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Probability and Expectation Inequalities

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Summary. This paper introduces a mathematical framework within which a wide variety of known and new inequalities can be viewed from a common perspective. Probability and expectation inequalities of the following types are considered: (a) $P(Z \in A) \geqq P(Z' \in A)$ for some class of sets A, (b) $\mathscr{E} k(Z) \geqq \mathscr{E} k(Z')$ for some class of functions k, and (c) $\mathscr{E} l(Z) \geqq \mathscr{E} k(Z')$ for some class of pairs of functions l and k. It is shown, sometimes using explicit constructions of Z and Z', that, in several cases, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c); included here are cases of normal and elliptically contoured distributions. A case where (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) is studied and is expressed in terms of "n-monotone" functions for (some of) which integral representations are obtained. Also, necessary and sufficient conditions for (c) are given.

1. Introduction

Let Z and Z' be n-dimensional random vectors and consider probability and expectation inequalities of the following forms:

(C1) $P(Z \in A) \geqq P(Z' \in A), \quad A \in \mathscr{A}$,

(C2) $\mathscr{E} k(Z) \geqq \mathscr{E} k(Z'), \quad k \in \mathscr{F}$,

(C3) $\mathscr{E} l(Z) \geqq \mathscr{E} k(Z'), \quad (l, k) \in \mathscr{G}$,

where $\mathscr{A}$ is a class of (Borel) subsets of $\mathbb{R}^n$, $\mathscr{F}$ is a class of real (measurable) functions on $\mathbb{R}^n$, and $\mathscr{G}$ is a class of pairs of (measurable) functions on $\mathbb{R}^n$. When the classes $\mathscr{A}, \mathscr{F}, \mathscr{G}$ are progressively richer, then conditions (C1), (C2), (C3) are progressively stronger. Specifically, if $1_{\mathscr{A}} \subset \mathscr{F}$, i.e. $1_A \in \mathscr{F}$ for all $A \in \mathscr{A}$, then (C1) $\Leftarrow$ (C2); and if $\mathscr{F} \subset \{k\colon (k, k) \in \mathscr{G}\} = \mathscr{F}_{\mathscr{G}}$, then (C2) $\Leftarrow$ (C3). The more

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interesting questions are therefore those which lead from (C1) to (C2) to (C3), and these are the object of this paper.

There is a vast literature on the question of describing conditions on the distributions of Z and Z' which guarantee (C1) for specific classes $\mathscr{A}$ of sets. A recent survey is in the book by Tong [17]. This question is touched only peripherally in this paper in Sect. 3. The first question considered in this paper is, given a class of sets $\mathscr{A}$, to describe a class of functions $\mathscr{F}$, depending of course on $\mathscr{A}$, for which (C1) $\Rightarrow$ (C2); if such a class $\mathscr{F}$ contains $1_{\mathscr{A}}$, then in fact (C1) $\Leftrightarrow$ (C2). The second question is given a class of functions $\mathscr{F}$, to describe a class $\mathscr{G}$ of pairs of functions for which (C2) $\Rightarrow$ (C3). If, furthermore, $\mathscr{F} \subset \mathscr{F}_{\mathscr{G}}$, then (C2) $\Leftrightarrow$ (C3). Clearly this equivalence holds for the class $\mathscr{G}_{\mathscr{F}}$ defined by what may be called *the separation approach:*

$$\begin{aligned}\mathscr{G}_{\mathscr{F}} = \{(l,k)\colon\ & l \geqq m \geqq k \text{ for some } m \in \mathscr{F}, \\ & \text{and the expectations in (C3) are defined}\},\end{aligned} \tag{1.1}$$

which is most useful when there is a direct description of $\mathscr{G}_{\mathscr{F}}$. Following is a brief description of the results of the type

$$\text{(C1)} \Rightarrow \text{(C2)} \Leftrightarrow \text{(C3)} \tag{1.2}$$

$$\text{(C1)} \Leftrightarrow \text{(C2)} \Leftrightarrow \text{(C3)} \tag{1.3}$$

available in the literature and of those derived here. Most of the known results can be found in the book of Marshall and Olkin [7].

Kemperman [6] and Kamae, Krengel and O'Brien [5] established (1.3) (for more general partially ordered spaces) with $\mathscr{A}$ the class of all measurable increasing sets (i.e. $a \in A$, $a \leqq b \Rightarrow b \in A$), $\mathscr{F}$ the class of all measurable increasing functions for which the expectations in (C2) exist, and $\mathscr{G} = \mathscr{G}_{\mathscr{F}}$ the class of all pairs of functions l, k satisfying $l(z) \geqq k(z')$, $z \geqq z'$, for which the expectations in (1.3) are defined - provided the "separating" increasing function m defined, for instance, by $m(x) = \sup\{k(y),\ x \geqq y\}$ is measurable (this is always true in $\mathbb{R}^1$ but may fail even in $\mathbb{R}^2$). Nevius, Proschan and Sethuraman [8] established (1.3) (also for more general partially ordered spaces) with $\mathscr{A}$ the class of all measurable Schur-convex sets, $\mathscr{F}$ the class of all measurable Schur-convex functions for which the expectations in (C2) exist, and $\mathscr{G} = \mathscr{G}_{\mathscr{F}}$ the class of all pairs of functions l, k satisfying $l(z) \geqq k(z')$ for all z which majorize z' (see, for instance, [7]) and for which the expectations in (C3) are defined. Both of these results are put in common perspective in Sect. 4, where a more general result of the type (1.3) is obtained via a generalization of a theorem due to Strassen [14]. As a special case, we have (1.3) with $\mathscr{A}$ the class of all sets homothetic to a symmetric convex compact set S, $\mathscr{G}$ the class of all pairs of functions l, k satisfying $l(z) \geqq k(z')$, $\|z\|_S \geqq \|z'\|_S$, for which the expectations in (C3) exist, and $\mathscr{F} = \mathscr{F}_{\mathscr{G}}$, where $\|z\|_S$ is the norm corresponding to S (see Example 4.4).

