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Central Limit Theorems for Local Martingales

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Introduction

This paper is involved with the following problem. Given a sequence of local martingales, say (M_n), under which conditions on the *quadratic variations* $([M_n])$, can we state the convergence in distribution of the (M_n) sequence towards a continuous gaussian martingale limit?

"Convergence in distribution" means here the weak convergence of the $(\mathscr{L}(M_n))$ sequence on the space D of right continuous and left hand limited functions, $\mathscr{L}(M_n)$ being the probability measure induced on D by M_n (i.e. the distribution or law of the M_n process).

In preceding works, the author has investigated an analogous problem for *locally square integrable* local martingales (in short, "locally square integrable martingales"). In such case we were interested in finding out conditions on the $(\langle M_n\rangle)$ sequence of *associated increasing processes* to insure the (M_n)'s convergence in distribution.

It is a well known fact (c.f. [9]) that for a local martingale M the associated increasing process $\langle M\rangle$ exists if and only if M is locally square integrable. On the contrary, $[M]$ always exists and, furthermore, $[M]$ is easier to calculate than $\langle M\rangle$ when both processes exist. Thus the problem with which we will deal below is a very natural one.

In the first paragraph, we will explain some notations. Paragraph two is devoted to state the main results of this paper. Proofs of these results are given in paragraph three. Paragraph four contains some particular cases of the main theorems. The last paragraph gives a complementary result for locally square integrable martingales. The Appendix contains the recall of a classical Tightness Criterion used in the paper.

I. Notations – Preliminaries

1. We will begin by defining some notations. We consider a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, and a filtering family $\mathbb{F}=(\mathfrak{F}_t, t\in\mathbb{R}_+)$, of sub σ-algebras satisfying the usual Dellacherie's conditions (viz. $\mathbb{F}$ is right continuous, and every σ-algebra is complete).

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Processes are considered to be applications X from $\Omega\times\mathbb{R}_+$ to $[-\infty\ \infty]$ such that for all $t\in\mathbb{R}_+$, the function $\omega\mapsto X(\omega,t)$ is measurable.[1]

The $\mathbb{F}$-predictable σ-algebra $\mathfrak{P}(\mathbb{F})$ on $\Omega\times\mathbb{R}_+$ is generated by $\mathbb{F}$-adapted processes with left continuous trajectoires. So an $\mathbb{F}$-predictable process is a $\mathfrak{P}(\mathbb{F})$-measurable function from $\Omega\times\mathbb{R}_+$ to $[-\infty,\infty]$ endowed with its borelian σ-algebra.

Martingales are considered as in [9], i.e. we always suppose that their trajectories are right continuous and possess left hand side limits at every point of $\mathbb{R}_+$. The following notations are standard:

$\mathcal{M}[\mathbb{F},\mathbb{P}]$: the space of uniform integrable $(\mathbb{F},P)$-martingales

$\mathcal{M}^2[\mathbb{F},\mathbb{P}]$: the space of $(\mathbb{F},\mathbb{P})$-martingales M such that $\sup_{t\in\mathbb{R}_+}\mathbb{E}(M^2(t))<\infty$. This space will be called *the space of square integrable martingales.*

$\mathcal{V}_+[\mathbb{F},\mathbb{P}]$: the set of increasing processes A, $\mathbb{F}$-adapted, with right continuous paths, such that $A(0)=0$ ($\mathbb{P}$-a.s.) and $\mathbb{E}(A(\infty))<\infty$.

This set is called the set of *integrable increasing processes.* $\mathcal{V}[\mathbb{F},\mathbb{P}]=\mathcal{V}_+[\mathbb{F},\mathbb{P}]-\mathcal{V}_+[\mathbb{F},\mathbb{P}]$, is the set of processes of *integrable variations.*

Let $\mathcal{X}[\mathbb{F},\mathbb{P}]$ be one of the set of processes introduced above. A process X is *locally* in $\mathcal{X}[\mathbb{F},\mathbb{P}]$ if there exists a sequence (T_n) of $\mathbb{F}$-stopping times such that $T_n\uparrow\infty$ $\mathbb{P}$-a.s. as $n\uparrow\infty$ and $X^{T_n}\in\mathcal{X}[\mathbb{F},\mathbb{P}]$ for all $n\in\mathbb{N}$; where the notation X^T stands for the process $X^T(t)=X(T\wedge t)$, T being a $\mathbb{F}$-stopping time, $t\in\mathbb{R}_+$. We write $\mathcal{X}^{\text{loc}}[\mathbb{F},\mathbb{P}]$ for the set of process X locally in $\mathcal{X}[\mathbb{F},\mathbb{P}]$; we will add a subscript 0 (i.e. $\mathcal{X}_0^{\text{loc}}[\mathbb{F},\mathbb{P}]$) for the subset of elements $X\in\mathcal{X}^{\text{loc}}[\mathbb{F},\mathbb{P}]$ such that $X(0)=0$.

For a process V of $(\mathbb{F},\mathbb{P})$-local integrable variation (i.e. $V\in\mathcal{V}^{\text{loc}}[\mathbb{F},\mathbb{P}]$) we denote by $\tilde{V}$ its *predictable compensator* (or dual predictable projection, c.f. [9]).

If $M\in\mathcal{M}^{\text{loc}}[\mathbb{F},\mathbb{P}]$, $[M]$ denotes the *quadratic variation process* (c.f. [9]) and for $M\in\mathcal{M}^{2,\text{loc}}[\mathbb{F},\mathbb{P}]$, $\langle M\rangle=\widetilde{[M]}$ denotes the associated (predictable) increasing process. If $M,N\in\mathcal{M}^{\text{loc}}[\mathbb{F},\mathbb{P}]$ we put $[M,N]=\frac{1}{4}([M+N]-[M-N])$ and if M, $N\in\mathcal{M}^{2,\text{loc}}[\mathbb{F},\mathbb{P}]$ we call $\langle M,N\rangle$ the process $\widetilde{[M,N]}$.

As usual, the notation $D=D(\mathbb{R}_+,\mathbb{R})$ stands for the Polish space of right continuous and left hand limited functions from $\mathbb{R}_+$ to $\mathbb{R}$ endowed with the customary Skorokhod's topology (see [1, 13, 18]). For every $x\in D$, we denote by $x(t-)$ the left hand limit at $t\in\mathbb{R}_+$ and by $\Delta x(t)=x(t)-x(t-)$ the jump at $t\in\mathbb{R}_+$.

We will also define a mapping * from D to D by

$$x^*(t)=\sup_{s\leqq t}|x(s)|.$$

The function x^* defined in this way is an increasing one so we can extend its domain to $[-\infty,\infty]$ by putting $x^*(\infty)=\lim_{t\uparrow\infty}x^*(t)=\sup_s|x(s)|$.

