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Markov Processes with Identical Last Exit Distributions

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Summary. Let X and Y be two transient locally Hunt Markov processes. If X and Y enjoy the same last exit distributions from compact sets, then Y is equivalent to a time change of X by the inverse of a strictly increasing continuous additive functional. This result can also be interpreted (with natural auxiliary hypotheses) as a statement in potential theory involving equilibrium measures.

0. Introduction

In 1962, Blumenthal, Getoor and McKean published a paper entitled "Markov Processes with Identical Hitting Distributions" [3] in which they proved the following theorem:

(B-G-McK) *Let X and $\tilde{X}$ be standard processes so that for each compact subset K of E_Δ, $P_K(x,\cdot)=\tilde{P}_K(x,\cdot)$ for all x. There exists a continuous additive functional A_t of X_t which is strictly increasing and finite on $[0,\zeta)$ so that if T_t is the right continuous inverse of A_t, then $X(T_t)$ and $\tilde{X}_t$ have the same joint distributions.*

Here, $P_K(x,\cdot)$ is the distribution of the *first hitting time* of K starting from the point x. This remarkable theorem has been proved again by Chacon and Jamison in quite a different style from the original proof by Blumenthal, Getoor and McKean (see [4] for their proof under the additional assumptions that there are no holding points and the lifetimes are infinite).

The importance of *last exit times* in the theory of Markov processes has become increasingly apparent in recent years (the reader may wish to consult the following small sampler of papers: [1, 5, 8, 9]), and it seems altogether natural to formulate the following question: if two Markov processes enjoy the

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same last exit distributions, then after a time change, are they equivalent in distribution? This question may also be phrased in terms of potential theory (see Sect. 2) and is a natural one there also. The answer to the question is yes (modulo appropriate hypotheses), and the proof is given in Sect. 1. The presentation of these results is intended to be accessible to a reader familiar with standard processes as discussed in [2]. We have given complete references to any results we use which are not in [2].

The reader may recall that (B-G-McK) was proved by:

(1) Reducing to the case where X and $\tilde{X}$ are transient by killing the processes;

(2) Producing a continuous additive functional A_t as described above so that the potential kernel of $X(T_t)$, $U_A f(x) = E^x \int f(X(T_t))\,dt$, is the same as the potential kernel of $\tilde{X}_t$, $\tilde{U} f(x) = \tilde{E}^x \int f(\tilde{X}_t)\,dt$;

(3) Observing that any two transient processes with the same potential kernels have the same resolvents and therefore have the same joint distributions;

(4) And finally "pasting" these results together to obtain the result for the original (possibly recurrent) processes X and $\tilde{X}$.

Our proof is "basically" the same, but is a little simpler. To ensure that the last exit times are finite, we need to make a transience hypothesis, which eliminates the need for steps (1) and (4) above. In step (2) we find it more convenient to produce an additive functional $\tilde{A}_t$ of $\tilde{X}_t$ and an additive functional A_t of X_t so that $\tilde{U}_{\tilde{A}} = U_A$. We have also used the techniques of "projections" of raw additive functionals (rather than the connections between natural additive functionals and their potentials as described in Chaps. IV and V of [2]) since they seem to provide more natural proofs. An appendix has been provided at the end of the paper giving detailed definitions and references to the results about projections of raw additive functionals used herein.

1. The Main Theorem

Let E denote a locally compact space with countable base with a metric d chosen so that closed, bounded sets are compact. Adjoin a point Δ to E as the point at infinity if E is non-compact and as an isolated point if E is compact to obtain a compact metrizable space E_Δ. Let $\mathbf{E}_\Delta$ denote the sigma-algebra of Borel sets of E_Δ. We assume that $X = (\Omega, \mathbf{F}, \mathbf{F}_t, X_t, \theta_t, P^x)$ is a standard process on $(E_\Delta, \mathbf{E}_\Delta)$, by which we mean ([2], p. 45):

(i) Ω is the space of right continuous paths in E, and $\mathbf{F}$ and $\mathbf{F}_t$ are the natural Borel field and filtration on Ω generated by the coordinate maps X_t, and completed in the usual manner.

(ii) X_t is a normal strong Markov process with lifetime $\zeta = \inf\{t > 0\colon X_t = \Delta\}$.

(iii) X_t is quasi-left-continuous (i.e. if (T_n) is an increasing sequence of $(\mathbf{F}_t)$-optional times with limit T, then almost surely $X(T_n)$ converges to X_T on $\{T < \zeta\}$).