In Sect. 2, (1.2) is established for $n=2$ with $\mathscr{A}$ the class of all closed symmetric rectangles, $\mathscr{F}$ the class of all functions $k(x,y)$ of the form $f(x^2+y^2)$ with f a nonincreasing and convex function on $[0,\infty)$, and $\mathscr{G}$ defined as in (2.11). For $n>2$, it is shown that (C1) $\Rightarrow$ (C2) with $\mathscr{A}$ the class of all closed symmetric rectangles and $\mathscr{F}$ the class of all functions $k(z)$ of the form $f(\|z\|^2)$

with f an n-monotone function on $[0, \infty)$. Section 2 includes also, in Lemma 2.3, integral representations for certain n-monotone functions which may be of independent interest. Even though only the representations of the form (2.16) or (2.18) are used here, we also include the very interesting form (2.17) suggested to us by Daryl Daley.

For two-dimensional random vectors $Z=(X, Y)$ and $Z'=(X', Y')$ with common marginal distributions, Cambanis, Simons and Stout [3] and Tchen [16] established that (C1) $\Leftrightarrow$ (C2) with $\mathscr{A}$ the class of all principal lower ideals $(-\infty, x] \times (-\infty, y]$ in $\mathbb{R}^2$, and $\mathscr{F}$ the class of all quasi-monotone functions, i.e. functions k which satisfy the inequalities

$$k(x_1, y_1)+k(x_2, y_2) \geqq k(x_1, y_2)+k(x_2, y_1), \quad x_1 \leqq x_2, \; y_1 \leqq y_2, \tag{1.4}$$

for which the expectations in (C2) are defined and which satisfy certain minor regularity conditions. Thus (1.3) holds true with $\mathscr{G}=\mathscr{G}_{\mathscr{F}}$, but we have been unable to obtain a direct description of $\mathscr{G}_{\mathscr{F}}$. When $(l, k) \in \mathscr{G}_{\mathscr{F}}$, the functions l and k are separated by a quasi-monotone function, and thus

$$l(x_1, y_1)+l(x_2, y_2) \geqq k(x_1, y_2)+k(x_2, y_1), \quad x_1 \leqq x_2, \; y_1 \leqq y_2. \tag{1.5}$$

If we denote by $\mathscr{G}$ the (larger than $\mathscr{G}_{\mathscr{F}}$) class of pairs of functions (l, k) which satisfy (1.5) (plus appropriate regularity conditions) and for which the expectations in (C3) are defined, then in general (C1) does not imply (C3). However, in the special cases where Z and Z' have normal or elliptically contoured distributions, it is shown in Sect. 3 that (1.3) *is* valid. It is not currently known whether (1.3) is valid for other classes of bivariate distributions. A generalization from two to higher dimensions is also described in Sect. 3 (Theorem 3.3).

Higher dimensional generalizations of the results in [3, 16], described in the previous paragraph, have been obtained by Bergmann [1] and Rüschendorf [10]. They established (C1) $\Leftrightarrow$ (C2) with $\mathscr{A}$ the class of all principal upper ideals $[z, \infty)$ in $\mathbb{R}^n$ ($z \in \mathbb{R}^n$), and $\mathscr{F}$ in [1] the class of all functions of the form $k(x_1, \ldots, x_n)=f_1(x_1) \ldots f_n(x_n)$ where each f_i is nonnegative and nondecreasing, and in [10] the class of all right continuous functions on $\mathbb{R}^n$ which are quasi- or Δ-monotone as functions of any k of the n variables, $1 \leqq k \leqq n$, and for which the expectations in (C2) are defined and finite. Rüschendorf [10] also established (C1) $\Rightarrow$ (C2) with $\mathscr{A}$ as before and $\mathscr{F}$ the class of all right continuous functions which are Δ-monotone as functions of all n variables and vanish when any variable approaches $-\infty$, and for which the expectations in (C2) are defined and finite. Bergmann [1] also established (C1) $\Leftrightarrow$ (C2) with $\mathscr{A}$ the class of all principal lower ideals $(-\infty, z]$ in $\mathbb{R}^n$ ($z \in \mathbb{R}^n$), and $\mathscr{F}$ the class of all functions of the form $k(x_1, \ldots, x_n)=f_1(x_1) \ldots f_n(x_n)$ where each f_i is nonpositive and nondecreasing. Of course, the above results imply (1.2) or (1.3) (accordingly) with $\mathscr{G}=\mathscr{G}_{\mathscr{F}}$, and it is an open problem to obtain a direct description of $\mathscr{G}_{\mathscr{F}}$ or to find a larger class $\mathscr{G}$ with simple description (cf. previous paragraph), for which the results are valid for special classes of distributions such as normal or elliptically contoured.