If X is a process with paths in D, we denote by $\mathcal{L}(X)$ its distribution on $(D,\mathfrak{B})$ where $\mathfrak{B}$ is the borelian σ-algebra of D. If (X_n) is a sequence of processes with trajectories in D, we say that this sequence is *tight* (*resp. C-tight*) if the

[1] We identify indistinguishable processes

associated sequence $(\mathscr{L}(X_n))$ is tight on D (resp. if $(\mathscr{L}(X_n))$ is tight and every limit point is concentrated on $C = C(\mathbb{R}_+, \mathbb{R})$, the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$). We say that (X_n) converges *in distribution or in law* to X as $n \uparrow \infty$ (we write $X_n \xrightarrow[n\uparrow\infty]{\mathscr{L}} X$) if the sequence $(\mathscr{L}(X_n))$ converges weakly towards $\mathscr{L}(X)$.

Weak convergence on D will be denoted by the symbol "$\xrightarrow{w}$". The symbol "$\xrightarrow{\mathbb{P}}$" means *convergence in probability*.

A dot between two process, e.g. $Y \cdot X$, means stochastic integration (of the first process with respect to the second one) when this operation has a meaning. If $X \in \mathscr{V}^{\mathrm{loc}}[\mathbb{F}, \mathbb{P}]$ and Y is a positive process, $Y \cdot X$ is just the Stieltjes integral

$$Y \cdot X(t) = \int_{]0,t]} Y(s)\, dX(s); \qquad (t \in \mathbb{R}_+).$$

In general, we refer the reader to Meyer's theory on Stochastic Integrals ([9]).

2. Now, let us consider $M \subset \mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}, \mathbb{P}]$ and $\varepsilon > 0$. We can construct the following increasing processes:

$$\alpha^\varepsilon[M](t) = \sum_{s \leq t} |\Delta M(s)|\, I_{[|\Delta M(s)| > \varepsilon]},$$

$$\sigma^\varepsilon[M](t) = \sum_{s \leq t} |\Delta M(s)|^2\, I_{[|\Delta M(s)| > \varepsilon]} \qquad (t \in R_+).$$

It was shown in [10] (see also [14]) that $\alpha^\varepsilon[M]$ is locally integrable. We also recall that the process $[M]^{1/2}$ is locally integrable (see [3]). The process $\sigma^\varepsilon[M]$ is locally integrable if $M \in \mathscr{M}_0^{2,\mathrm{loc}}[\mathbb{F}, \mathbb{P}]$, but it is not the case when M is just a local martingale. The local integrability of $\alpha^\varepsilon[M]$ implies that the process

$$A^\varepsilon[M](t) = \sum_{s \leq t} \Delta M(s)\, I_{[|M(s)| > \varepsilon]} \qquad (t \in \mathbb{R}_+)$$

has local integrable variations, so its predictable compensator $\tilde{A}^\varepsilon[M]$ exists. Now put $\overline{M}^\varepsilon = A^\varepsilon[M] - \tilde{A}^\varepsilon[M]$. The local martingale $\underline{M}^\varepsilon = M - \overline{M}^\varepsilon$ satisfies $(\Delta \underline{M}^\varepsilon)^*(\infty) \leq 2\varepsilon$ in the general case and $(\Delta \underline{M})^*(\infty) \leq \varepsilon$ when M is quasi-left-continuous (see [14, 16]). In the quasi-left-continuous case we also have $[\overline{M}^\varepsilon] = \sigma^\varepsilon[M]$ (see [14, 16]), and $\overline{M}^\varepsilon$ is orthogonal to $\underline{M}^\varepsilon$.

3. Let us consider a sequence of filtering families of sub σ-algebras of $\mathscr{F}$, say $(\mathbb{F}_n)$, $\mathbb{F}_n = (\mathscr{F}_{n,t};\, t \in \mathbb{R}_+)$, $(n \in \mathbb{N})$.

We suppose that each $\mathbb{F}_n$ satisfies the usual Dellacherie's conditions on $(\Omega, \mathscr{F}, \mathbb{P})$.

We shall deal with local martingales sequences (M_n) in the following context. Each M_n belongs to $\mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$, $(n \in \mathbb{N})$, so our processes are all defined on the *same* probability space, adapted to different filtrations $\mathbb{F}_n$. This is not a real restriction. In fact, if the M_n's are respectively defined on different filtered probability spaces $(\Omega_n, \mathscr{F}_n, \mathbb{F}_n, \mathbb{P}_n)$, one can consider the product space $\Omega = \prod_{\mathbb{N}} \Omega_n$ with the product σ-algebras $\mathscr{F} = \bigotimes_{\mathbb{N}} \mathscr{F}_n$ and $\mathscr{F}_t = \bigotimes_{\mathbb{N}} \mathscr{F}_{n,t}$ $(t \in \mathbb{R}_+)$, and the product probability measure $\mathbb{P} = \bigotimes_{\mathbb{N}} \mathbb{P}_n$. We define then a new sequence (X_n) by $X_n = M_n \circ \Pi_n$, where $\Pi_n\colon \Omega \to \Omega_n$ is the canonical projection $(n \in \mathbb{N})$. Clearly $\mathscr{L}(X_n) = \mathscr{L}(M_n)$ and it is easy to derive that $X_n \in \mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}, \mathbb{P}]$ if and only if

$M_n \in \mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}_n]$ and in this case, $[X_n]=[M_n]\circ \Pi_n$, $\mathscr{L}([X_n])=\mathscr{L}([M_n])$ (see Additional References [AR 2], § X-2 by example).

4. *Definition.* Let $(M_n)\in\prod_{\mathbb{N}} \mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$.[2]

1) We say that (M_n) satisfies the *Asymptotic Rarefaction of Jumps condition of the first type* (A.R.J. (1) condition) if

(1) $\widetilde{([\overline{M}_n^\varepsilon]^{1/2}+[\underline{M}_n^\varepsilon, \overline{M}_n^\varepsilon]^*)}(t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0, \quad$ for all $\varepsilon>0$, $t\in\mathbb{R}_+$.

We say that (M_n) satisfies the *strong A.R.J. (1)* condition if

(2) $\tilde{\alpha}^\varepsilon[M_n](t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0 \quad$ for all $\varepsilon>0$, $t\in\mathbb{R}_+$.

2°) The second type of ARJ conditions involves locally square integrable martingales. Let $(M_n)\in\prod_{\mathbb{N}} \mathscr{M}_0^{2,\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$.

We say that $(M_n; n\in\mathbb{N})$ satisfies the *Asymptotic Rarefaction of Jumps Condition of the second type* (ARJ(2) condition) if

(3) $(\langle\overline{M}_n^\varepsilon\rangle+\widetilde{[\underline{M}_n^\varepsilon, \overline{M}_n^\varepsilon]^*})(t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0, \quad$ for all $\varepsilon>0$, $t\in\mathbb{R}_+$.

We say that $(M_n; n\in\mathbb{N})$ satisfy the *strong ARJ(2) condition* if

(4) $\tilde{\sigma}^\varepsilon[M_n](t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0, \quad$ for all $\varepsilon>0$, $t\in\mathbb{R}_+$.

Finally, $(M_n, n\in\mathbb{N})$ satisfy the Lindeberg condition (L-condition) if for all $\varepsilon>0$, and all $t\in\mathbb{R}_+$

(5) $\mathbb{E}(\sigma^\varepsilon[M_n](t)) \xrightarrow[n\uparrow\infty]{} 0$.