Recall that a standard process is called a Hunt process if $X(T_n)$ converges to X_T on $\{T<\infty\}$ almost surely whenever (T_n) is an increasing sequence of $(\mathbf{F}_t)$-optional times with limit T. We shall call a standard process *locally Hunt* if we can find a sequence (E_n) of open sets with compact closures increasing to E so that whenever (T_k) is an increasing sequence of $(\mathbf{F}_t)$-optional times with limit T so that for some m, $X(T_k)\in E_m\cup\{\Delta\}$ for all k, then $X(T_k)$ converges to X_T on $\{T<\infty\}$. Let $N_n=\inf\{t\colon X_t\in E_n^c\}$. It follows from IV-T28 of [7] that the locally Hunt process X_t has left limits on $(0,N_n]$ almost surely. Since $\lim_{n\to\infty} N_n=\zeta$ almost surely, X_t has left limits on $(0,\zeta)$ almost surely. Finally, if $\Lambda=\{\omega\colon N_n(\omega)=\zeta(\omega)$ for some $n\}$ and $\Gamma=\{\omega\colon X_t(\omega)$ has left limits on $(0,\infty)\}$, then $P^x(\Gamma\cap\Lambda)=P^x(\Lambda)$ for all x.

Let K be a compact set, and define $L_K=\sup\{t>0\colon X_t\in K\}$ $(\sup\emptyset=0)$. If $L_K<\infty$ a.s. for all compact sets K contained in E, then X will be called *transient*. In this paper we shall deal exclusively with transient locally Hunt processes. This class is larger than the class of transient Hunt processes (e.g. it includes the case of Brownian motion killed the first time it leaves the open unit ball), but is smaller than the class of all standard processes (e.g. it excludes the process which performs uniform motion around a circle and which dies when it reaches a distinguished point 0).

We now assume that we are given two transient locally Hunt processes

$$X=(\Omega,\mathbf{F},\mathbf{F}_t,X_t,\theta_t,P^x) \quad\text{and}\quad Y=(\Omega,\mathbf{G},\mathbf{G}_t,Y_t,\theta_t,Q^x) \quad\text{on } (E_\Delta,\mathbf{E}_\Delta).$$

The processes are named differently simply to aid the reader. Only the measures and the completions of the sigma-algebras on Ω change. Our goal in this section is to prove the following theorem.

(1.1) **Theorem.** *Let X and Y be two transient locally Hunt processes on $(E_\Delta,\mathbf{E}_\Delta)$ so that $P^x(f(X_{L_K-});L_K>0)=Q^x(f(Y_{L_K-});L_K>0)$ for all bounded functions f on E and for all bounded open sets K contained in E. There exists a strictly increasing continuous additive functional A_t so that if T_t is the right continuous inverse of A_t, then (Y_t,Q^x) and $(X(T_t),P^x)$ have the same joint distributions.*

Remark. Our assumptions imply that X_{L_K-} and Y_{L_K-} exist almost surely. We observed that X_{L_K-} exists on $\{L_K<\zeta\}$. Suppose $P^x(L_K=\zeta,\ N_n<\zeta$ for all $n)>0$ for some x. Let $T(n)=\inf\{t\geq N_n\colon X_t\in K\}$. Since N_n increases to ζ, we have that $T=\lim_{n\to\infty} T(n)=\zeta$ on $\{L_K=\zeta\}$. Then $X(T(n))$ does not converge to $X(T)$ with positive probability, in contradiction to the locally Hunt assumption (Note that $X(T(n))\in\bar{K}\cup\{\Delta\}$.) So it must be that $P^x(\{L_K=\zeta\}-\Lambda)=0$, and we conclude that X_{L_K-} also exists on $\{L_K=\zeta\}$.

To begin, let (q_i) be a countable collection of points which is dense in E. If q is a point in E, let $B_r(q)$ be the open ball of radius r about q (so $B_r(q)=\{x\in E\colon d(q,x)<r\}$), and let $L_r(q)=\sup\{t>0\colon X_t\in B_r(q)\}$ $(\sup\emptyset=0)$. For the next lemma, recall that $\mathbf{F}^0=\sigma\{X_t\colon t\geq 0\}=\sigma\{Y_t\colon t\geq 0\}$: no completions!