A final comment on the methods used here: The results in Sect. 2 are established through straight analysis. Sections 3 and 4 use a novel and powerful approach which may be called *the surrogate approach* and which is described there. The surrogate approach makes it possible to establish results of the type (1.3) for specific classes of distributions via special constructions, i.e. in an elementary way, and to our knowledge, this is the first time this is done in the literature (see Sect. 3 and 4). Also, in conjunction with a generalization of a theorem by Strassen [14], the surrogate approach gives results of the type (1.3) in a general context (see Sect. 4); in a special context, this has been used earlier by Kemperman [6] and by Kamae, Krengel and O'Brien [5].

2. n-monotone Functions

In this section, we develop inequalities for expectations of n-monotone functions (to be defined below) of the squares of the moduli of n-dimensional random vectors. We begin with the case $n=2$ (Theorem 2.1) and then proceed to the general case $n \geqq 2$ (Theorem 2.2). In the process of establishing Theorem 2.2, we develop an integral representation for certain n-monotone functions (Lemma 2.3) which may be of independent interest.

Theorem 2.1. *Suppose $Z=(X, Y)$ and $Z'=(X', Y')$ are bivariate random vectors for which*

$$P(|X| \leqq a, |Y| \leqq b) \geqq P(|X'| \leqq a, |Y'| \leqq b), \qquad a \geqq 0,\ b \geqq 0. \tag{2.1}$$

Then for every nonincreasing convex function f on $[0, \infty)$,

$$\mathscr{E} f(X^2+Y^2) \geqq \mathscr{E} f(X'^2+Y'^2). \tag{2.2}$$

Proof. Condition (2.1) is equivalent to saying that $aX'^2 \vee bY'^2$ is stochastically larger than $aX^2 \vee bY^2$ for $a \geqq 0$, $b \geqq 0$, where $u \vee v$ denotes the maximum of u and v. Thus for any bounded nonincreasing function h on $[0, \infty)$,

$$\mathscr{E} h(aX^2 \vee bY^2) \geqq \mathscr{E} h(aX'^2 \vee bY'^2), \qquad a \geqq 0,\ b \geqq 0, \tag{2.3}$$

and, consequently,

$$\begin{aligned} &\mathscr{E} \int_0^{\pi/2} h\left(\frac{X^2}{\cos^2\theta} \vee \frac{Y^2}{\sin^2\theta}\right) \sin\theta\cos\theta\, d\theta \\ &\quad \geqq \mathscr{E} \int_0^{\pi/2} h\left(\frac{X'^2}{\cos^2\theta} \vee \frac{Y'^2}{\sin^2\theta}\right) \sin\theta\cos\theta\, d\theta. \end{aligned} \tag{2.4}$$

Now with the substitution of $(x^2+y^2)u$ for $(x^2/\cos^2\theta) \vee (y^2/\sin^2\theta)$, the integral $\int_0^{\pi/2} h[(x^2/\cos^2\theta) \vee (y^2/\sin^2\theta)] \sin\theta\cos\theta\, d\theta$ simplifies to $(\frac{1}{2}) \int_1^{\infty} h((x^2+y^2)u)u^{-2}\, du$. Thus (2.2) holds for functions f of the form $f(s) = \int_1^{\infty} h(su)u^{-2}\, du$, $s \geqq 0$. But, according to Lemma 2.1 below, the class of such functions coincides with the class of *bounded* nonincreasing convex functions. The unwanted restriction of boundedness is easily removed by truncation: If f is any nonincreasing convex function, then $f \vee (-n)$ is a bounded nonincreasing convex function whose limit, as $n \to \infty$, is f. Then (2.2) follows by means of the monotone convergence theorem. □

Lemma 2.1. *A function f on $[0, \infty)$ is a bounded nonincreasing convex function if and only if it is of the form*

$$f(s) = \int_1^\infty \frac{h(su)}{u^2}\, du, \qquad s \geqq 0, \tag{2.5}$$

with h nonincreasing and bounded on $[0, \infty)$, or equivalently if and only if it is of the form

$$f(s) = f(\infty) + \int_{[0,\infty)} \left(1 - \frac{s}{u}\right)_+ d\mu(u), \qquad s \geqq 0, \tag{2.6}$$

with μ a finite measure on $[0, \infty)$.

Proof. Both characterizations follow from the characterization

$$f(s) = f(\infty) + \int_s^\infty g(u)\, du, \qquad s > 0,$$

$$f(0) \geqq f(\infty) + \int_0^\infty g(u)\, du,$$

with g nonnegative, nonincreasing and integrable over $(0, \infty)$ (cf. Roberts and Varberg [9], pp. 9–10), and the straightforward steps are omitted. Each of the functions g, h can be expressed in terms of the other as follows:

$$h(0) = f(0), \qquad h(s) = f(\infty) + s g(s) + \int_s^\infty g(u)\, du = f(\infty) - \int_s^\infty u\, dg(u), \qquad s > 0,$$

$$g(s) = \frac{h(s)}{s} - \int_s^\infty \frac{h(u)}{u^2}\, du = -\int_s^\infty \frac{dh(u)}{u^2}, \qquad s > 0,$$

and the measure μ and function h can be expressed in terms of each other as follows:

$$d\mu(s) = -dh(s) \qquad \text{on } [0, \infty),$$

i.e. μ is the Lebesgue-Stieltjes measure corresponding to (the right continuous version of) $-h$, and

$$h(s) = f(\infty) + \mu\{(s, \infty)\}, \qquad s \geqq 0. \quad \square$$

Only characterization (2.5) is used here in the proof of Theorem 2.1. The characterization (2.6) is known (see [7], pp. 448–449) and has been included here only for completeness (in view of its generalization in Lemma 2.3).