We are now going to prove a version of the Central Limit Theorem for local martingales under the ARJ(1) condition. But let us give before the relations between the ARJ conditions.

5. **Proposition.** 1) *Let* $(M_n)\in\prod_{\mathbb{N}} \mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$. *Then the following implication holds: strong* ARJ(1)$\Rightarrow$ARJ(1).

2) *Let* $(M_n, n\in\mathbb{N})\in\prod_{\mathbb{N}} \mathscr{M}_0^{2,\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$. *Then the following implications hold:*
L-condition$\Rightarrow$*strong* ARJ(2)$\Rightarrow$*strong* ARJ(1) *and* ARJ(2); ARJ(2)$\Rightarrow$ARJ(1)
if each M_n *is quasi-left continuous, then*
L-condition$\Rightarrow$strong ARJ(2)$\Leftrightarrow$ARJ(2)$\Rightarrow$strong ARJ(1).

II. Main Results

I would now like to state the main results of this paper, proofs of which will be given in the next paragraph.

[2] i.e. for every $n\in\mathbb{N}$, $M_n\in\mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$

We fix a continuous increasing real function A, $A(0)=0$. We know (see [5, 11, 6]) that there exists a filtered probability space $(\Omega^A, \mathscr{F}^A, \mathbb{F}^A, \mathbb{P}^A)$ – a canonical one – and a continuous $(\mathbb{F}^A, \mathbb{P}^A)$-martingale M *(unique in law)* such that $\langle M\rangle = A$. These notations will hold throughout this paper. Therefore M is a gaussian centered continuous martingale of covariance function $K(s,t) = A(s\wedge t)$, $(s,t)\in\mathbb{R}_+\times\mathbb{R}_+$. (see [11, 6]).

1. **Proposition.** *Let* $(M_n)\in\prod_{\mathbb{N}}\mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}_n,\mathbb{P}]$ *be such that*

$$(\Delta M_n)^*(\infty)\leqq c_n, \quad \textit{where } c_n\downarrow 0 \quad \textit{as} \quad n\uparrow\infty.$$

Consider the two following relations.

(1) $\langle M_n\rangle(t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} A(t) \qquad (\forall t\in\mathbb{R}_+)$.

(2) $[M_n](t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} A(t) \qquad (\forall t\in\mathbb{R}_+)$.

If (1) (*respectively* (2)) *holds, then relation* (2) (*resp.* (1)) *is also valid and* $M_n \xrightarrow[n\uparrow\infty]{\mathscr{L}} M$.

2. **Theorem** (Central Limit Theorem for Local Martingales). *Let*

$$(M_n)\in\prod_{\mathbb{N}}\mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}_n,\mathbb{P}]$$

and let us suppose that

(1) (M_n) *satisfy the* $ARJ(1)$ *condition*

(2) $[M_n](t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} A(t) \qquad (\forall t\in\mathbb{R}_+)$

then

$$M_n \xrightarrow[n\uparrow\infty]{\mathscr{L}} M.$$

III. Proofs

We shall first discuss tightness. With this aim, we state here a useful criterion on tightness proved in [16] (see also [AR1]).

1. **Proposition.** *Let* (X_n) *be a sequence of processes with trajectories in D and such that, for all* $n\in\mathbb{N}$, X_n *is adapted to the filtering family* $\mathbb{F}_n$.

Suppose that for all $N\in\mathbb{N}$ *and all* $\varepsilon,\eta>0$,

(1) *there exists* $a>0$ *(depending on N and* η*) such that*

$$\sup_{n\in\mathbb{N}}\mathbb{P}(\sup_{t\in[0,N]}|X_n(t)|>a)\leqq\eta$$

(2) *there exists* $\delta>0$ *(depending on* ε,η*) and* $n_0\in\mathbb{N}$ *such that for every sequence* (T_n) *of random variables in which each* T_n *is an* $\mathbb{F}_n$*-stopping time bounded by* N, *we*

have

$$\sup_{n \geqq n_0} \mathbb{P}\Big(\sup_{\substack{T_n \leqq s \leqq T_n + \delta \\ s \in [0, N]}} |X_n(s) - X_n(T_n)| > \varepsilon \Big) \leqq \eta.$$

Then (X_n) *is tight.*

Proof. C.f. [16], Proposition II.1.3 and Remark II.1.4.1).

Now we shall introduce the concept of domination between processes adapted to a filtering family $\mathbb{F}$. This notion was introduced by Lenglart in [6, 7].

2. *Definition.* Let X, Y be two $\mathbb{F}$-adapted, positive processes with trajectories in D. Suppose also that Y is increasing. We say that X *is* $\mathbb{F}$*-dominated by* Y, we write $X \prec Y$, if for all finite stopping time T of $\mathbb{F}$ we have

$$\mathbb{E}(X(T)) \leqq \mathbb{E}(Y(T)).$$

3. **Lemma.** *Let* X, Y *be as in the preceding definition and* $X \prec Y$.

a) *If* Y *is predictable, then for all stopping time* T, *and for all* $\varepsilon, \eta > 0$

$$\mathbb{P}(X^*(T) > \varepsilon) \leqq \frac{1}{\varepsilon} \mathbb{E}(Y(T) \wedge \eta) \qquad + \mathbb{P}(Y(T) > \eta).$$

b) *If* Y *is not predictable, but if* $(\Delta Y)^*(\infty) < c$, *with* c *a positive constant, then*

$$\mathbb{P}(X^*(T) > \varepsilon) \leqq \frac{1}{\varepsilon} \mathbb{E}(Y(T) \wedge (\eta + c)) + \mathbb{P}(Y(T) > \eta)$$

for all finite stopping time T *and all* $\varepsilon, \eta > 0$.

Proof. Part a) was proved by Lenglart in [7].

Part b) is only a slight modification of the first result, (see [16] for more details). □

The following corollary is an elementary application of the preceding lemma.

4. **Corollary.** *Let* (X_n), (Y_n) *be two sequences of processes such that for each* $n \in \mathbb{N}$, X_n *and* Y_n *are positive*, $\mathbb{F}_n$*-adapted, right-continuous and increasing. Suppose that* $X_n \prec Y_n$ $(n \in \mathbb{N})$ *and that one of the following hypothesis is true*

(a) Y_n *is* $\mathbb{F}_n$*-predictable* $(n \in \mathbb{N})$, *or*

(b) $(\Delta Y_n)^*(\infty) \leqq c$ $(n \in \mathbb{N})$, *where* $c > 0$ *is a constant. Then, if* $Y_n(t) \xrightarrow[n \uparrow \infty]{\mathbb{P}} 0$ $(t \in \mathbb{R}_+)$, *we have* $X_n(t) \xrightarrow[n \uparrow \infty]{\mathbb{P}} 0$.

