called the *tail σ-field* of $\mathfrak{X}$.

By $\tilde{P}(t,(x,s), \quad A\times G):=P_x(X_t\in A)\mathbb{1}_G(s+t) \quad (A\in\mathscr{B}\cap E_m), \quad G\in\mathscr{B}\cap[0,\infty);$ $s,t\in[0,\infty))$ a transition function $\tilde{P}$ on $\tilde{E}_m=E_m\times[0,\infty)$ is determined, which is stochastically continuous and Fellerian. (See [18].) Thus it generates a Hunt process $\tilde{\mathfrak{X}}=(\tilde{X}_t, \tilde{\mathscr{M}}_t, \tilde{\theta}_t, \tilde{P}_{(x,s)})$ on $\tilde{E}_m$, the so called *space-time process* for $\mathfrak{X}$. The first component of $\tilde{\mathfrak{X}}$ has the same finite-dimensional distributions as $\mathfrak{X}$ and thus it can be identified with $\mathfrak{X}$. In this sense it follows that

$$\tilde{P}_{(x,s)}(\tilde{X}_t=(X_t, s+t))=1 \qquad ((x,s)\in\tilde{E}_m).$$

Let h be a nonnegative function on $\tilde{E}_m$. We say that h is *parabolic* for $\mathfrak{X}$, if it is harmonic in the sense of [28] for the space-time process $\tilde{\mathfrak{X}}$, i.e. if

– h is $\tilde{\mathfrak{X}}$-universally measurable,

– $h(x,s)=E_x h(X_{\tau_U}, \tau_U+s)$ for every open neighbourhood $U\subset\tilde{E}_m$ of (x,s) with compact closure in $\tilde{E}_m$, where τ_U denotes the first hitting time of $\tilde{E}_m\setminus U$ $((x,s)\in\tilde{E}_m)$.

Obviously the constant function $\mathbb{1}$ is parabolic.

A parabolic function h is said to be *minimal*, if there is no parabolic function h_0 with $0<h_0<h$ unless $h_0=ch$ for $c\in(0,1)$. Bounded parabolic functions are continuous (see [20]). The constant function $\mathbb{1}$ is minimal if and only if $\mathscr{T}$ is trivial. More generally, the relation

$$Z\to h(x,s)=E_x Z\circ\theta_s \tag{17}$$

establishes a one-to-one correspondence between the set of all bounded $\mathscr{T}$-measurable random variables Z and the set of all bounded parabolic functions h. The inverse formula of (17) is given by $Z=\lim_{t\to\infty} h(X_t, t)$.

For the proof we recall that by virtue of the finite life time of $\mathfrak{X}$ a bounded nonnegative function h on $E_m\times[0,\infty)$ is parabolic for $\mathfrak{X}$ if and only if it is invariant, i.e. if

$$E_x h(X_t, s+t)=h(x,s) \qquad (x\in E_m;\ s,t\geqq 0) \tag{18}$$

holds (see e.g. [20], p. 41).

Now the assertion can by verified analogously to the proof of the same proposition for discrete time Markov processes in [30], Chapter V.

Formula (18) implies that $(h(X_t, s+t))_{t\geqq 0}$ is a martingale with respect to P_x $(x\in E_m, s\geqq 0)$. Indeed we have

$$E_x(h(X_{t+u}, s+t+u)\,|\,X_v, v\leqq t)=E_{X_t}h(X_u, s+t+u)=h(X_t, s+t)$$
$$(x\in E_m;\ s,t,u\geqq 0). \tag{19}$$

If h is continuous, nonnegative and satisfies the equation

$$D_m D_p h + \frac{\partial h}{\partial s} = 0, \qquad D_p^- h(0, s) = 0$$

then h is parabolic for the quasi-diffusion $\mathfrak{X}$ (see [18, 20]). (The boundary condition $D_p^- h(0, s) = 0$ is necessary because $0 \in E_m$ and 0 is reflecting.)

In particular for every $\mu \geqq \lambda_0$ the function h_μ defined by

$$h_\mu(x, s) = \exp(-\mu s)\, \varphi(x, \mu) \qquad ((x, s) \in \tilde{E}_m)$$

is parabolic (see [20]).

The Coincidence Property

We say that a quasi-diffusion $\mathfrak{X}$ has the *coincidence property* if two independent versions $\mathfrak{X}^{(1)}$, $\mathfrak{X}^{(2)}$ of $\mathfrak{X}$, starting in arbitrary states $x, y \in E_m$ respectively, coincide $P_{(x,y)}$-a.s. at some time. (Here and in the sequel $P_{(x,y)}$ denotes the product measure $P_x \oplus P_y$; $x, y \in E_m$.) More precisely, putting

$$\sigma(\omega) = \inf\{t \geqq 0 \mid X_t^{(1)}(\omega) = X_t^{(2)}(\omega)\}$$

the coincidence property means $P_{(x,y)}(\sigma < \infty) = 1$ $(x, y \in E_m)$.

In [16] it was proved that a birth and death process with intensities a_k, b_k has the coincidence property if and only if

$$\sum_{k=1}^{\infty} (a_k m_k)^{-1} \sum_{r=0}^{k} m_r (w_{k+1} - w_r) = \infty$$

with

$$w_m = \sum_{i=0}^{m-1} (a_i m_i)^{-1} \sum_{j=0}^{i} m_j \quad \text{and} \quad w_0 = 0$$

where m_k $(k \in \mathbb{N})$ are defined as in the example above.

In terms of the corresponding speed measure m and scale p this criterion can be written as follows (see [25]):

$$\int_0^L \int_0^x \int_s^x m(t)\, p(dt)\, m(ds)\, p(dx) = \infty.$$

A simplification of this analytical criterion will be given in Theorem 1 below.

The coincidence property for random walks on $\mathbb{N}$ has been considered in [23].

3. Quasi-Diffusions with Trivial Tail σ-Field

The theorem below characterizes the quasi-diffusions $\mathfrak{X}$ with trivial tail σ-field $\mathscr{T}$ by different objects connected with $\mathfrak{X}$: by the parabolic functions, by the trajectories, by the functions m and p and by the spectrum σ_2 of $\mathbb{D}_m\mathbb{D}_p$ in $\mathbb{L}_2(m)$.