There are *unbounded* nonincreasing convex functions f which cannot be expressed in the integral form (2.5) with h nonincreasing and necessarily unbounded, e.g. $f(s) = -s$, $s \geqq 0$. An analogue of Lemma 2.1 can be established for nonincreasing convex functions f defined on the *open* interval $(0, \infty)$. Boundedness is not essential on $(0, 1]$, but is on $[1, \infty)$. See Lemma 2.3 below.

Likewise, Theorem 2.1 can be modified to cover functions f defined on $(0, \infty)$ which are nonincreasing and convex. Such functions can be approximated from below by functions of the type described in Theorem 2.1; and through use of the monotone convergence theorem, we can obtain:

Corollary 2.1. *If, in addition to* (2.1), $P(Z=(0,0))=0$, *then* ($P(Z'=(0,0))=0$ *and*) (2.2) *holds for each nonincreasing convex function* f *on* $(0,\infty)$ *for which the expectations contained therein exist.*

It is apparent from the nature of assumption (2.1), appearing in Theorem 2.1, that inequality (2.2) can be extended to

$$\mathscr{E} f(\alpha X^2+\beta Y^2) \geqq \mathscr{E} f(\alpha X'^2+\beta Y'^2), \qquad \alpha \geqq 0, \ \beta \geqq 0.$$

There are, of course, many nonincreasing convex functions f to which Theorem 2.1 or its corollary is applicable. As an example the assumptions of Theorem 2.1 imply $\mathscr{E} R^\alpha \leqq \mathscr{E} R'^\alpha$, $0<\alpha \leqq 2$, where $R^2=X^2+Y^2$ and $R'^2=X'^2+Y'^2$, while the assumptions of Corollary 2.1 permit the conclusion $\mathscr{E} R^\alpha \geqq \mathscr{E} R'^\alpha$, $\alpha<0$.

The value of Theorem 2.1 and its corollary depends, of course, upon the reasonableness of assumption (2.1), an inequality of type (C1). Theorem 2.1 of Das Gupta et al. [4] states easily checked conditions under which this inequality holds for pairs of related elliptically contoured distributions, as well as conditions under which assumption (2.12) holds in Theorem 2.2 below and in its corollary.

The requirements in Theorem 2.1 that f be nonincreasing and convex are both necessary for the generality of the theorem: *If f is a function on* $[0,\infty)$ *which satisfies* (2.2) *whenever* (2.1) *holds and the expectations make sense, then f must be nonincreasing and convex.*

Proof. The need for f to be nonincreasing can be seen by considering nonstochastic Z and Z' of the form $(x,0)$ and $(x'0)$, $0 \leqq x \leqq x' < \infty$. Now suppose f is nonincreasing and satisfies (2.2) for all $Z=(X,Y)$ and $Z'=(X',Y')$ satisfying (2.1). For $s>0$ and $\rho \in (0,1]$, let $Z'=s^{\frac{1}{2}} V$ and $Z=s^{\frac{1}{2}} V \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}^{\frac{1}{2}}$, where V is uniformly distributed on the unit circle. Since Z and Z' are elliptically contoured vectors which satisfy (2.1) (cf. Theorem 2.1 of [4]), inequality (2.2) holds, which translates into

$$f(s) \leqq \pi^{-1} \int_{-1}^{1} (1-u^2)^{-\frac{1}{2}} f(s[1+u\rho])\, du, \qquad s>0, \ \rho \in (0,1]. \tag{2.7}$$

Replacing s by $s-1/n$ and letting $n \to \infty$ yields

$$f(s-) \leqq \pi^{-1} \int_{-1}^{1} (1-u^2)^{-\frac{1}{2}} f(s[1+u\rho])\, du, \qquad s>0, \ \rho \in (0,1],$$

which, in turn, due to the monotonicity of f, yields

$$f(s-)\leqq \pi^{-1}\int_{-1}^{0}(1-u^2)^{-\frac{1}{2}}f(s-s\rho)\,du+\int_{0}^{1}(1-u^2)^{-\frac{1}{2}}f(s+)\,du$$
$$=\tfrac{1}{2}f(s-s\rho)+\tfrac{1}{2}f(s+)$$

for $s>0$ and $\rho\in(0,1]$. Letting $\rho\downarrow 0$, we obtain $f(s+)\geqq f(s-)$, $s>0$, which establishes the continuity of f on $(0,\infty)$.

Now suppose f is not convex so that for some $0\leqq a<b$, we have $f(a)>f(b)$ and

$$f(a)+f(b)<2f\left(\frac{a+b}{2}\right). \tag{2.8}$$

Consider lines $t=ms+c$, $a\leqq s\leqq b$, of negative slope $m=(f(b)-f(a))/(b-a)$. For large values of c, the line $t=ms+c>f(s)$ over the entire iterval $[a,b]$. Let c decrease until the line first touches the graph of f at some point in the interval $[a,b]$, and let s_0 be the smallest such point of contact with this line. (Since f is continuous on $(0,\infty)$, both c and s_0 are well-defined.) Due to (2.8), s_0 is in the open interval (a,b). Setting $s=s_0$ and $\rho=(1-a/s_0)\wedge(b/s_0-1)$, so that $0\leqq\rho\leqq 1$ and $a\leqq s(1+u\rho)\leqq b$ for $-1\leqq u\leqq 1$, we obtain from inequality (2.7):

$$f(s_0)\leqq \pi^{-1}\int_{-1}^{1}(1-u^2)^{-\frac{1}{2}}f(s_0[1+u\rho])\,du$$
$$\leqq \pi^{-1}\int_{-1}^{1}(1-u^2)^{-\frac{1}{2}}(ms_0[1+u\rho]+c)\,du$$
$$=ms_0+c=f(s_0).$$