(1.2) **Lemma.** *The map $(t,r,\omega)\to 1_{\{0<L_r(q)\leq t\}}$ is $\mathbf{B}(\mathbb{R}^+)\times\mathbf{B}(\mathbb{R}^+)\times\mathbf{F}^0$-measurable.*

Proof. The map $\omega\to L_r(q)$ is $\mathbf{F}^0$-measurable since $B_r(q)$ is open. Since $r\to L_r(q)$ is left continuous, $(r,\omega)\to L_r(q)$ is $\mathbf{B}(\mathbb{R}^+)\times\mathbf{F}^0$-measurable. Thus $(r,\omega)\to 1_{\{0<L_r(q)\leq t\}}$

is $\mathbf{B}(\mathbb{R}^+)\times\mathbf{F}^0$-measurable for each t. Since this last process is right continuous in t, the result follows.

Thus if we set

$$A_t=\sum_j 2^{-j}\int_0^1 1_{\{0<L_r(q_j)\leqq t\}}\,dr,$$

$A_t\in\mathbf{B}(\mathbb{R}^+)\times\mathbf{F}^0$. Moreover, A_t is increasing, right continuous and satisfies $A_{t+s}=A_t+A_s\circ\theta_t$ (by virtue of the fact that $L_r(q_j)\circ\theta_t=(L_r(q_j)-t)^+$). We call such a process A_t a *raw* additive functional. For a discussion of additive functionals, see [2] and [11]. If we set $u(x)=P^x(A_\infty)$ (resp. $v(x)=Q^x(A_\infty)$), then $u(x)$ (resp. $v(x)$) is excessive for X (resp. Y) and is the potential of a natural additive functional B_t of X (resp. C_t of Y) as described in [2]. For our purposes, it is important to know that B_t (resp. C_t) is exactly the dual predictable projection of A_t with respect to the measure P^x (resp. Q^x) (see the Appendix for references and a discussion of these terms). Therefore, B_t is characterized as the unique increasing predictable process so that

$$P^x\int Z_t\,dA_t=P^x\int Z_t\,dB_t$$

for all bounded $(\mathbf{F}_t)$-predictable processes Z_t (and C_t is characterized similarly). This fact permits the proof of the following important result.

(1.3) **Proposition.** *B_t (resp. C_t) is continuous a.s. P^x (resp. Q^x) for all x in E.*

Proof. Since B_t is predictable, $J=\{(t,\omega)\colon B_t(\omega)\neq B_{t-}(\omega)\}$ is a predictable set. Assume J is not P^x-evanescent. By the section theorem ([7]), we may choose an $(\mathbf{F}_t)$-predictable time T with $P^x(B_T-B_{T-})>0$. But this is simply

$$P^x\int 1_{[T]}(s)\,dB_s=P^x\int 1_{[T]}(s)\,dA_s=\sum_j 2^{-j}P^x\int_0^1 1_{\{0<L_r(q_j)=T\}}\,dr,$$

(where $[T]=\{(t,\omega)\colon t=T(\omega)<\infty\}$).

The only way this last expression can be positive is if, for some j,

$$P^x(0<L_r(q_j)=L_{r+s}(q_j)=T)>0 \tag{1.4}$$

for some positive r and for some positive s. Note that $T<\zeta$ on $\{0<L_r(q_j)=T\}$ by the local Hunt character of X. Therefore, $X_T=X_{T-}$ on $\{0<L_r(q_j)=T\}$ since T is predictable. This fact together with (1.4) imply that X_T is in the intersection of the boundaries of $B_r(q_j)$ and $B_{r+s}(q_j)$. But this intersection is empty! Thus J must be P^x-evanescent, and the proposition is proved for B_t. The same argument applies to C_t.

(1.5) **Proposition.** *B_t (resp. C_t) is strictly increasing a.s. P^x (resp. Q^x) for all x in E.*

Proof. As in (1.3), we shall prove the result for B_t, and the same reasoning will apply to C_t. Let $R=\inf\{t>0\colon B_t>0\}$, let $p(x)=P^x(e^{-R})$, and let $F=\{x\colon p(x)=1\}$. Then F is the support of the continuous additive functional B_t, and we need to show $F=E$. So assume x is in $E-F$, and define $T_r(q_j)=\inf\{t>0\colon X_t\in B_r(q_j)\}$.

Now $R>0$ a.s. P^x and

$$0=P^x\int 1_{(0,R]}(s)\,dB_s=P^x\int 1_{(0,R]}(s)\,dA_s=\sum_j 2^{-j}\int_0^1 P^x(0<L_r(q_j)\leqq R)\,dr,$$

the second equality holding because the set $(0,R]$ is predictable. This implies that for each q_j with $d(q_j,x)<1$,

$$\text{(1.6)}\qquad \lambda\{r\colon d(q_j,x)<r<1,\ P^x(T_r(q_j)\circ\theta_R<\infty)\neq 1\}=0$$

where λ is Lebesgue measure.