If L is an entrance boundary then an easy calculation shows that (v) and therefore (i)–(vi) hold. If L is a natural boundary then (v) may but need not hold (see the examples below). Thus Feller's boundary classification can be refined by the theorem below, indeed, the quasi-diffusions with natural boundary L are divided into two classes according to the convergence or divergence of the integral in (v).

By the way if L is accessible then non of the properties (ii)–(vi) holds.

Theorem 1. *Let $\mathfrak{X}$ be a quasi-diffusion on $[0, L)$ with speed measure m and scale p. The boundary 0 is assumed to be reflecting, L is assumed to be inaccessible $\left(\int_0^L m\,dp = \infty\right)$. Then the following properties are equivalent:*

(i) *The tail σ-field $\mathcal{T}$ is trivial,*

(ii) *The constant function $\mathbb{1}$ is minimal parabolic,*

(iii) *The process $\mathfrak{X}$ has the coincidence property,*

(iv) $\lim\limits_{l\uparrow L,\, l\in E_m} E_x(\tau_l - E_x\tau_l)^2 = \infty \quad (x \in E_m)$,

(v)[1] $\int_0^L (p(L-0)-p(x))^2\, dm^2(x) = \infty$,

(vi) *The spectrum σ_2 of $\mathbb{D}_m\mathbb{D}_p$ contains a sequence $(\lambda_n)_{n\in N}$ of pairwise different real numbers λ_n such that $\sum_{n=0}^{\infty} \lambda_n^{-2} = \infty$ (with the notation $0^{-2} = \infty$), i.e. $\mathbb{R}_0 = (\mathbb{D}_m\mathbb{D}_p)^{-1}$ is not defined or does not belong to the class $\mathcal{S}_2$ of Hilbert-Schmidt-operators.*

If $\mathfrak{X}$ is the Brownian motion on $[0, \infty)$ with zero drift, which is reflected at zero then $L = \infty$ is a natural boundary and (v) holds because $p(x) = x$ and therefore $p(L-0) = \infty$.

If $\mathfrak{X}$ is a diffusion on $[0, 1)$ with reflecting boundary 0 and

$$\mathbb{D}_m\mathbb{D}_p = (1+x)\frac{d^2}{dx^2} + (1+x)(\exp x)\frac{d}{dx}, \quad \text{i.e.,}$$

$$dm(x) = (1+x)^{-1}\exp((\exp x) - 1)\,dx, \qquad dp(x) = \exp(1-\exp x)\,dx,$$

then $L = \infty$ is a natural boundary and (v) does not hold (see [25]).

Proof of the Theorem. At first let us give some more properties of the hitting times τ_l. Using (4) and (8) we get for $l \in E_m$

$$E_0\tau_l^2 = \frac{\partial^2}{\partial\lambda^2}(\varphi^{-1}(l, \lambda)) = 2\left(\int_0^l m\,dp\right)^2 - 2\int_0^l\int_0^z\int_0^s m(t)\,dp(t)\,dm(s)\,dp(z)$$

and thus we have

[1] If $p(L-0) = \infty$ then the integral is defined to be equal ∞.

$$D_0^2\tau_l = E_0(\tau_l - E_0\tau_l)^2 = 2\int_0^l\int_s^l\int_s^z m(t)\,dp(t)\,dp(z)\,dm(s). \tag{20}$$

From (5) and (8) it follows immediately that

$$D_0^2\tau_l = \sum_{i=0}^{\infty} \lambda_j^{-2}(l) \qquad (l\in E_m). \tag{21}$$

Now let $(l_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence from E_m with $l_0=0$ and $\lim_{n\uparrow\infty} l_n = L$. We use the notation $\tau_n = \tau_{l_n}$ and $n(y) = \min\{n\in\mathbb{N} \mid l_n \geqq y\}$. For every $x\in E_m$ the variables $\tau_{n(x)}$, $(\tau_{k+1}-\tau_k)$ $(k\geqq n(x))$ are independent with respect to P_x. This is an easy consequence of the strong Markov property of $\mathfrak{X}$ and the property $X_{\tau_l} = l$ P_x-a.s. $(l\geqq x)$ (see (1)). Thus for every $n\geqq n(x)$ the hitting time τ_n is the sum of random variables which are independent with respect to P_x:

$$\tau_n = \tau_{n(x)} + \sum_{k=n(x)}^{n-1} (\tau_{k+1}-\tau_k) \qquad P_x\text{-a.s.} \tag{22}$$

If we put $x=l_1$, $l=l_2$ then in particular we have

$$D_0^2\tau_l = D_0^2\tau_x + D_0^2(\tau_l-\tau_x), \qquad \text{i.e.}$$

$$D_x^2\tau_l = E_x(\tau_l - E_x\tau_l)^2 = D_0^2\tau_l - D_0^2\tau_x$$
$$= 2\int_x^l\int_s^l\int_s^z m(t)\,dp(t)\,dp(z)\,dm(s). \tag{23}$$

Let us recall that the partial sums (Z_n) of a sequence of independent random variables on a probability space $(\Omega, \mathscr{A}, P)$ is said to be *essentially divergent*, if no sequence (A_n) of real numbers exists such that $(Z_n - A_n)$ is P-a.s. converging to a finite random variable (see e.g. [24]).

(iv) $\Rightarrow$ (iii): Assume that (iv) holds.

Lemma 4. *For every $x\in E_m$ the sequence (τ_n) defined by* (22) *is essentially divergent with respect to P_x.*

Proof. The function $\varphi(y,.)$ is entire and has strictly negative zeros only (see Sect. 2). Thus the right hand side of (7) is holomorphic in $\{z\in\mathbb{K} \mid \operatorname{Re} z \geqq -\delta\}$ for some $\delta>0$. The function $z\to E_x\exp(-z\tau_l)$ is also holomorphic in $\{z\in\mathbb{K} \mid \operatorname{Re} z>0\}$ and continuous on the corresponding closure. In particular we have

$$E_x\exp(-i\mu\tau_l) = \frac{\varphi(x,i\mu)}{\varphi(l,i\mu)} \qquad (\mu\in\mathbb{R};\ x,l\in E_m,\ x\leqq l).$$

(With i here we mean the imaginar unit.)