5. **Lemma.** *Let* $M \in \mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}, \mathbb{P}]$ *and* $\varepsilon > 0$. *Then there exists constants* k *and* K, *positive, such that*

$$k \, \widetilde{[\overline{M}^\varepsilon]^{1/2}} \prec \widetilde{(\overline{M}^\varepsilon)^*} \prec K \, \widetilde{[\overline{M}^\varepsilon]^{1/2}}$$

and

$$\widetilde{(\overline{M}^\varepsilon)^*} \prec 2 \, \tilde{\alpha}^\varepsilon [M]; \quad \widetilde{[\underline{M}^\varepsilon, \overline{M}^\varepsilon]^*} \prec 4 \, \varepsilon \, \tilde{\alpha}^\varepsilon [M].$$

Proof. It suffices to show that

$$(\widetilde{\overline{M}^{\varepsilon}})^* \prec 2\,\tilde{\alpha}^{\varepsilon}[M] \quad \text{and} \quad [\underline{M}^{\varepsilon}, \overline{M}^{\varepsilon}]^* \prec 4\,\varepsilon\,\tilde{\alpha}^{\varepsilon}[M].$$

The other relation is a consequence of the Burkholder-Davis-Gundy's inequality.

It is clear that

$$|A^{\varepsilon}[M]| \leqq \alpha^{\varepsilon}[M]$$

and also

$$|\tilde{A}^{\varepsilon}[M]| \leqq \tilde{\alpha}^{\varepsilon}[M]$$

thus

$$|A^{\varepsilon}[M] - \tilde{A}^{\varepsilon}[M]| \leqq \alpha^{\varepsilon}[M] + \tilde{\alpha}^{\varepsilon}[M]$$

but the right member of this inequality is an increasing process and we get $(\overline{M}^{\varepsilon})^* \leqq \alpha^{\varepsilon}[M] + \tilde{\alpha}^{\varepsilon}[M]$.

From this relation we obtain, from the definition of the predictable compensator,

$$(\overline{M}^{\varepsilon})^* \prec 2\,\tilde{\alpha}^{\varepsilon}[M]$$

and

$$(\widetilde{\overline{M}^{\varepsilon}})^* \prec 2\,\tilde{\alpha}^{\varepsilon}[M].$$

Finally,

$$[\underline{M}^{\varepsilon}, \overline{M}^{\varepsilon}](t) = \sum_{s \leqq t} \Delta \underline{M}^{\varepsilon}(s)\, \Delta \overline{M}^{\varepsilon}(s), \qquad (t \in \mathbb{R}_+)$$

and

$$|[\underline{M}^{\varepsilon}, \overline{M}^{\varepsilon}](t)| \leqq 2\,\varepsilon \sum_{s \leqq t} |\Delta \overline{M}^{\varepsilon}(s)|, \qquad (t \in \mathbb{R}_+).$$

If T is totally inaccessible stopping time,

$$|\Delta \overline{M}^{\varepsilon}(T)| = |\Delta M(T)|\, I_{[|\Delta M(T)| > \varepsilon]} = \Delta\,\alpha^{\varepsilon}[M](T).$$

If T is a predictable stopping time,

$$\begin{aligned} |\Delta \overline{M}^{\varepsilon}(T)| &\leqq |\Delta M(T)|\, I_{[|\Delta M(T)| > \varepsilon]} + \mathbb{E}^{\mathscr{F}_{T^-}} |\Delta M(T)|\, I_{[|\Delta M(T)| > \varepsilon]} \\ &\leqq \Delta\,\alpha^{\varepsilon}[M](T) + \Delta\,\tilde{\alpha}^{\varepsilon}[M](T) \end{aligned}$$

then

$$\sum_{s \leqq t} |\Delta \overline{M}^{\varepsilon}(s)| \leqq \alpha^{\varepsilon}[M](t) + \tilde{\alpha}^{\varepsilon}[M](t) \qquad (t \in \mathbb{R}_+)$$

and we easily obtain the two following relations

$$[\underline{M}^{\varepsilon}, \overline{M}^{\varepsilon}]^* \prec 4\varepsilon\tilde{\alpha}^{\varepsilon}[M],$$
$$\widetilde{[\underline{M}^{\varepsilon}, \overline{M}^{\varepsilon}]}^* \prec 4\varepsilon\tilde{\alpha}^{\varepsilon}[M].$$

6. **Lemma.** *If* $M \in \mathscr{M}^{2,\mathrm{loc}}[\mathbb{F}, \mathbb{P}]$ *and* $\varepsilon > 0$ *is given, then*

$$\tilde{\alpha}^{\varepsilon}[M] \prec \varepsilon^{-1}\tilde{\sigma}^{\varepsilon}[M].$$

Proof. This lemma is evident from the inequality

$$\alpha^{\varepsilon}[M] \leqq \varepsilon^{-1}\sigma^{\varepsilon}[M].$$

7. **Lemma.** *Let* $M \in \mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}, \mathbb{P}]$ *with* $(\Delta M)^*(\infty) \leqq c$, *where c is a positive constant, (this implies that M is locally square integrable). Then*

(1) $([M] - \langle M\rangle)^{*2} \prec 16c^2[M]$,
(2) $([M] - \langle M\rangle)^{*2} \prec 16c^2\langle M\rangle$.

Proof. Put $L = [M] - \langle M\rangle$. L is a local martingale locally bounded because

$$(\Delta L)^*(\infty) \leqq 2c^2.$$

Moreover

$$\Delta L = \Delta[M] - \Delta\langle M\rangle \leqq \Delta[M] + \Delta\langle M\rangle$$

and we get

$$\Delta L^2 \leqq 2c^2(\Delta[M] + \Delta\langle M\rangle)$$

But L is a compensated sum of jumps, so $[L] = \sum_{s<.} \Delta L^2(s)$ and then

$$[L] \leqq 2c^2([M] + \langle M\rangle).$$

Now by Dobb's inequality

$$L^{*2} \prec 4[L]$$

then

$$L^{*2} \prec 16c^2[M]$$

and furthermore

$$L^{*2} \prec 16c^2\langle M\rangle. \quad \square$$

8. *Proof of Proposition I.5.* 1) Strong ARJ(1)⇒ARJ(1) is merely a consequence of Lemma 5 and Corollary 4.

2) L-condition⇒strong ARJ(2) is merely a consequence of the definition of predictable compensators and the fact that L^1-convergence implies convergence in probability.

Strong ARJ(2)$\Rightarrow$strong ARJ(1) is a consequence of Lemma 6 and Corollary 4. Strong ARJ(2)$\Rightarrow$ARJ(2) follows from Corollary 4 and the relation $\langle \overline{M}_n^\varepsilon \rangle \prec 3\tilde{\sigma}^\varepsilon[M_n]$, ($n\in\mathbb{N}$, $\varepsilon>0$) proved in [16] (also see [14]). ARJ(2)$\Rightarrow$ARJ(1) follows from the inequality $[\overline{M}^\varepsilon] \leqq \langle \overline{M}^\varepsilon \rangle^{1/2} \leqq \langle \overline{M}^\varepsilon \rangle^{1/2}$ [3]

$$\langle \overline{M}_n^\varepsilon \rangle = \tilde{\sigma}^\varepsilon[M_n] \quad (n\in\mathbb{N}, \varepsilon>0), \ [\underline{M}_n^\varepsilon, \overline{M}_n^\varepsilon]=0,$$

so ARJ(2) and strong ARJ(2) conditions are equivalent. □

9. *Definition.* Let $(X_n, n\in\mathbb{N})$, $(Y_n; n\in\mathbb{N})$ be two sequences of processes with trajectories in D. We say that they are *contiguous* if for all $t\in\mathbb{R}_+$, $(X_n - Y_n)^*(t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0$