So let $T_n=\inf\{t>0\colon d(X_t,x)\leqq n^{-1}\}$. We are interested in examining the sequence of times (S_n) defined by $S_n=R+T_n\circ\theta_R$. This sequence increases to some optional time S. Note that $S\leqq\zeta$ and $S<\infty$ a.s. P^x by (1.6). Since $X(S_n)\in\overline{B_1(x)}\cup\{\Delta\}$ for all n, the local Hunt character of X_t implies that $S<\zeta$. Thus $\lim_{n\to\infty} X(S_n)=X(S)=x$ a.s. P^x. But this contradicts the transience hypothesis (for an application of the strong Markov property tells us that the process will always return to x as time progresses). Thus, $F=E$, and this concludes the proof of the proposition.

Finally, we come to the desired result.

Proof of Theorem (1.1). Let f be a positive continuous function on E. Then

$$\begin{aligned}P^x\int f(X_t)\,dB_t&=P^x\int f(X_{t-})\,dB_t=P^x\int f(X_{t-})\,dA_t\\&=\sum_j 2^{-j}\int_0^1 P^x(f(X_{L_r(q_j)-});\ L_r(q_j)>0)\,dr\\&=\sum_j 2^{-j}\int_0^1 Q^x(f(X_{L_r(q_j)-});\ L_r(q_j)>0)\,dr\\&=Q^x\int f(X_{t-})\,dC_t=Q^x\int f(X_t)\,dC_t\end{aligned}$$

(*reasons:* the first and sixth equalities by continuity of B_t and C_t; the second and fifth equalities by predictability of $f(X_{t-})$; the fourth equality by hypothesis; the third equality by definition). If we denote by S_t (resp. T_t), the right continuous inverse of B_t (resp. C_t), we have that S_t and T_t are continuous and strictly increasing, since B_t and C_t are, and

$$\text{(1.7)}\qquad P^x\int f(X_{S_t})\,dt=Q^x\int f(Y_{T_t})\,dt.$$

But $(X(S_t),P^x)$ and $(Y(T_t),Q^x)$ are strong Markov processes on $(E_\Delta,\mathbf{E}_\Delta)$ ([2], p. 212), with bounded resolvents $(U^a)_{a\geqq 0}$ and $(V^a)_{a\geqq 0}$, respectively. Equation (1.7) implies $U^0=V^0$. It is simple to verify (see e.g. [2], p. 238) that this implies $U^a=V^a$ for all nonnegative a. Therefore, $(X(S_t),P^x)$ and $(Y(T_t),Q^x)$ have the same joint distributions. Since T_t is a strictly increasing continuous additive functional of $(Y(T_t),Q^x)$, there is a strictly increasing continuous additive functional D_t of $(X(S_t),P^x)$ so that if U_t is the right continuous inverse of D_t, then $(X(S_{U(t)}),P^x)$ and $(Y(T_{C(t)})=Y_t,Q^x)$ have the same joint distributions. It is easy to check that $X(S_{U(t)})$ is a time-change of X_t by a strictly increasing continuous additive functional, and this completes the proof.

It is perhaps worth noting that the conclusion of Theorem (1.1) remains true if in the hypothesis of (1.1) one replaces the condition "bounded open sets" with "compact sets." For suppose

(1.8) $P^x(f(X_{L_K-}); L_K>0)=Q^x(f(Y_{L_K-}); L_K>0)$ for all compacts sets K.

Let G be any bounded open set, and let K_n be a sequence of compact sets increasing to G. Then $\{(t,\omega): X_t(\omega)\in K_n\}$ increases to $\{(t,\omega): X_t(\omega)\in G\}$, and it follows that $\lim_{n\to\infty} L_{K_n}=L_G$ a.s. and

$$\lim_{n\to\infty} f(X_{L_{K_n}-})\,1_{\{L_{K_n}>0\}}=f(X_{L_G-})\,1_{\{L_G>0\}} \quad \text{a.s.}$$

for every continuous bounded function f. Thus (1.8) implies the hypothesis of Theorem (1.1) is true.