Let $x\in E_m$ and (A_n) a sequence of real numbers such that $(\tau_n - A_n)$ converges P_x-a.s. to a finite limit. Then the functions

$$E_x\exp(-i\mu(\tau_n - A_n)) = \frac{\exp(i\mu A_n)\,\varphi(x,i\mu)}{\varphi(l_n,i\mu)} \qquad (\mu\in\mathbb{R})$$

converge pointwise as $n\to\infty$ to a characteristic function $\rho(.)$. From (4) it follows that

$$|\varphi(l, i\mu)|^2 = \prod_{j=0}^{\infty} \left(1 + \frac{\mu^2}{\lambda_j^2(l)}\right) \geqq 1 + |\mu|^2 \sum_{j=0}^{\infty} \lambda_j^{-2}(l).$$

Now (iv) and (21) imply $\rho(.) \equiv 0$ which is a contradiction. $\lrcorner$

As in Sect. 2 let $\mathfrak{X}^{(1)}$, $\mathfrak{X}^{(2)}$ be two independent versions of $\mathfrak{X}$ on a probability space $(\tilde{\Omega}, \tilde{\mathscr{A}}, P_{(x,y)})$ $(x, y \in E_m)$ where $P_{(x,y)}$ denotes the product measure $P_x \otimes P_y$. Let $\tau_k^{(i)}$ be the hitting time of l_k by $\mathfrak{X}^{(i)}$ $(i=1,2)$. If we abbreviate

$$T_{j,n} = (\tau_n^{(1)} - \tau_n^{(2)}) - (\tau_j^{(1)} - \tau_j^{(2)}) \qquad (j, n \in \mathbb{N},\ j < n)$$

then for all $x, y \in E_m$ the variable $T_{j_0,n}$ with $j_0 := n(x \vee y)$ is a sum of independent random variables

$$[(\tau_{k+1}^{(1)} - \tau_{k+1}^{(2)}) - (\tau_k^{(1)} - \tau_k^{(2)})] \qquad (j_0 < k \leqq n-1)$$

having symmetric distributions with respect to $P_{(x,y)}$.

Lemma 5. *For every $x, y \in E_m$ the sequence $(T_{j_0,n})_{n > j_0}$ is essentially divergent with respect to $P_{(x,y)}$.*

Proof. The proof can be provided analogously to the proof of Lemma 4 if we use

$$E_{(x,y)} \exp(-i\mu(T_{j_0,n} - A_n)) = \frac{\exp(i\mu A_n)\, |\varphi(l_{j_0}, i\mu)|^2}{|\varphi(l_n, i\mu)|^2}. \quad \lrcorner$$

Now it follows from a theorem of Lévy ([24], p. 147) that $P_{(x,y)}(T_{j_0,n} \geqq 0$ infinitely often) equals either 0 or 1.

By the symmetry of the distributions of $T_{j_0,n}$ this probability has to be 1. Another theorem of Lévy ([24], p. 147) implies that $P_{(x,y)}(T_{j_0,n} \geqq c$ infinitely often$) = 1$ for every $c \in R$. By virtue of $\tau_n^{(1)} - \tau_n^{(2)} = \tau_{j_0}^{(1)} - \tau_{j_0}^{(2)} + T_{j_0,n}$ we have $P_{(x,y)}(\tau_n^{(1)} - \tau_n^{(2)} > 0$ infinitely often$) = 1$ and, by symmetry $P_{(x,y)}(\tau_n^{(1)} - \tau_n^{(2)} < 0$ infinitely often$) = 1$.

In particular we get with the notation

$$\tau = \inf\{t \geqq 0 \mid (x-y)(X_t^{(1)} - X_t^{(2)}) \leqq 0\}$$

that

$$P_{(x,y)}(\tau < \infty) = 1 \qquad (x, y \in E_m). \tag{24}$$

Lemma 6. *If $P_{(x,y)}(\tau < \infty) = 1$ $(x, y \in E_m)$ then the coincidence property holds: $P_{(x,y)}(\sigma < \infty) = 1$ $(x, y \in E_m)$.*

Proof. We show that the event $A = \{\sigma = \infty, \tau < \infty\}$ has the probability zero with respect to every $P_{(x,y)}$. Let be $x, y \in E_m$. We can assume $x < y$. Choose $\omega \in A$. Then we have

$$X_{\tau-0}^{(1)}(\omega) \leqq X_{\tau-0}^{(2)}(\omega) \quad \text{and} \quad X_\tau^{(1)}(\omega) > X_\tau^{(2)}(\omega). \tag{25}$$

Assume $X_{\tau-0}^{(1)}(\omega) = X_{\tau-0}^{(2)}(\omega) = l$. If $X_\tau^{(1)}(\omega) > l$ and $X_\tau^{(2)}(\omega) < l$ then from the $\mathscr{C}^*$-continuity it follows that l has to be an isolated point of E_m. This implies that $X_{\tau-s}^{(1)}(\omega) = X_{\tau-s}^{(2)}(\omega) = l$ for sufficiently small but positive s, i.e. $\sigma(\omega) < \tau(\omega)$ holds.