10. **Lemma** (Approximation procedure). *Let* $(M_n; n\in\mathbb{N}) \in \prod_{\mathbb{N}} \mathcal{M}_0^{\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$ *and suppose that this sequence satisfies the* ARJ(1) *condition.*
Then for every real sequence $(c_k; k\in\mathbb{N})$ *such that* $c_k \downarrow 0$ *as* $k\uparrow\infty$ *there is a sequence* $(n_k; k\in\mathbb{N})$ *of positive integers and a sequence* $(N_k) \in \prod_{k\in\mathbb{N}} \mathcal{M}_0^{2,\mathrm{loc}}[\mathbb{F}_{n_k}, \mathbb{P}]$ *such that*

(1) $(\Delta N_k)^*(\infty) \leqq c_k$,

(2) $(M_{n_k}; k\in\mathbb{N})$ *and* $(N_k, k\in\mathbb{N})$ *are contiguous;*

(3) $([M_{n_k}], k\in\mathbb{N})$ *and* $([N_k]; k\in\mathbb{N})$ *are contiguous.*

Proof. Let $c_k\downarrow 0$ and put $\varepsilon_k = c_k/2$.
By Lemma 5 and Corollary 4, the ARJ(1) condition implies both

(4) $(\overline{M}_n^{\varepsilon_k})^*(t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0$,

(5) $[\overline{M}_n^{\varepsilon_k}](t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0$ and $[\underline{M}_n^{\varepsilon_k}, \overline{M}_n^{\varepsilon_k}]^*(t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0$ for all $t\in\mathbb{R}_+$, all $k\in\mathbb{N}$.

Put $\rho_c(x, y) = \sum_{N\geqq 1} \frac{1}{2^N} \frac{(x-y)^*(N)}{1+(x-y)^*(N)}$, for $x, y\in D$ and denote $e(k, n)$ the sum

$$e(k, n) = \mathbb{E}(\rho_c([M_n], [\underline{M}_n^{\varepsilon k}])) + \mathbb{E}(\rho_c(M_n, \underline{M}_n^{\varepsilon k}))$$

(4) and (5) imply $e(k, n) \xrightarrow[n\uparrow\infty]{} 0$ for all $k\in\mathbb{N}$.
Choose n_k such that

$$e(k, n_k) \leqq \frac{1}{2^k} \quad (k\in\mathbb{N}).$$

Put $N_k = \underline{M}_{n_k}^{\varepsilon_k}$

$(n_k; k\in\mathbb{N})$ and $(N_k; k\in\mathbb{N})$ satisfy the required conditions. □

11. **Lemma** (Tightness criterion for bounded jumps local martingales). *Let* $(M_n) \in \prod_{\mathbb{N}} \mathcal{M}_0^{\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$ *such that* $(\Delta M_n)^*(\infty) \leqq c_n$ *for all* $n\in\mathbb{N}$.
Suppose that

(1) $c_n \downarrow 0$ *as* $n\uparrow\infty$,

(2) *the sequence* $([M_n]; n\in\mathbb{N})$ *is C-tight. Therefore* (M_n) *is C-tight.*

---

[3] If B is an adapted increasing process locally integrable and ϕ a real concave function, positive and increasing we have $\widetilde{\phi(B(t))} \leqq \phi(\tilde{B}(t))$ whenever $\phi\circ B$ is locally integrable

Proof. We will prove that $(M_n; n\in\mathbb{N})$ is D-tight. C-tightness follows then from the relation

$$W_C^N(M_n, \delta) \leqq 2\, W_D^N(M_n, \delta) + c_n;\ (N, n)\in\mathbb{N}^2$$ [4]

(cf, [13]) and the fact that $c_n \downarrow 0$.

Now consider a local martingale $M\in\mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}, \mathbb{P}]$ such that $(\Delta M)^*(\infty)\leqq c$ with c a positive constant. Then $(\Delta[M])^*(\infty)\leqq c^2$ and from the relation $M^2 \prec [M]$.

We get by Lemma 3

$$(3)\quad \mathbb{P}(M^*(T)>\varepsilon \leqq \frac{1}{\varepsilon^2}\mathbb{E}([M](T)\wedge(\eta+c^2))+\mathbb{P}([M](T)>\eta)$$

for all finite stopping time T and $\varepsilon, \eta>0$.

Now, given a finite stopping time T of $\mathbb{F}$, put $\mathscr{G}_t=\mathscr{F}_{T+t}$, $L(t)=M(T+t)-M(T)$ $(t\in\mathbb{R}_+)$. $L\in\mathscr{M}_0^{2,\mathrm{loc}}[\mathbb{G}, \mathbb{P}]$, and from (3) we obtain

$$(4)\quad \mathbb{P}(\sup_{T\leqq s\leqq T+\delta}|M(s)-M(T)|>\varepsilon)\leqq\frac{\eta+c^2}{\varepsilon^2}+\mathbb{P}([M](T+\delta)-[M](T)>\eta)$$

for all ε, δ, $\eta>0$.

Condition (1) of Proposition 1 can be easily obtained using (3) as well as the C-tightness of $([M_n]; n\in\mathbb{N})$.

Finally let $N\in\mathbb{N}$ and consider a sequence $(T_n; n\in\mathbb{N})$ of random variables as in Proposition 1. Hence T_n is an $\mathbb{F}_n$-stopping time bounded by N $(n\in\mathbb{N})$.

Therefore we find

$\mathbb{P}([M_n](T_n+\delta)-[M_n](T_n)>\eta)\leqq\mathbb{P}(W_C^{2N}([M_n], \delta)>\eta)$ and from (4)

$$(5)\quad \mathbb{P}(\sup_{\substack{T_n\leqq s\leqq T_n+\delta\\ s\in[0,N]}}|M_n(s)-M_n(T_n)|>\varepsilon)\leqq\frac{n+c^2}{\varepsilon^2}+\mathbb{P}(W_C^{2N}([M_n], \delta)>\eta)$$

for all ε, δ, $\eta>0$.