This can only happen if

$$f(s_0[1+u\rho])=ms_0(1+u\rho)+c, \qquad -1\leqq u\leqq 1,$$

which is impossible (for negative u) due to the way s_0 is defined. Thus f must be convex. □

We remark that the random variables R^2 and R'^2, associated with the random vectors $Z=s^{\frac{1}{2}}V\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}^{\frac{1}{2}}$ and $Z'=s^{\frac{1}{2}}V$ (used in this proof), are not stochastically ordered since $\mathscr{E}R^2=\mathscr{E}R'^2=s$. Thus condition (2.1) *can* hold without R^2 being stochastically smaller than R'^2. It follows, of course, that condition (2.1) can hold without (2.2) holding for every nonincreasing function f.

Finally it should be pointed out that the argument used in establishing Theorem 2.1 shows that inequality (C2) holds for all functions k of the form

$$k(x,y)=\int_{0}^{\pi/2}F_\theta\left(\frac{|x|}{\cos\theta}\vee\frac{|y|}{\sin\theta}\right)d\mu(\theta),$$

where $F_\theta(r)$ is jointly measurable in (θ,r) and nonincreasing in r for each fixed θ, μ is a measure on the open interval $(0,\pi/2)$, and the indicated integral exists and is finite. By choosing, for instance, $F_\theta(r)=h(r)\,g(\theta)$ with h bounded and nonincreasing and g bounded and $\geqq 0$ (e.g. $g(\theta)=(\sin\theta)^n(\cos\theta)^m$) and μ Lebesgue measure, we can generate a large class of symmetric as well as nonsymmetric functions $k(x,y)$. The choice $g(\theta)=\sin\theta\cos\theta$ gives Theorem 2.1 for bounded nonincreasing convex functions of x^2+y^2; $k(x,y)=f(x^2+y^2)$.

We have thus shown in Theorem 2.1 that (C1) $\Rightarrow$ (C2) with $\mathscr{A}$ the class of all closed symmetric rectangles and

$$\mathscr{F}=\{k(x,y)=f(x^2+y^2), f\colon [0,\infty)\to \mathbb{R}^1 \text{ nonincreasing and convex}\}.$$

Using the separation approach, we obtain (C1) $\Rightarrow$ (C2) $\Leftrightarrow$ (C3) with

$$\begin{aligned}\mathscr{G}=\{(l,k)\colon\, & l(x,y)=g(x^2+y^2),\ k(x,y)=f(x^2+y^2),\ g\geqq h\geqq f,\\ & h\colon [0,\infty)\to\mathbb{R}^1 \text{ nonincreasing and convex}\}.\end{aligned}$$

In order to obtain a direct description of the class $\mathscr{G}$, we note that functions f and g on the positive real line can be separated by a convex function h, $g\geqq h\geqq f$, if and only if

$$\lambda g(s)+(1-\lambda)g(t)\geqq f[\lambda s+(1-\lambda)t], \qquad s\leqq t,\ 0\leqq\lambda\leqq 1, \tag{2.9}$$

and then the convex separating function h can be defined (not necessarily uniquely) by

$$h(u)=\inf\left\{\frac{t-u}{t-s}g(s)+\frac{u-s}{t-s}g(t), s<u\leqq t\right\}.$$

Also, this choice of h, or some simple modification of it, is nonincreasing if and only if

$$g(s)\geqq f(t), \qquad s\leqq t. \tag{2.10}$$

We thus have (1.2) with

$$\begin{aligned}\mathscr{G}=\{(l,k)\colon\, & l(x,y)=g(x^2+y^2),\ k(x,y)=f(x^2+y^2)\colon\\ & g,f \text{ satisfy (2.9) and (2.10)}\}.\end{aligned} \tag{2.11}$$

Theorem 2.1 can be generalized to higher dimensional vectors, and this is done in Theorem 2.2, where the following terminology is used. For $2\leqq n<\infty$, a function f defined on $[0,\infty)$ or $(0,\infty)$ is said to be *n-monotone* if its k^{th} order divided differences are of alternating signs for $1\leqq k\leqq n$, of nonpositive sign for odd k and of nonnegative sign for even k. (Thus $[x_0,x_1;f]$, defined by $(f(x_0)-f(x_1))/(x_0-x_1)$, is nonpositive for distinct x_0 and x_1 in the domain of f; $[x_0,x_1,x_2;f]$, defined by $([x_0,x_1;f]-[x_1,x_2;f])/(x_0-x_2)$, is nonnegative for distinct x_0, x_1, and x_2; etc.) It follows from Theorem A, page 238, of Roberts and Varberg [9] that f is n-monotone iff (i) it is nonincreasing on its domain, (ii) it is $(n-2)$-times continuously differentiable on $(0,\infty)$ with