Thus we have $X_\tau^{(1)}(\omega)=l$ or $X_\tau^{(2)}(\omega)=l$, let us assume the second equality. Putting $X_\tau^{(1)}(\omega)=l_1$ we have $l_1>l$ and (by Lemma 1b)) $(l,l_1)\subset[0,L)\setminus E_m$. From Lemma 3 it follows that there exists a strictly increasing sequence (t_n) with $t_n\uparrow\tau(\omega)$ such that $X_{t_n}^{(1)}(\omega)=l$. By virtue of $t_n<\tau(\omega)$ we have

$$l=X_{t_n}^{(1)}(\omega)<X_{t_n}^{(2)}(\omega) \qquad (n\in\mathbb{N})$$

and because (l,l_1) does not contain an element of E_m we get

$$X_{t_n}^{(2)}(\omega)\geqq l_1 \qquad (n\in\mathbb{N}).$$

This contradicts $X_{\tau-0}^{(2)}(\omega)=l<l_1$. Thus the assumption of equality in (25) was false, i.e., we have

$$X_{\tau-0}^{(1)}(\omega)<X_{\tau-0}^{(2)}(\omega), \qquad X_\tau^{(1)}(\omega)>X_\tau^{(2)}(\omega). \tag{25'}$$

Now from (25′) and the $\mathscr{C}^*$-continuity (see Lemma 1b)) it follows that

$$X_{\tau-0}^{(1)}(\omega)<X_\tau^{(1)}(\omega), \qquad X_{\tau-0}^{(2)}(\omega)>X_\tau^{(2)}(\omega). \tag{26}$$

(If we assume $X_\tau^{(1)}(\omega)\leqq X_{\tau-0}^{(1)}(\omega)$ then $X_\tau^{(2)}(\omega)<X_\tau^{(1)}(\omega)<X_{\tau-0}^{(2)}(\omega)$, i.e., $(X_\tau^{(2)}(\omega),X_{\tau-0}^{(2)}(\omega))$ contains an element of E_m. This contradicts Lemma 1b). The second inequality follows on the same way.)

Analogous conclusions imply

$$X_{\tau-0}^{(1)}(\omega)=X_\tau^{(2)}(\omega)=a_1, \qquad X_{\tau-0}^{(2)}(\omega)=X_\tau^{(1)}(\omega)=a_2.$$

Put $I=(a_1,a_2)$. Then $\tau(\omega)=\tau_I^{(1,j)}(\omega)=\sigma_I^{(2,k)}(\omega)$ for some $j,k\in N$ where $\tau_I^{(1,j)}$ denotes the j-th jump of $\mathfrak{X}^{(1)}$ over I from below and $\sigma_I^{(2,k)}$ the k-th jump of $\mathfrak{X}^{(2)}$ over I from above.

The distribution functions of $\tau_I^{(1,j)}$, $\sigma_I^{(2,k)}$ under $P_{(x,y)}$ are absolutely continuous, therefore the probability of the event that both processes $\mathfrak{X}^{(1)}$, $\mathfrak{X}^{(2)}$ jump at the same time equals zero, i.e., $P_{(x,y)}(A)=0$ holds.

(The absolutely continuity was shown by Rösler [33] for processes with strictly increasing speed measure m. His arguments hold also for general m.) Thus the lemma is proved. $\lrcorner$

From (24) and Lemma 6 it follows that $P_{(x,y)}(\sigma<\infty)=1$ $(x,y\in E_m)$, i.e., (iii) holds.

(iii) $\Rightarrow$ (ii): Let h be a bounded parabolic function and define $\tilde{\mathscr{F}}_t=\sigma[X_u^{(1)},X_u^{(2)}|u\leqq t]$ to be the σ-field generated by $\{X_u^{(1)},X_u^{(2)}|u\leqq t\}$. Then for every $s\geqq 0$ the processes $(h(X_t^{(i)},s+t),\tilde{\mathscr{F}}_t)_{t\geqq 0}$ $(i=1,2)$ are martingales with respect to $P_{(x,y)}$ $(x,y\in E_m)$.

Indeed, choose $A=A_1\cap A_2$ with $A_i\in\sigma[X_u^{(i)}|u\leqq t]$ $(i=1,2)$. Then from the independence of $\mathfrak{X}^{(1)}$ and $\mathfrak{X}^{(2)}$ under $P_{(x,y)}$ it follows that (put for example $i=1$)

$$\begin{aligned}\int_A h(X_{t+v}^{(1)},s+t+v)\,dP_{(x,y)}&=\int_{A_1} h(X_{t+v}^{(1)},s+t+v)\,dP_x\int_{A_2}dP_y\\&=\int_{A_1} h(X_t^{(1)},s+t)\,dP_x\,P_y(A_2)=\int_A h(X_t^{(1)},s+t)\,dP_{(x,y)} \qquad \text{(see (19))}.\end{aligned}$$

Recall that the set of all such A generates $\tilde{\mathscr{F}}_t$.

The random variable σ defined in Sect. 2 is a $(\tilde{\mathscr{F}}_t)$-stopping time. Now a stopping theorem for martingales (see [27], VI. 14.), property (iii) and $X_\sigma^{(1)}=X_\sigma^{(2)}$ $P_{(x,y)}$-a.s. $(x, y \in E_m)$ imply

$$h(x,s)=E_{(x,y)}\,h(X_\sigma^{(1)}, s+\sigma)=E_{(x,y)}\,h(X_\sigma^{(2)}, s+\sigma)=h(y,s) \qquad (s\geqq 0, x, y\in E_m).$$

This means that for every $s\in[0,\infty)$ the function $h(.,s)$ is constant. Now it follows from (18) that h is constant. Therefore the constant function $\mathbb{1}$ is minimal parabolic.

(ii) $\Rightarrow$ (i): Let be $A\in\mathscr{T}$. The function h_A defined by

$$h_A(x,s)=P_x(\theta_s A) \qquad (x\in E_m, s\geqq 0)$$

is bounded. Because $A\in\mathscr{F}_t^\infty$ for every $t\geqq 0$ it follows from the Markov property that

$$h_A(X_t,t)=P_{X_t}(\theta_t A)=P_x(A\,|\,X_u, u\leqq t) \qquad P_x\text{-a.s.} \qquad (x\in E_m). \tag{27}$$

Thus $(h_A(X_t,t))_{t\geqq 0}$ is a bounded martingale with respect to $P_x\,(x\in E_m)$. The stopping theorem for martingales mentioned above implies that h_A is parabolic (the universal measurability follows from the $\tilde{\mathfrak{X}}$-excessivity of h_A, see e.g. [28]). By virtue of (ii) h_A has to be constant. From (27) and the convergence theorem [27], VI. 6. we have by virtue of $A\in\mathscr{F}_t^\infty\,(t\geqq 0)$:

$$\lim_{t\to\infty} h_A(X_t,t)=P_x(A\,|\,X_u, u\in[0,\infty))=\mathbb{1}_A \quad P_x\text{-a.s.} \qquad (x\in E_m).$$

This means $h_A\equiv 0$ or $h_A\equiv 1$, i.e. (i) holds.