2. The Connection with Potential Theory

The quantity $v_f(x)=P^x(f(X_{L_K-}); L_K>0)$ has an interpretation in terms of the potential theory of the process if we admit certain (natural) auxiliary hypotheses. Chung [5] gave an interpretation of $v_f(x)$ in terms of the equilibrium measure of the process X assuming that the potential kernel $u(x,y)$ of the process X satisfies certain analytic conditions. Getoor and Sharpe [8] gave a similar interpretation assuming that X has a dual $\hat{X}$ (as in Chap. VI of [2]), and Meyer [9] gave the interpretation under assumptions slightly weaker than duality. We very briefly indicate the result of Getoor and Sharpe in our setting. For further historical comments, consult the paper of Chung.

(2.1) **Theorem.** *Suppose X and $\hat{X}$ are transient locally Hunt processes in duality with respect to a sigma-finite excessive reference measure $m(dx)$. If K is bounded, then $v_f(x)=\int u(x,y)\,f(y)\,p_K(dy)$, where $u(x,y)$ is the potential density, and p_K is the equilibrium measure of K.*

Proof. As observed in Sect. 1, $v_f(x)=E^x\int f(X_{t-})\,dD_t$, where D_t is the dual predictable projection of $1_{\{0<L_K\leqq t\}}$. Thus, by the representation theory of Revuz [10], $v_f(x)=\int u(x,y) f(y)\,z(dy)$ for some measure $z(dy)$. But $v_1(x)=P^x(T_K<\infty)=P_K 1(x)=U p_K(x)$, where p_K is the equilibrium measure of K ([2], VI-4), and therefore $z=p_K$.

Therefore, Theorem (1.1) may be restated as follows:

(2.2) **Theorem.** *Suppose X and Y are two transient locally Hunt processes, each possessing a dual (or satisfying the auxiliary analytic hypotheses of Chung, or those of Meyer). Suppose X (resp. Y) has potential kernel U and equilibrium measures p_K (resp. potential kernel V and equilibrium measures q_K). If $Ufp_K(x)=Vfq_K(x)$ for all bounded functions f on E and for all bounded open sets K contained in E, then the class of excessive functions for X coincides with the class of excessive functions for Y. Thus X and Y have the same potential theories.*

Proof. Simply observe that if two processes X and Y are related by a time-change by a strictly increasing continuous additive functional as in Theorem (1.1), then they have the same excessive functions.

3. Two Examples

We first write down the equilibrium measures of balls for Brownian motion in $\mathbb{R}^3$. If X is Brownian motion in $\mathbb{R}^3$, it is in duality with itself with respect to Lebesgue measure on $\mathbb{R}^3$, so the discussion in Section 2 applies. Let L be the last exit time of X_t from the ball B of radius r centered about q. Then $P^x(f(X_{L-}); L>0)=\int|x-y|^{-1}f(y)\,p(dy)$, where p is the equilibrium measure of the ball. Since X exits B continuously, p charges only the boundary of B, and the spherical symmetry of the Brownian motion implies that $p(dy)$ is simply Lebesgue surface measure on the skin of the ball normalized so that $\int|x-y|^{-1}p(dy)=1$ for all x in B.

Now let X be a linear diffusion on the interval $(0,1)$ so that $X_{\zeta-}\in\{0,1\}$. Recall that X is determined by a *scale function* $s(x)=P^x(X_{\zeta-}=1)$ and a *speed measure* $m(dx)$ (we assume $0<s(x)<1$). A well-known theorem states that by a change of scale and time, X may be transformed into a Brownian motion which is killed the first time it leaves $(0,1)$. This fact is contained in Theorem (1.1) (it is, of course, a "simpler" result than Theorem (1.1) since the massive Markov machinery was used to prove (1.1)). For $s(X_t)$ is a local martingale on $[0,\zeta)$, and it is easy to compute the quantity $P^x(T_{r,q}<\infty)$, where $T_{r,q}=\inf\{t>0: X_t\in B_r(q)\}$. Using this and $s(x)$, one can compute the last exit distribution of the process from a ball. Therefore, the scale function completely determines the last exit distributions. If we let $Y_t=s(X_t)$, and $Q^{s(x)}=P^x$, then (Y_t, Q^x) has scale function x (*natural* scale). Therefore, Y_t can be time-changed into a Brownian motion since Brownian motion also has scale function x.(This same result can be proved with the Blumenthal, Getoor, and McKean Theorem by using an analogous argument.)

4. Similarly

One may attempt to find other collections of "last" times which determine the process up to time-change (and which, therefore, determine the potential theory of the processes). We indicate an example below. In this section, for simplicity, we shall let X and Y be Hunt processes (with the same notations as in Sect. 1).