$$(-1)^k f^{(k)}(s)\geqq 0, \qquad s>0,\ k=1,\dots,n-2,$$

and (iii) $(-1)^{n-2}f^{(n-2)}$ is nonincreasing and convex on $(0,\infty)$. For future reference, we note that (iii) is equivalent to: (iii′) $(-1)^{n-2}f^{(n-2)}$ is locally absolutely continuous with a nonpositive and nondrecreasing (Radon-Nikodym) derivative $(-1)^{n-2}f^{(n-1)}$. A function f defined on $[0,\infty)$ or $(0,\infty)$ is said to be ∞-*monotone* if it is n-monotone for all n, i.e. if f is nonincreasing on its domain, and f is infinitely differentiable on $(0, \infty)$ with $(-1)^k f^k(s) \geqq 0$, $s > 0$, $k \geqq 1$. In Lemma 2.3, an integral representation is obtained for all bounded n-monotone functions defined on $[0, \infty)$, $2 \leqq n \leqq \infty$. A well-known related notion is that of complete monotonicity. A function f defined on $[0, \infty)$ or $(0, \infty)$ is called completely monotone if it is continuous on its domain, and it is infinitely differentiable on $(0, \infty)$ with $(-1)^k f^{(k)}(s) \geqq 0$, $s > 0$, $k \geqq 0$. Thus a completely monotone function is ∞-monotone, and if f is ∞-monotone on $[0, \infty)$ or $(0, \infty)$, then $-f^{(1)}$ is completeley monotone on $(0, \infty)$. Completely monotone functions on $[0, \infty)$ are Laplace transforms of finite measures on $[0, \infty)$, and completely monotone functions on $(0, \infty)$ are Laplace transforms of (not necessarily finite) measures on $[0, \infty)$ for which the Laplace transform is finite on $(0, \infty)$. (See Widder [18].)

Theorem 2.2. *If the random vectors* $Z = (Z_1, \ldots, Z_n)$ *and* $Z' = (Z'_1, \ldots, Z'_n)$, $n \geqq 2$, *satisfy*

$$P(|Z_1| \leqq a_1, \ldots, |Z_n| \leqq a_n) \geqq P(|Z'_1| \leqq a_1, \ldots, |Z'_n| \leqq a_n), \quad a_1 \geqq 0, \ldots, a_n \geqq 0, \tag{2.12}$$

and if f is an n-monotone function on $[0, \infty)$, *then*

$$\mathscr{E} f\left(\sum_1^n Z_i^2\right) \geqq \mathscr{E} f\left(\sum_1^n Z_i'^2\right). \tag{2.13}$$

Remarks. 1. Theorem 2.2 can be viewed as an extension to higher dimensions of a well-known result in one dimension, provided one interprets a 1-monotone function as a nonincreasing function.

2. Some examples of ∞-monotone functions on $[0, \infty)$, to which Theorem 2.2 is applicable are: $e^{-\alpha s} (\alpha > 0)$, $-s^\alpha (0 < \alpha \leqq 1)$, $(s+a)^\alpha (\alpha < 0,\ a > 0)$, $-\log(s+a)(a > 0)$.

3. An example of an n-monotone function which is not $(n+1)$-monotone is the function f defined by $f(s) = ((1-s) \vee 0)^{n-1}$, $s \geqq 0\, (n \geqq 2)$.

Before we prove Theorem 2.2, we must gather together a number of facts, some of which we state in the form of lemmas.

Lemma 2.2. *If h is a bounded function and the random vector* $(V_1, \ldots, V_n)$, $n \geqq 2$, *is uniformly distributed on the surface of the n-dimensional unit sphere, then*

$$\mathscr{E} h\left(\frac{x_1^2}{V_1^2} \vee \ldots \vee \frac{x_n^2}{V_n^2}\right) |V_1 \ldots V_n| = \frac{\Gamma(n/2)}{\pi^{n/2}(n-2)!} \int_1^\infty \frac{(u-1)^{n-2}}{u^n} h(r^2 u)\, du, \tag{2.14}$$

for every real vector $(x_1, \ldots, x_n)$, *where* $r^2 = x_1^2 + \ldots + x_n^2$.

Proof. For $n = 2$, (2.14) is established in the proof of Theorem 2.1. We now assume (2.14) is true when n in (2.14) is replaced by $n-1$, and proceed to establish its validity for $n\, (n \geqq 3)$. Using the facts that V_n has density

$$f_{V_n}(v) = \frac{\Gamma(n/2)}{\pi^{\frac{1}{2}} \Gamma[(n-1)/2]} (1-v^2)^{\frac{n-3}{2}}, \qquad -1 < v < 1,$$

that, conditioned on $V_n=v$, $(1-v^2)^{-\frac{1}{2}}(V_1,\ldots,V_{n-1})$ is uniformly distributed on the $(n-1)$-dimensional unit sphere (see, for unstance, Lemma 2 in [2]), and therefore that (by the induction hypothesis) the conditional expectation of $h[(x_1^2/V_1^2)\vee\ldots\vee(x_n^2/V_n^2)]\,|V_1\ldots V_n|$ given $V_n=v$ is

$$\frac{\Gamma[(n-1)/2]}{\pi^{(n-1)/2}(n-3)!}\,v(1-v^2)^{\frac{n-1}{2}}\int_1^\infty\frac{(u-1)^{n-3}}{u^{n-1}}\,h\left(\frac{(r^2-x_n^2)\,u}{1-v^2}\vee\frac{x_n^2}{v^2}\right)du,$$

we obtain (after some minor simplifications)

$$\begin{aligned}\mathscr{E}h\left(\bigvee_1^n\frac{x_i^2}{V_i^2}\right)\prod_1^n|V_i|&=\int_{-1}^1\mathscr{E}\left(h\left(\bigvee_1^n\frac{x_i^2}{V_i^2}\right)\prod_1^n|V_i|\,\middle|\,V_n=v\right)f_{V_n}(v)\,dv\\&=\frac{\Gamma(n/2)}{\pi^{n/2}(n-3)!}\int_1^\infty\frac{(u-1)^{n-3}}{u^{n-1}}\int_0^1 h\left(\frac{(r^2-x_n^2)\,u}{1-v^2}\vee\frac{x_n^2}{v^2}\right)v(1-v^2)^{n-2}\,dv\,du.\end{aligned}\tag{2.15}$$