(i) $\Rightarrow$ (iv): Assume (iv) does not hold, i.e., by (23) and (9),

$$\lim_{l\uparrow L} D_0^2\,\tau_l^2<\infty. \tag{28}$$

For every $x\in E_m$ and every k with $l_k\geqq x$ the random variables τ_k, $(\tau_{j+1}-\tau_j)_{j\geqq k}$ are independent with respect to P_x (see above). We have for every $x\in E_m$

$$\begin{aligned}\tau_n-E_0\,\tau_n=&\sum_{j=n(x)}^{n-1}(\tau_{j+1}-\tau_j-E_0(\tau_{j+1}-\tau_j))\\ &+(\tau_{n(x)}-E_0\,\tau_{n(x)}) \quad P_x\text{-a.s.}\end{aligned}$$

Thus by virtue of (28) and (23) we can conclude that $(\tau_n-E_0\,\tau_n)$ converges P_x-a.s. and in P_x-square mean $(x\in E_m)$. The limit is denoted by Z.

For all $n\in\mathbb{N}$ and $s>0$ the equation $\tau_n\circ\theta_s=\tau_n-s$ holds on $\{\tau_n>s\}$. By virtue of $\lim_{n\to\infty}\tau_n=\tau_L=\infty$ (P_x-a.s., $x\in E_m$) we have $\{\tau_n>s\}\uparrow\Omega$ (P_x-a.s., $x\in E_m$) for $n\to\infty$ and all $s>0$. It follows that $Z\circ\theta_s=\lim_{n\to\infty}(\tau_n\circ\theta_s-E_0\,\tau_n)=\lim_{n\to\infty}(\tau_n-s-E_0\,\tau_n)=Z-s$ P_x-a.s. $(x\in E_m)$ for every $s>0$.

This implies that Z is $\mathscr{F}_s^\infty$-measurable for every $s>0$, i.e., Z is $\mathscr{T}$-measurable. Because $D_0^2 Z=\lim_{n\to\infty} D_0^2(\tau_n-E_0\,\tau_n)>0$ the variable Z is nondegenerate, i.e., $\mathscr{T}$ is nontrivial, and thus (i) does not hold.

(iv) $\Rightarrow$ (v): Assume (v) is not true. Then $p(L-0)<\infty$ and integration by parts yields for every $l\in[0,L)$

$$\int_0^l (p(L-0)-p(x))^2\, m^2(dx)$$
$$=(p(L-0)-p(l))^2\, m^2(l)+2\int_0^l (p(L-0)-p(x))\cdot m^2(x)\, p(dx). \tag{29}$$

By assumption the left hand side and therefore all terms of this equation are bounded with respect to $l\in E_m$. Thus the inequality

$$\int_0^l (p(L-0)-p(x))\, m^2(x)\, p(dx) \geqq \int_0^l (p(l)-p(x))\, m^2(x)\, p(dx)$$
$$=\int_0^l (p(l)-p(x))\, m(x)\int_0^x m(ds)\, p(dx)$$
$$=\int_0^l\int_s^l (p(l)-p(x))\, m(x)\, p(dx)\, m(ds)$$
$$=\int_0^l\int_s^l\int_x^l p(dt)\, m(x)\, p(dx)\, m(ds)$$
$$=\int_0^l\int_s^l\int_s^t m(x)\, p(dx)\, p(dt)\, m(ds)=D_0^2\,\tau_l$$

(see (20)) implies

$$\lim_{l\uparrow L} D_0^2\,\tau_l<\infty.$$

This contradicts (iv).

(v) $\Rightarrow$ (iv): Firstly let us remark that for every nonnegative continuous function $f(.)$ on $[0,L)$ it holds

$$2\int_0^l f(t)\, m(t-0)\, m(dt) \leqq \int_0^l f(t)\, m^2(dt)$$
$$\leqq 2\int_0^l f(t)\, m(t+0)\, m(dt) \qquad (l\in E_m). \tag{30}$$

(The first and the last integrals are understood in the Lebesgue's sense. The proof of (30) is left to the reader.)

From (30) it follows

$$\tfrac{1}{4}\int_0^L (p(L-0)-p(x))^2\, m^2(dx) \leqq \int_0^L\int_x^L (p(L-0)-p(t))\, p(dt)\, m(x+0)\, m(dx)$$
$$\leqq \int_0^L\int_x^L (p(L-0)-p(t))\, m(t)\, p(dt)\, m(dx)$$
$$=\int_0^L\int_s^L\int_s^z m(t)\, p(dt)\, p(dz)\, m(ds).$$

Now it follows from (v), (20) and (23) that (iv) holds.

(v) $\Rightarrow$ (vi): Assume that (vi) is not true. Then the spectrum σ_2 of the self-adjoint operator $\mathbb{D}_m\mathbb{D}_p$ consist of a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues with $\sum_{n=0}^{\infty}\lambda_n^{-2}<\infty$ and $\lambda_0=\max\sigma_2<0$.

Thus the operator $\mathbb{R}_0=(\mathbb{D}_m\mathbb{D}_p)^{-1}$ exists and has the spectrum $(\lambda_n^{-1})_{n\in\mathbb{N}}$. It follows from Lemma 2 that $p(L-0)<\infty$. With the notation $r_0(x,y):=(p(L-0)-p(x\vee y))$ we have

$$\mathbb{R}_0 f(x)=\int_0^L r_0(x,y)\,f(y)\,m(dy).$$

This follows from the general theory of the operator $\mathbb{D}_m\mathbb{D}_p$ and can be also proved by direct calculations (see e.g. [15]). For the trace of $\mathbb{R}_0^2$ we thus have (see e.g. [11], III §10)

$$\begin{aligned}\infty>\sum_{n=0}^{\infty}\lambda_n^{-2}&=\int_0^L\left(\int_0^L r_0(x,y)\,r_0(y,x)\,m(dy)\right)m(dx)\\ &=\int_0^L(p(L-0)-p(x))^2(m(x+0)+m(x-0))\,m(dx).\end{aligned}\tag{31}$$

Using (30) we get a contradiction to (v).