Condition (2) of Proposition 1 is obtained using (5) as well as the C-tightness of $([M_n], n\in\mathbb{N})$ and this completes the proof. □

12. **Proposition.** *Let* $(M_n)\in\prod_{\mathbb{N}}\mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$ *and suppose that*

(1) (M_n) *satisfies the* ARJ(1) *condition,*

(2) $([M_n])$ *is C-tight.*

Then (M_n) *is C-tight.*

Proof. Let $(M_{n_r}; r\in\mathbb{N})$ be an arbitrary subsequence. Call $L_r=M_{n_r}$, $(r\in\mathbb{N})$ and consider a real sequence $(c_k; k\in\mathbb{N})$ such that $c_k\downarrow 0$ as $k\uparrow\infty$. Apply lemma 10 to (L_r) and (c_k) and denote as $(r_k; k\in\mathbb{N})$ and $(N_k; k\in\mathbb{N})$ respectively the integer and local martingale sequences constructed using this lemma. Hence $([N_k])$ is

[4] see the Appendix

contiguous with $([L_{r_k}])$ and the C-tightness of the last sequence implies C-tightness of the first (c.f. [1]). Then by the preceding lemma (N_k) is C-tight since $(\Delta N_k)^*(\infty) \leqq c_k$ and $c_k \downarrow 0$. But we also have that (N_k) and (L_{r_k}) are contiguous, making (L_{r_k}) C-tight and from this we obtain C-tightness of the entire (M_n) sequence. □

13. *Remark.* If $M \in \mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}, \mathbb{P}]$ such that $(\Delta M)^*(\infty) \leqq c$, with c constant >0, is a well known fact that the process defined by $E^\lambda[M](t) = \exp(\lambda M(t) - \phi_c(\lambda)\langle M\rangle(t))$ $(t \in \mathbb{R}_+)$ is a positive supermartingale for all $\lambda \geqq 0$, where

$$\phi_c(\lambda) = (\exp(\lambda c) - 1 - \lambda c)\, c^{-2} \text{ (c.f. [12, 8]).}$$

Now consider $(M_n) \in \prod_N \mathscr{M}_0^{\mathrm{loc}}[\mathbb{F}_n, \mathbb{P}]$ in which $(\Delta M_n)^*(\infty) \leqq c$ for all $n \in \mathbb{N}$. Suppose that for all $n \in \mathbb{N}$, $t \in \mathbb{R}_+$:

(1) $\langle M_n\rangle(t) \leqq a(t)$; where a is an increasing and positive real function.

Since $E^\lambda[M_n]$ is a supermartingale and $E^\lambda[M_n]\,(0) = 1$ for all $n \in \mathbb{N}$, we obtain

$$\mathbb{E}(E^\lambda[M_n](t)) \leqq 1 \qquad (\lambda, t \in \mathbb{R}_+, n \in \mathbb{N})$$

but

$$\mathbb{E}(\exp(\lambda M_n(t) - \phi_c(\lambda)\, a(t))) \leqq \mathbb{E}(E^\lambda[M_n](t))$$

and

(2) $\mathbb{E}(\exp(\lambda M_n(t)) \leqq \exp \phi_c(\lambda)\, a(t), \qquad (\lambda, t \in \mathbb{R}_+, n \in \mathbb{N}).$

From (1) and (2) is clear that for all $t \in \mathbb{R}_+$, the sequences $(M_n(t);\, n \in \mathbb{N})$ and $(M_n^2(t) - \langle M_n\rangle(t);\, n \in \mathbb{N})$ are uniformly integrable. We shall use this fact in the proof of Proposition II.1.

14. *Proof of Proposition II.1.* Suppose (1) to hold. Then for all $t \in \mathbb{R}_+$, $c_n^2 \langle M_n\rangle(t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0$. Hence by Lemma 7 and Corollary 4,

$$([M_n] - \langle M_n\rangle)^*(t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0 \qquad (t \in \mathbb{R}_+)$$

i.e. $([M_n])$ and $(\langle M_n\rangle)$ are contiguous sequences and therefore

$$[M_n](t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} A(t) \qquad (t \in \mathbb{R}_+).$$

If (2) holds, we can apply Lemma 7 and Corollary 4 once again since $(\Delta[M_n])^*(\infty) \leqq c_n^2$. In this way $([M_n])$ and $(\langle M_n\rangle)$ are also contiguous and relation (1) follows from (2).

Let us prove the second part of the proposition. Define, for all $n \in \mathbb{N}$

$$R_n = \inf\{t > 0 : \langle M_n\rangle(t) > A(t) + 1\} \qquad (\inf \phi = +\infty).$$

This is a predictable stopping time of $\mathbb{F}_n$ and (1) implies that

$$\mathbb{P}(R_n < t) \xrightarrow[n\uparrow\infty]{} 0 \quad \text{for all } t \in \mathbb{R}_+.$$

Choose a sequence $(R_{nk};\, k\in\mathbb{N})$ announcing R_n, i.e.

$$R_{nk}\uparrow R_n \quad \text{as} \quad k\uparrow\infty \quad \text{and} \quad R_{nk}<R_n \quad \text{on} \quad \{0<R_n<\infty\}.$$

Construct a sequence $(k_n,\, n\in\mathbb{N})$ of positive integers such that

$$\mathbb{P}\left(R_n - R_{nk_n} > \frac{1}{2^n}\right) < \frac{1}{3^n}$$

hence

(3) $\mathbb{P}(R_{nk_n}<t) \xrightarrow[n\uparrow\infty]{} 0 \qquad \text{for all } t\in\mathbb{R}_+$

and

(4) $\langle M_n\rangle(\omega, t) \leqq A(t)+1 \qquad \forall(\omega, t)\in [\![0, R_{nk_n}]\!]$.

If we put down

(5) $N_n = M_n^{R_{nk_n}}$.

We find

$$\mathbb{P}((N_n - M_n)^*(t) > \varepsilon) \leqq \mathbb{P}(R_{nk_n}<t),$$
$$\mathbb{P}((\langle N_n\rangle - \langle M_n\rangle)^*(t) > \varepsilon) \leqq \mathbb{P}(R_{nk_n}<t).$$

Thus (N_n) and (M_n) (respectively $(\langle N_n\rangle)$ and $(\langle M_n\rangle)$) are contiguous. Furthermore, $(\Delta N_n)^*(\infty) \leqq c_n \leqq c_0$ and by (4), $\langle N_n\rangle(t) \leqq A(t)+1$ for all $t\in\mathbb{R}_+$, $n\in\mathbb{N}$. Then by applying Remark 13 we obtain that

(6) $(N_n(t), n\in\mathbb{N})$ *and* $(N_n^2(t)-\langle N_n\rangle(t);\, n\in\mathbb{N})$ *are uniformly integrable sequences of random variables, for all* $t\in\mathbb{R}_+$.

Now, condition (2) together with the jumps condition imply that (M_n) is C-tight (Lemma 11). We shall prove that the set of limit points of $(\mathscr{L}(M_n))$ is reduced to a single point; $\{\mathscr{L}(M)\}$, so that $M_n \xrightarrow[n\uparrow\infty]{\mathscr{L}} M$ (M being the gaussian continuous martingale introduced in II).

We suppose that (M_{n_k}) is an arbitrary subsequence converging in distribution. Call P the weak limit of $(\mathscr{L}(M_{n_k}))$.

Consider the canonical projections over D:

$$X(w, t) = w(t),\ w\in D,\ t\in\mathbb{R}_+; \qquad X = (X(t);\, t\in\mathbb{R}_+).$$

Define $\mathfrak{B}_t$ as being $\bigcap_{s>t} \sigma(X(u);\, u\leqq s)$, $(t\in\mathbb{R}_+)$. Put

$$\mathfrak{B} = \mathfrak{B}_\infty = \bigvee_{t\in\mathbb{R}_+} \mathfrak{B}_t, \qquad \mathbb{B} = (\mathfrak{B}_t;\, t\in\mathbb{R}_+).$$

The distribution $\mathscr{L}(M)$ is characterized in the following way. A probability measure Q on $(D, \mathfrak{B})$ is equal to $\mathscr{L}(M)$ if and only if

(7) $Q(C)=1$,

(8) X is a $(\mathbb{B}, Q)$-martingale,

(9) $X^2 - A$ is a $(\mathbb{B}, Q)$-martingale.