With the change of variable $v\to y$: $\dfrac{u(r^2-x_n^2)}{1-v^2}\vee\dfrac{x_n^2}{v^2}=r^2y$, the inner integral in (2.15) simplifies to

$$\frac{u^{n-1}}{r^{2(n-1)}}\int_{\frac{u(r^2-x_n^2)+x_n^2}{r^2}}^\infty h(r^2y)\left\{\frac{x_n^2(r^2y-x_n^2)^{n-2}}{u^{n-1}}+(r^2-x_n^2)^{n-1}\right\}y^{-n}\,dy,$$

and (2.15) becomes

$$\frac{\Gamma(n/2)}{\pi^{n/2}(n-3)!}\,r^{-2(n-1)}\int_1^\infty\frac{h(r^2y)}{y^n}\cdot\int_1^{\frac{r^2y-x_n^2}{r^2-x_n^2}}(u-1)^{n-3}\left\{\frac{x_n^2(r^2y-x_n^2)}{u^{n-1}}+(r^2-x_n^2)^{n-1}\right\}du\,dy.$$

The inner integral equals

$$\begin{aligned}&x_n(r^2y-x_n^2)\frac{1}{n-2}\left(1-\frac{r^2-x_n^2}{r^2y-x_n^2}\right)^{n-2}+(r^2-x_n^2)^{n-1}\frac{1}{n-2}\left(\frac{r^2y-x_n^2}{r^2-x_n^2}-1\right)^{n-2}\\&=\frac{1}{n-2}\,r^{2(n-1)}(y-1)^{n-2},\end{aligned}$$

and thus (2.15) becomes

$$\frac{\Gamma(n/2)}{\pi^{n/2}(n-2)!}\int_1^\infty\frac{(y-1)^{n-2}}{y^n}\,h(r^2y)\,dy.\quad\square$$

By using the same argument used in proving Theorem 2.1, together with Lemma 2.2, one can readily establish (2.13) for functions f defined on $[0,\infty)$ of the form

$$f(s)=\int_1^\infty\frac{(u-1)^{n-2}}{u^n}\,h(us)\,du,\qquad s\geqq 0,\tag{2.16}$$

where h is bounded and nonincreasing. The class of such functions is characterized in Lemma 2.3, which follows.

Lemma 2.3. *A function f on $[0, \infty)$ is bounded n-monotone if and only if it is of the form* (2.16) *with h bounded nonincreasing on* $[0, \infty)$, *or equivalently if and only if it is of the form*

$$f(s)=f(\infty)+\int_{[0,\infty)}\left(1-\frac{s}{u}\right)_+^{n-1} d\mu(u), \qquad s\geqq 0, \tag{2.17}$$

with μ a finite measure on $[0, \infty)$. *Also a function f on* $(0, \infty)$ *is n-monotone on* $(0, \infty)$ *and bounded on* $[1, \infty)$ *if and only if it is of the form*

$$f(s)=s\int_s^\infty \frac{(v-s)^{n-2}}{v^n} h(v)\,dv, \qquad s>0, \tag{2.18}$$

with h nonincreasing on $(0, \infty)$ *and bounded on* $[1, \infty)$, *or equivalently if and only if it is of the form* (2.17) *on* $(0, \infty)$, *with μ a σ-finite measure on* $(0, \infty)$, *which is finite on* $[1, \infty)$.

Proof. We shall prove the second statement. Since the right-hand side of (2.18) is equal to the integral in (2.16) when $s>0$, the first statement is easily inferred from the second.

Suppose f is of the form (2.18) with h nonincreasing on $(0, \infty)$ and bounded on $[1, \infty)$. It is clear from (2.16) that f is bounded on $[1, \infty)$, and from (2.18) that f is $(n-2)$-times continuously differentiable on $(0, \infty)$ with

$$f^{(k)}(s)=(-1)^{k-1}\frac{(n-2)!}{(n-k)!}\int_s^\infty \frac{(v-s)^{n-k-2}}{v^n}[kv-(n-1)s]\,h(v)\,dv,$$
$$s>0,\ 1\leqq k\leqq n-2. \tag{2.19}$$

Also, $f^{(n-2)}$ is locally absolutely continuous with a (Radon-Nikodym) derivative

$$f^{(n-1)}(s)=(-1)^{n-2}(n-1)!\int_s^\infty \frac{h(v)}{v^n}dv+(-1)^{n-1}(n-2)!\,s^{-(n-1)}h(s) \quad \text{a.e. on } (0,\infty),$$

which, after integrating by parts, simplifies to the version we will use:

$$f^{(n-1)}(s)=(-1)^n(n-2)!\int_s^\infty v^{-(n-1)}\,dh(v), \qquad s>0. \tag{2.20}$$

Clearly $f^{(n-1)}(\infty)$ $(=\lim_{s\to\infty} f^{(n-1)}(s))$ exists and equals zero. In fact,

$$s^{k-1}f^{(k)}(s)\to 0 \quad \text{as } s\to\infty,\ 1\leqq k\leqq n-1. \tag{2.21}$$