(vi) $\Rightarrow$ (v): If (v) does not hold, we have $p(L-0)<\infty$ and

$$\int_0^L(p(L-0)-p(x))^2\,m^2(dx)<\infty.\tag{32}$$

Therefore the integrals in (29) converge to a finite limit if $l\uparrow L$ and thus the limit $\lim_{l\uparrow L}(p(L-0)-p(l))\,m(l)=:C$ exists and is finite. If $C>0$ then for every $l\in E_m$ sufficiently near to L we have by virtue of (30) and the assumption $\int_0^L m\,dp=\infty$:[2]

$$\begin{aligned}\int_l^L(p(L-0)-p(x))^2\,m^2(dx)&\geqq 2\int_l^L(p(L-0)-p(x))^2\,m(x-0)\,m(dx)\\ &\geqq C\int_l^L(p(L-0)-p(x))\,m(dx)=C\int_l^L(m(s)-m(l-0))\,p(ds)\\ &\geqq C\left(\int_l^L m\,dp-m(l-0)\,p(L-0)\right)=\infty.\end{aligned}$$

This contradicts (32). Therefore $C=0$ holds, and this implies that the spectrum σ_2 is discrete and consists only of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ without finite accumulation point (see [15], §11). It now follows from Lemma 2 that $\lambda_0<0$ holds. In particular the trace formula

$$\sum_{n=0}^{\infty}\lambda_n^{-2}=\int_0^L(p(L-0)-p(x))^2(m(x+0)+m(x-0))\,m(dx)\tag{33}$$

holds (compare (31) and the cited place in [11]).

---

[2] The integrals $\int_l^L \ldots\, \nu(dx)$ are defined to be equal $\int_{l-0}^L \ldots\, \nu(dx)$ ($\nu=m$ or m^2).

By virtue of (32) the integral in (33) converges. This can be seen as follows. By the first inequality of (30) it suffices to consider the integral

$$\int_0^L (p(L-0)-p(x))^2(m(x+0)-m(x-0))\,m(d\,x)$$
$$=\sum(p(L-0)-p(x))^2(\Delta\,m(x))^2 \tag{34}$$

where the sum runs over all x such that $\Delta\,m(x)=m(x+0)-m(x-0)>0$. But this sum is less or equal

$$\sum(p(L-0)-p(x))^2\,\Delta\,m(x)(m(x+0)+m(x-0))$$
$$=\sum(p(L-0)-p(x))^2(m^2(x+0)-m^2(x-0))\leqq\int_0^L(p(L-0)-p(x))^2\,m^2(d\,x).$$

Thus the integral in (34) and therefore also the integral in (33) are finite. This means that (vi) does not hold.

The proof of the theorem is finished. ┘

Remark. In the proof above we have shown that

$$\int_0^L (p(L-0)-p(s))^2\,m^2(d\,s)<\infty$$

is equivalent with the P_x-a.s. convergence of $(\tau_l-E_0\,\tau_l)$ as $l\uparrow L$ to a $\mathcal{T}$-measurable nontrivial random variable Z with finite variation $\lim\limits_{l\uparrow L} D_x^2\,\tau_l$. This result has been proved for diffusions also by Fristedt, Orey in [10] in another way. Moreover, these authors have shown in [10] that Z generates the σ-field $\mathcal{T}$ and that $t-E_0\,\tau_{X_t}$ converges to Z P_x-a.s. $(x\in E_m)$.

4. Minimality of Factorizing Parabolic Functions

As an application of the Theorem 1, in particular of the characterization (vi), we can decide, if the factorizing parabolic functions $(x,\mu)\to h_\mu(x,s)=\exp(-\mu s)\,\varphi(x,\mu)(\mu\geqq\lambda_0=\sup\sigma_2)$ are minimal parabolic. This is substantial for studying the Martin boundary of the space-time process corresponding to $\mathfrak{X}$. We formulate this application in the following

Theorem 2. *Let $\mathfrak{X}$ be a quasi-diffusion on $[0,L)$ with speed measure m and scale p. Assume 0 is reflecting and L is inaccessible $\left(\int_0^L m\,d\,p=\infty\right)$. Then the following properties hold:*

(i) *The parabolic function h_{λ_0} is minimal parabolic,*

(ii) *If L is natural and σ_2 contains pairwise different nonzero λ_n $(\lambda_n\geqq 1)$ with $\sum_{n=1}^{\infty}\lambda_n^{-2}=\infty$ then for every $\mu>\lambda_0$ the parabolic function h_μ is minimal parabolic,*

(iii) *If L is natural and σ_2 does not contain pairwise different nonzero λ_n $(n\geqq 1)$ with $\sum_{n=1}^{\infty}\lambda_n^{-2}=\infty$ then for no $\mu>\lambda_0$ the parabolic function h_μ is minimal parabolic,*

(iv) *If L is entrance then for no $\mu>\lambda_0$ the parabolic function h_μ is minimal parabolic.*

Remark. If L is accessible, then h_{λ_0} is also minimal parabolic but the functions $h_\mu\,(\mu>\lambda_0)$ are not. The minimal parabolic functions for this case (and for the case that L is an entrance boundary) have been described in [19, 20].