Remark that (9) is equivalent to:
(9') X is a $(\mathbb{B}, F(Q))$-martingale, where $F(Q)$ is the image measure of Q by the mapping $F: D \to D$ defined by $w \mapsto w^2 - A$, $(w \in D)$. (This mapping is continuous when restricted to C.)

Then let us prove that P satisfies (7), (8), (9'). By a contiguity argument,

(10) $\mathscr{L}(N_{n_k}) \xrightarrow[k\uparrow\infty]{w} P$ (N_{n_k} constructed by (5)).

For the same reason, $\langle N_{nk}\rangle(t) \xrightarrow[k\uparrow\infty]{\mathbb{P}} A(t)$ $(t \in \mathbb{R}_+)$, and since A is continuous and increasing, and $\langle N_{n_k}\rangle$ is increasing, this implies that

(11) $(\langle N_{n_k}\rangle - A)^*(t) \xrightarrow[k\uparrow\infty]{\mathbb{P}} 0$.

Thus the sequences $(N_{nk}^2 - A;\ k \in \mathbb{N})$ and $(N_{nk}^2 - \langle N_{n_k}\rangle;\ k \in \mathbb{N})$ are contiguous. The probability P satisfy (7) because (M_n) is C-tight. Hence the mapping F is P-continuous, i.e. continuous over a set of probability one. Then by the Continuous Mapping Principle (see [1]),

$$\mathscr{L}(F(N_{n_k})) \xrightarrow[k\uparrow\infty]{w} F(P)$$

and also, by a contiguity argument,

(12) $\mathscr{L}(N_{n_k}^2 - \langle N_{n_k}\rangle) \xrightarrow[k\uparrow\infty]{w} F(P)$.

Finally, (6), (10) and (12) imply that P satisfies (8) and (9'). This is a classical argument: for each $s, t \in \mathbb{R}_+$ and for each bounded, continuous, $\mathfrak{B}_s$-measurable function $h: D \to \mathbb{R}$,

$$\int_D hX(t)\, d(\mathscr{L}(N_{n_k})) \xrightarrow[k\uparrow\infty]{} \int_D hX(t)\, dP$$

$$\int_D hX(t)\, d(\mathscr{L}(N_{n_k}^2 - \langle N_{n_k}\rangle)) \xrightarrow[k\uparrow\infty]{} \int_D hX(t)\, dF(P) \qquad \text{(see [16] for example).}$$

The proof is now complete. □

15. *Proof of Theorem II.2.* By Proposition 12, (M_n) is C-tight.

Let (M_{nk}) be an arbitrary subsequence converging in distribution.

Let (c_l) be a real sequence such that $c_l \downarrow 0$ as $l \uparrow \infty$. Apply Lemma 10 to construct a sequence $(k_l;\ l \in \mathbb{N})$ of integers and $(N_l;\ l \in \mathbb{N}) \in \prod_{l \in \mathbb{N}} \mathscr{M}_0^{2,\mathrm{loc}}[\mathbb{F}_{n_{kl}}, \mathbb{P}]$ so that $(\Delta N_l)^*(\infty) \leq c_l$; $(M_{n_{kl}};\ l \in \mathbb{N})$ and $(N_l, l \in \mathbb{N})$ (respectively $(M_{n_{kl}};\ l \in \mathbb{N})$ and $(N_l;\ l \in \mathbb{N})$) are contiguous.

In this way, $[N_l](t) \xrightarrow[l\uparrow\infty]{\mathbb{P}} A(t)$ $(t \in \mathbb{R}_+)$ by contiguity. Applying Proposition II.1 to the sequence (N_l) we obtain

$$N_l \xrightarrow[l\uparrow\infty]{\mathscr{L}} M.$$

By contiguity,

$$M_{n_{kl}} \xrightarrow[l\uparrow\infty]{\mathscr{L}} M.$$

That is, from *every* convergent subsequence we can get a further subsequence converging in distribution towards the *same* limit M. That actually means that

$$M_n \xrightarrow[n\uparrow\infty]{\mathscr{L}} M. \quad \square$$

IV. Two Particular Cases

1. Let us consider point process sequences $(T_m^n; m\in\mathbb{N},\ n\in\mathbb{N})$ according to Bremaud and Jacod [2] with their respective counting processes $(N_n;\ n\in\mathbb{N})$ defined by

$$N_n(t)=\begin{cases} m \text{ si } T_m^n \leqq t < T_{m+1}^n \\ \infty \text{ si } T_\infty^n \leqq t, \end{cases} \quad \text{where } T_\infty^n=\lim_m T_m^n,\ (n\in\mathbb{N},\ t\in\mathbb{R}_+).$$

We will suppose that each N_n is *non explosive* (i.e. $T_\infty^n=\infty$) and adapted to $\mathbb{F}_n$. Call $\overset{c}{N}_n$ the compensated counting process, i.e.

$$\overset{c}{N}_n = N_n - \tilde{N}_n \qquad (n\in\mathbb{N}).$$

$(\Delta\overset{c}{N}_n)^*(\infty)\leqq 2 (n\in\mathbb{N})$, so $\overset{c}{N}_n$ is in fact a locally square integrable martingale. Now we shall deal with stochastic integrals with respect to $\overset{c}{N}_n$ that are local martingales not locally square integrable. Let us consider a sequence of processes $(Y_n;\ n\in\mathbb{N})$ in which each Y_n is $\mathbb{F}_n$-predictable and

$$(1) \qquad \int_{]0,t]} |Y_n(s)|\, d\tilde{N}_n(s) < \infty \qquad (t\in\mathbb{R}_+).$$

In this case, the stochastic integral $Y_n\cdot\overset{c}{N}_n$ is an element of $\mathscr{M}_o^{\text{loc}}[\mathbb{F}_n, \mathbb{P}]$ (c.f. [2]), for all $n\in\mathbb{N}$. Now suppose that N_n processes are quasi-left-continuous, i.e. $\tilde{N}_n$ processes are continuous or, equivalently, the stopping times T_m^n are totally inaccessible. In this case,

$$[Y_n\cdot N_n] = Y_n^2\cdot N_n = \sum_{T_m^n\leqq\,\cdot} Y_n^2(T_m^n)$$

(because $\Delta N_n(t)=0$ or 1), and for all $\varepsilon>0$, $n\in\mathbb{N}$

$$\begin{aligned} \alpha^\varepsilon[Y_n\cdot\overset{c}{N}_n] &= \sum_{T_m^n\leqq\,\cdot} |Y_n(T_m^n)|\, I_{[|Y_n(T_m^n)|>\varepsilon]} \\ &= |Y_n|\, I_{[|Y_n(\cdot)|>\varepsilon]}\cdot N_n \end{aligned}$$

(recall that $\Delta(Y_n\cdot\overset{c}{N}_n)=Y_n\Delta N_n$).