Let D be a countable collection of points which is dense in E, and let $B_r(q)$ denote the ball of radius r about q. Let $G=\{(p,q): p\in D, q\in D\}$. Enumerate the pairs in G: (p_1,q_1), (p_2,q_2), etc. Let $M_r(j)=\sup(t: X_{t-}\in B_r(p_j), X_t\in B_r(q_j)\}$, and set

$$A_t=\sum_j 2^{-j}\int_0^1 e^{-r}\,1_{\{0<M_r(j)\leqq t\}}\,1_{\{\overline{B_r(p_j)}\cap\overline{B_r(q_j)}=\emptyset\}}\,dr.$$

Let B_t (resp. C_t) denote the dual predictable projection of A_t for X (resp. Y). The fact that B_t is continuous boils down to the observation that $P^x(0<M_r(j) = T)\,1_{\{\overline{B_r(p_j)} \cap \overline{B_r(q_j)} = 0\}} = 0$ for all predictable times T since $M_r(j)$is contained in the jumps of the process X_t (provided the balls do not intersect). It may happen, however, that B_t and C_t do not increase (e.g. in the case of diffusions!). The reader may easily formulate hypotheses on the process (perhaps involving the Lévy system) and an analogue of Theorem (1.1) in this situation.

Appendix

Let **P** denote the sigma algebra on $R^+ \times \Omega$ generated by the left continuous processes which are adapted to the filtration $(\mathbf{F}_t)$: **P** is called the *predictable* sigma algebra. Sets B in **P** are called predictable sets and **P**-measurable processes $Z_t(\omega)$ are called predictable processes. If T is any $(\mathbf{F}_t)$-stopping time so that $[T, \infty) = \{(t, \omega): t \geqq T(\omega)\} \in \mathbf{P}$, then T is a predictable time (see IV-47 through IV-78 in [13] for a discussion of these objects).

If $Z_t(\omega) \in \mathbf{B}(\mathbb{R}^+) \times \mathbf{F}$ is a bounded process, the P^x-predictable projection of Z is the process ${}^pZ_t$ (unique up to P^x-indistinguishability) satisfying ${}^pZ_T = E^x(Z_T | \mathbf{F}_{T-})$ for every $(\mathbf{F}_t)$-predictable time T (Appendice 1, Thm. 6, [14]). (We refer to the discussion in Appendice 1 rather than Chapitre VI since $(\mathbf{F}_t)$ is not P^x-complete for any x, and hence does not satisfy what are known as the "usual conditions".) If $A_t \in \mathbf{B}(\mathbb{R}^+) \times \mathbf{F}$ is any process with right continuous increasing paths so that $E^x A_\infty < \infty$, then the P^x-dual predictable projection of A is defined to be the unique predictable process A_t^p with right continuous increasing paths satisfying $E^x \int {}^pZ_t\, dA_t = E^x \int Z_t\, dA_t^p$ for all bounded processes $Z_t \in \mathbf{B}(\mathbb{R}^+) \times \mathbf{F}$ (Appendice 1, Thm. 12, [14]). Notice that the predictable and dual predictable projections depend on the underlying probability measure. In the Markovian framework, however, one can choose versions of ${}^pZ$ and A^p so that they are the P^x-predictable projection of Z and the P^x-dual predictable projection of A for all x (assuming appropriate integrability conditions: boundedness of Z and A certainly suffices; see (3.12) and (3.32) in [15]). If A_t has the additive property: $A_{t+s} = A_t + A_s \circ \theta_t$, then the last sentence of (3.18) in [15] states that A^p also has this property: $A_{t+s}^p = A_t^p + A_s^p \circ \theta_t$. Thus the dual predictable projection of a raw additive functional is an additive functional. The interested reader may wish to consult [12] also, where it is shown (in detail) that the dual optional projection of a raw additive functional is an additive functional. Note: in general, if $A_\zeta \neq A_{\zeta-}$, it may be that $A_\zeta^p \neq A_{\zeta-}^p$.

Returning to the notation in Sect. 1, we have that $u(x) = P^x(A_\infty)$ is the bounded potential of a natural additive functional B_t. We may also write $u(x) = P^x(A_\infty^p)$. We showed in (1.3) that A_t^p is continuous, so $A_t^p = B_t$ (since two natural additive functionals with the same finite potential must be equal [2]).

The reader may wish to consult the related references: [11], Chaps. IV and III; [2], pp. 299–302; [7]; [8].

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