Example. If $\mathfrak{X}$ is a Brownian motion on $[0,\infty)$ with zero drift and which is reflected at zero, then $\lambda_0=0$ and $h_\mu(x,s)=\exp(-\mu s)\cosh(\sqrt{2\mu}\,x)$. The boundary $L=\infty$ is natural and by virtue of $p(x)=x$ we have $p(\infty-0)=\infty$. Thus the corollary implies that every $h_\mu\,(\mu\geqq 0)$ is minimal parabolic in this case. Woo has shown in [36] that for the Brownian motion on $(-\infty,\infty)$ with zero drift a parabolic function is minimal if and only if it is of the form

$$g_\mu(x,s)=\exp(-\mu s+\sqrt{2\mu}\,x) \qquad (\mu\in(-\infty,\infty))$$

(up to a multiplicative constant).

His proof makes use of analytical properties of the corresponding transition densities which are not available for general quasi-diffusions and thus it can not generalized to our case. Furthermore the Theorem 2 above shows that the results for the Brownian motion are not the model for all quasi-diffusions with natural boundary L. Indeed, if $\mathfrak{X}$ is the diffusion on $[0,\infty)$ given in the example after Theorem 1 then $L=\infty$ is a natural boundary and $\int_0^L (p(L-0)-p(x))^2\,m^2(dx)<\infty$.

Thus the corresponding factorizing parabolic functions $h_\mu\,(\mu>\lambda_0)$ are not minimal.

Proof of Theorem 2. We start with some preparation. Let be $\mu\geqq\lambda_0$. Then by $dm^{(\mu)}:=\varphi^2(.,\mu)\,dm$, $dp^{(\mu)}:=\varphi^{-2}(.,\mu)\,dp$ a new speed measure $m^{(\mu)}$ and a new scale $p^{(\mu)}$ on $[0,L)$ are defined (compare e.g. [9]). The corresponding quasi-diffusion is denoted by $\mathfrak{X}^{(\mu)}$. (All notions connected with $(m^{(\mu)}, p^{(\mu)})$ we mark with the superscript $^{(\mu)}$.)

Using the Lagrange identity for $D_m D_p$ (see [15]) we get

$$D_{m^{(\mu)}} D_{p^{(\mu)}}\, g=\frac{1}{\varphi(.,\mu)} D_m D_p(g\,\varphi(.,\mu))-\mu g$$

for all g such that the expressions are well defined. Therefore

$$P_x^{(\mu)}(X_t^{(\mu)}\in dy)=\frac{\exp(-\mu t)}{\varphi(x,\mu)} P_x(X_t\in dy)\,\varphi(y,\mu).$$

For details see [19, 20], the transformation $\mathfrak{X}\to\mathfrak{X}^{(\mu)}$ is a special case of transformation of Markov processes by multiplicative functionals.

It follows from this formula that a function h on $E_m\times[0,\infty)$ is parabolic with respect to $\mathfrak{X}^{(\mu)}$ if and only if $(x,s)\to\exp(-\mu s)\,\varphi(x,\mu)\,h(x,s)$ is parabolic with respect to $\mathfrak{X}$. Now the following Lemma is obvious.

Lemma 7. *Assume $\mu\geqq\lambda_0$. Then the function $(x,s)\to h_\mu(x,s)=\exp(-\mu s)\,\varphi(x,\mu)$ is minimal parabolic for $\mathfrak{X}$ if and only if the constant function $\mathbb{1}$ is minimal parabolic for $\mathfrak{X}^{(\mu)}$.*

Analogous statements for special Markov chains and for the Brownian motion has been used in [5, 23, 36] to study the minimality of factorizing parabolic functions for the corresponding processes.

Now the following lemma is the key for the successful application of Theorem 1 to the processes $\mathfrak{X}^{(\mu)}$.

Lemma 8. (i) *The spectrum* $\sigma_2^{(\mu)}$ *of* $\mathbb{D}_{m^{(\mu)}}\mathbb{D}_{p^{(\mu)}}$ *in* $\mathbb{L}_2(m^{(\mu)})$ *is the translation of* σ_2 *by* $-\mu$:

$$\sigma_2^{(\mu)}=\{\lambda-\mu \mid \lambda\in\sigma_2\},$$

(ii) *the boundary* L *is* (m,p)*-natural if and only if it is* $(m^{(\mu)},p^{(\mu)})$ *natural.*

The proof of part (i) easily follows from the definition of between $\mathbb{D}_{m^{(\mu)}}\mathbb{D}_{p^{(\mu)}}$ and $\mathbb{D}_m\mathbb{D}_p$. The proof of part (ii) is more complicate, see [19] or [20].

Now we are ready to prove the Theorem 2. Let $\mathfrak{X}$ be a quasi-diffusion on $[0,L)$ with speed measure m, scale p and $\int_0^L m\,dp=\infty$. Then $\mathfrak{X}^{(\lambda_0)}$ is a quasi-diffusion on $[0,L)$ and we have $\lambda_0^{(\lambda_0)}=\sup\sigma_2^{(\lambda_0)}=0$ by virtue of Lemma 8(i).

Thus $p^{(\lambda_0)}(L-0)=\infty$ (see Lemma 2) and this implies $\int_0^L m^{(\lambda_0)}\,dp^{(\lambda_0)}=\infty$, i.e., L is also $\mathfrak{X}^{(\lambda_0)}$-inaccessible.

Therefore Theorem 1 is applicable to the process $\mathfrak{X}^{(\lambda_0)}$. By virtue of $\lambda_0^{(\lambda_0)}=0$, $\mathfrak{X}^{(\lambda_0)}$ obviously fulfils (vi) of Theorem 1, and therefore it follows from this theorem and Lemma 7 that h_{λ_0} is minimal parabolic for $\mathfrak{X}$, i.e. (i) of Theorem 2 holds.

Let L be natural for $\mathfrak{X}$. By Lemma 8(ii) L is also natural for $\mathfrak{X}^{(\mu)}$. Thus Theorem 1 is applicable to $\mathfrak{X}^{(\mu)}$. Using Lemma 8(i) it is clear that the conditions in (ii) and (iii) of Theorem 2 are invariant under the transformation $(m,p)\to(m^{(\mu)},p^{(\mu)})$ for $\mu>\lambda_0$. Thus (ii) and (iii) of Theorem 2 immediately follow from the Theorem 1 and Lemma 7.