Therefore

$$\tilde{\alpha}^{\varepsilon}[Y_n \cdot \overset{c}{N_n}] = |Y_n|\, I_{[|Y_n(.)| > \varepsilon]} \cdot \tilde{N}_n.$$

2. **Corollary.** *Using the notations of Theorem II.2 and the preceding ones, suppose that*

(1) $$\int_{]0,t]} |Y_n(s)|\, I_{[|Y_n(s)| > \varepsilon]}\, d\tilde{N}_n(s) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0$$

for all $t \in \mathbb{R}_+$, $\varepsilon > 0$;

(2) $$\sum_{T_m^n \leqq t} Y_n^2(T_m^n) \xrightarrow[n\uparrow\infty]{\mathbb{P}} A(t), \qquad \text{for all } t \in \mathbb{R}_+$$

Then

$$Y_n \cdot \overset{c}{N_n} \xrightarrow[n\uparrow\infty]{\mathscr{L}} M$$

Remark that a sufficient condition to obtain (1) is:

(1′) $$\mathbb{E}\Big(\sum_{T_m^n \leqq t} |Y_n(T_m^n)|\, I_{[|Y_n(T_m^n)| > \varepsilon]}\Big) \xrightarrow[n\uparrow\infty]{} 0$$

for all $t \in \mathbb{R}_+$, and all $\varepsilon > 0$.

3. We will consider a sequence of discrete parameter filtering families of σ-algebras:

$$\mathbb{G}_n = (\mathscr{G}_{n,m};\, m \in \mathbb{N}), \qquad (n \in \mathbb{N}).$$

Let $(\xi_{n,m};\, n \in \mathbb{N},\, m \in \mathbb{N})$ be a double sequence of random variables such that for all $n \in \mathbb{N}$, $S_n \in \mathscr{M}[\mathbb{G}_n, \mathbb{P}]$ where

$$S_n(m) = \sum_{k=0}^{m} \xi_{n,k}, \qquad \xi_{n,0} = 0, \qquad (n,m) \in \mathbb{N}^2.$$

Put $\mathscr{F}_{n,t} = \mathscr{G}_{n,[nt]}$[5]; $\mathbb{F}_n = (\mathscr{F}_{n,t};\, t \in \mathbb{R}_+)$

$$M_n(t) = S_n([nt]), \qquad (n \in \mathbb{N}, t \in \mathbb{R}_+)$$

$M_n \in \mathscr{M}[\mathbb{F}_n, \mathbb{P}]$ and

$$\alpha^{\varepsilon}[M_n](t) = \sum_{k=0}^{[nt]} |\xi_{n,k}|\, I_{[|\xi_{n,k}| > \varepsilon]},$$

$$[M_n](t) = \sum_{k=0}^{[nt]} \xi_{n,k}^2, \qquad (n \in \mathbb{N}, t \in \mathbb{R}_+, \varepsilon > 0).$$

---

[5] here $[u]$ denotes the integer part of $u \in \mathbb{R}_+$

An elementary calculation yields the following result

$$\tilde{\alpha}^{\varepsilon}[M_n](t) = \sum_{k=1}^{[nt]} \mathbb{E}^{\mathscr{G}_{n,k-1}}(|\xi_{n,k}| I_{[|\xi_{n,k}|>\varepsilon]})$$

$\left(\text{by convention } \sum_{k=1}^{0} (\dots) = 0, \ (n \in \mathbb{N}, \ t \in \mathbb{R}_+, \ \varepsilon > 0)\right).$

4. **Corollary.** *In view of the above assumption, if moreover*

(1) $\sum_{k=1}^{[nt]} \mathbb{E}^{\mathscr{G}_{n,k-1}}(|\xi_{n,k}| I_{[|\xi_{n,k}|>\varepsilon]}) \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0,$

(2) $\sum_{k=0}^{[nt]} \xi_{n,k}^2 \xrightarrow[n\uparrow\infty]{\mathbb{P}} A(t);$ *for all* $\varepsilon > 0, \ t \in \mathbb{R}_+$

then

$$M_n \xrightarrow[n\uparrow\infty]{\mathscr{L}} M.$$

We can also change time in a more general way. The methods developped in [16] may yield more general results in this context.

V. Final Remarks

In [16] (c.f. also [14]) we have proved an approximation procedure analogous to III.10 for $(M_n; n \in \mathbb{N}) \in \prod_{\mathbb{N}} \mathscr{M}_0^{2,\text{loc}}[\mathbb{F}_n, \mathbb{P}]$ satisfying the ARJ (2) condition. This approximation procedure enables us to obtain the following complementary results to the Central Limit Theorem for Locally Square Integrable Martingales.

1. **Theorem.** *Let* $(M_n; (n \in \mathbb{N}) \in \prod_{\mathbb{N}} \mathscr{M}_0^{2,\text{loc}}[\mathbb{F}_n, \mathbb{P}]$ *and let us suppose that it satisfies the* ARJ (2) *condition. Consider the two following relations.*

(1) $\langle M_n \rangle(t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} A(t) \quad (\forall t \in \mathbb{R}_+),$

(2) $[M_n](t) \xrightarrow[n\uparrow\infty]{\mathbb{P}} A(t) \quad (\forall t \in \mathbb{R}_+).$

If (1) (*respectively* (2)) *holds, then relation* (2) (*resp.* (1)) *is also valid and*

$$M_n \xrightarrow[n\uparrow\infty]{\mathscr{L}} M.$$

The proof is straightforward from Proposition II.1. and the Approximation Procedure for Locally Square Integrable Martingales ([16], Lemma II.3.11; [14] Lemma 5).

Appendix

The Moduli W_C^N and W_D^N. Let $N \in \mathbb{N}$ and $x \in D$. The modulus $W_C^N(x, \delta)$ $(\delta > 0)$ is defined by

$$W_C^N(x,\delta) = \sup_{\substack{|t-s|\leqq \delta \\ s,t\in[0,N]}} |x(t)-x(s)|.$$

To define W_D^N, consider first the set $S(\delta)$, $(\delta>0)$, of all finte sets $\{t_i\}$ of points satisfying

$$0=t_0<t_1<\ldots<t_r=N$$
$$t_i-t_{i-1}>\delta, \qquad i=1,2,\ldots,r.$$

Thus

$$W_D^N(x,\delta) = \inf_{\{t_i\}\in S(\delta)} \max_{0<i\leqq r} \sup_{s,t\in[t_{i-1},t_i[} |x(t)-x(s)|.$$

If $(\Delta x)^*(\infty)\leqq c$, with $c>0$, then

$$W_C^N(x,\delta)\leqq 2\,W_D^N(x,\delta)+c.$$

This inequality is proved, by example, in [13] Chap. VII Lemma 6.4.

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Additional References

When this paper was finished the author learned about a recently published note of Aldous ([AR 1]) where a very general criterion on tightness is proved. Our proposition III.1 is a particular case of Aldous criterion.
Moreover, J. Jacod has recently published an excellent book on Stochastic Integration and related topics. The reader will find developped there the main results about Martingale Theory and the General Theory of Processes used in the present paper.

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