The part (iv) requires other methods and was proved in [18, 19]. $\lrcorner$

References

1. Blackwell, D., Freedman, D.: The tail σ-field of a Markov chain and a theorem of Orey. Ann. Math. Statist **38**, 1921–1925 (1964)
2. Blumenthal, R.M., Getoor, R.K.: Markov processes and potential theory. New York-San Francisco-London: Academic Press 1968
3. Cohn, H.: On the tail σ-field of the countable Markov chains. Rev. Roumaine Math. Pures Appl. **21**, 667–675 (1976)
4. Cohn, H.: On the invariant events of a Markov chain. Z. Wahrscheinlichkeitstheorie verw. Gebiete **248**, 81–96 (1979)
5. Doob, J.L., Snell, J.L., Williamson, R.E.: Application of boundary theory to sums of independent random variables. Contributions to Probab. and Statist. Stanford: Univ. Press 1960
6. Doetsch, G.: Einführung in Theorie und Anwendung der Laplace-Transformation. Basel: Birkhäuser 1958
7. Dym, H., McKean, H.P. jr.: Gaussian processes, function theory and the inverse spectral theorem. New York-San-Francisco-London: Academic Press 1976
8. Dynkin, E.B.: Markov processes. Berlin-Heidelberg-New York: Springer 1965
9. Feller, W.: Generalized second order differential operators and their lateral conditions. Ill. J. Math. **1**, 459–504 (1958)

10. Fristedt, B., Orey, S.: The tail σ-field of one-dimensional diffusions. Stochastic Analysis. New York-London: Academic Press 1978
11. Gochberg, I.G., Krein, M.G.: Vvedenije v teoriju lineinych njesamosoprjažennych operatorov v Gilbertovom prostranstve. Moskva: Nauka 1965 (in Russian)
12. Groh, J.: Über eine Klasse eindimensionaler Markovprozesse. Math. Nachr. **65**, 125–136 (1965)
13. Iosifescu, M.: On finite tail σ-algebras. Z. Wahrscheinlichkeitstheorie verw. Geb. **24**, 159–166 (1972)
14. Ito, K., McKean, H.P.: Diffusion processes and their sample paths. New York-Berlin: Academic Press/Springer 1965
15. Kac, I.S., Krein, M.G.: O spektralnych funkcii struny. (in Russian) App. to the russ. translation of Atkinson: Discrete and continuous boundary problems. Moskva: Mir 1968; see also Amer. Math. Soc. Trans. (2), **103**, 19–102 (1974)
16. Karlin, S., McGregor, J.L.: Coincidence properties of birth and death processes. Pacific J. Math. **9**, 1109–1140 (1959)
17. Karlin, S., McGregor, J.L.: The differential equations of birth and death processes and the Stieltjes moment problem. Trans. Amer. Math. Soc. **85**, 439–546 (1957)
18. Küchler, U.: On parabolic functions of one-dimensional quasi-diffusions. (to appear in Publ. of the RIMS, Kyoto (1980))
19. Küchler, U.: On asymptotic properties of the transition densities of quasi-diffusions. (to appear in RIMS, Kyoto (1980))
20. Küchler, U.: Über parabolische Funktionen für Quasidiffusionen und finale σ-Algebren für einige Klassen Markovscher Prozesse. Dresden: Technische Universität Dresden, Diss. B 1978
21. Küchler, U.: Über die σ-Algebra der asymptotischen Ereignisse bei diskreten Geburts- und Todesprozessen, Math. Nachr. **65**, 321–329 (1975)
22. Lai, T.L.: Space-time processes, parabolic functions and one-dimensional diffusions. Trans. Amer. Math. Soc. **175**, 409–430 (1974)
23. Lamperti, J., Snell, J.L.: Martin boundaries for certain Markov chains. J. Math. Soc. Japan **15**, 113–128 (1963)
24. Lévy, P.: Théorie de l'addition de variables aleatoires. Paris: Gauthier-Villars 1954
25. Lunze, U.: Eigenschaften der Übergangsdichte und der Trajektorien von Quasidiffusionsprozessen. Dresden: Technische Universität Dresden, Diss. A 1977
26. Mandl, P.: Analytical Treatment of one-dimensional Markov processes. Academic Prague and Springer Verlag Berlin 1968
27. Meyer, P.A.: Probability and Potentials. Waltham, Mass.-Toronto-London: Blaisdell 1966
28. Meyer, P.A.: Processus de Markov: la frontiere de Martin. Lecture notes in Math. **77**. Berlin-Heidelberg-New York: Springer 1968
29. Molčanov, A.S.: Granica Martina prjamogo proisvedenija markovskych processov. (in Russian) Sibirski Mat. Žurnal **XI**, 370–380 (1970)
30. Neveu, J.: Mathematical foundations of the probability theory. San Francisco: Holden Day 1966
31. Rösler, U.: Das $0-1$-Gesetz der terminalen σ-Algebren bei Harrisirrfahrten. Z. Wahrscheinlichkeitstheorie verw. Gebiete **37**, 227–242 (1977)
32. Rösler, U.: Eine Mischungseigenschaft von Zufallsvariablen. Z. Wahrscheinlichkeitstheorie verw. Gebiete **37**, 329–340 (1977)
33. Rösler, U.: Das $0-1$-Gesetz der terminalen σ-Algebra bei Fellerprozessen. Dissertation, Göttingen 1977
34. Schütze, D.: Über verallgemeinerte gewöhnliche Differentialoperatoren zweiter Ordnung mit nichtlokalen Randbedingungen und die von ihnen erzeugten Markovprozesse. Dresden: Technische Universität Dresden, Diss. A 1972
35. Watanabe, S.: On time-inversion of one-dimensional diffusion process. Z. Wahrscheinlichkeitstheorie verw. Gebiete **31**, 115–124 (1975)
36. Woo, K.Y.: Minimal invariant functions of space-time Wiener process. Trans. Amer. Math. Soc. **506**, 191–200 (1977)

Received November 13, 1978; in revised form October 19, 1979