This is obvious from (2.20) for $k=n-1$ and from (2.18) for $1\leqq k\leqq n-2$. Observe that $(-1)^{n-1}f^{(n-1)}$ is nonincreasing. Since h is nonincreasing, it follows by (2.20) that $(-1)^{n-1}f^{(n-1)}$ is nonnegative. Proceeding by backwards induction: From the nonnegativity of $(-1)^{n-1}f^{(n-1)}$, we infer that $(-1)^{n-2}f^{(n-2)}$ is nonincreasing. Since $f^{(n-2)}(\infty)=0$ (implied by (2.21)),

$(-1)^{n-2}f^{(n-2)}$ is nonnegative, etc. Thus $(-1)^j f^{(j)}$ is nonnegative for $j=1,\dots,n-1$, and $(-1)^{n-1}f^{(n-1)}$ is nonincreasing, which together say that f is n-monotone.

Conversely, suppose f is n-monotone on $(0,\infty)$ and bounded on $[1,\infty)$. Define h on $(0,\infty)$ by

$$h(v)=(n-1)\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\,v^k f^{(k)}(v),\qquad v>0, \tag{2.22}$$

which is nonincreasing because each term in the sum is nonincreasing (a consequence of f being n-monotone). Observe that h is of bounded local variation and

$$dh(v)=\frac{(-1)^{n-1}}{(n-2)!}\,v^{n-1}\,df^{(n-1)}(v),\qquad v>0, \tag{2.23}$$

i.e.

$$h(v)-h(s)=\frac{(-1)^{n-1}}{(n-2)!}\int_s^v u^{n-1}\,df^{(n-1)}(u),\qquad 0<s\leqq v<\infty. \tag{2.24}$$

Then for $s>0$ (using (2.24)),

$$\begin{aligned}
&s\int_s^\infty \frac{(v-s)^{n-2}}{v^n}(h(v)-h(s))\,dv\\
&=\frac{(-1)^{n-1}}{(n-2)!}\,s\int_s^\infty \frac{(v-s)^{n-2}}{v^n}\int_s^v u^{n-1}\,df^{(n-1)}(u)\,dv\\
&=\frac{(-1)^{n-1}}{(n-1)!}\int_s^\infty \{u^{n-1}-(u-s)^{n-1}\}\,df^{(n-1)}(u)\equiv F_{n-1}(s),
\end{aligned}$$

say, where we have applied Fubini's theorem for nonnegative functions (without knowing, as yet, that $F_{n-1}(s)$ is finite). In what follows, we shall need to use (2.21), which should be justified in the present context. This is done in Lemma 2.4 below for the k^{th} derivative of an arbitrary n-monotone function, $2\leqq k\leqq n-1$. The remainder of (2.21), for $k=1$, is valid in the present context since, by assumption, f is bounded on $[1,\infty)$.

Now, using integration by parts and (2.21) for $k=n-1$, we obtain

$$F_{n-1}(s)=-\frac{(-1)^{n-1}}{(n-1)!}\,s^{n-1}f^{(n-1)}(s)+F_{n-2}(s),$$

where

$$F_{n-2}(s)=\frac{(-1)^{n-2}}{(n-2)!}\int_s^\infty \{u^{n-2}-(u-s)^{n-2}\}\,f^{(n-1)}(u)\,du.$$

Proceeding by backwards induction, we are eventually led to

$$s\int_s^\infty \frac{(v-s)^{n-2}}{v^n}(h(v)-h(s))\,dv=f(s)-\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\,s^k f^{(k)}(s),$$

which, in view of (2.22), establishes (2.18). The boundedness of h on $[1, \infty)$ may be inferred from (2.18). The equivalence of the representations (2.18) and (2.17) follows easily by integration by parts, and the measure μ and function h can be expressed in terms of each other as follows: $d\mu(s) = -dh(s)/(n-1)$ on $(0, \infty)$, i.e. μ is the Lebesgue-Stieltjes measure corresponding to (the right continuous version of) $-h(s)/(n-1)$, and $h(s) = (n-1) f(\infty) + \mu\{(s, \infty)\}$, $s > 0$. □

If f is n-monotone on $[0, \infty)$ or $(0, \infty)$, then $(-1)^k f^{(k)}$ is nonnegative and nonincreasing on $(0, \infty)$ for $k = 1, \dots, n-1$, and hence

$$\begin{aligned} 0 &\leqq s(-1)^k f^{(k)}(s) \leqq 2(-1)^k \int_{s/2}^{s} f^{(k)}(u)\, du \\ &= 2(-1)^k \{f^{(k-1)}(s) - f^{(k-1)}(s/2)\}, \qquad s > 0,\ 1 \leqq k \leqq n-1. \end{aligned} \tag{2.25}$$

These inequalities permit us to describe the behavior of the derivatives of f as $s \to \infty$ and $s \downarrow 0$:

Lemma 2.4. *If f is n-monotone on $[0, \infty)$ or $(0, \infty)$ for some $n \geqq 3$, then*

$$s^{k-1} f^{(k)}(s) \to 0 \qquad \text{as } s \to \infty,\ 2 \leqq k \leqq n-1. \tag{2.26}$$

Proof. This follows by induction from (2.25), provided (corresponding to $k-1 = 1$) $f^{(1)}(s) - f^{(1)}(s/2) \to 0$. But this is the case since $f^{(1)}$ is nonpositive and nondecreasing. □

Lemma 2.5. *If f is n-monotone on $[0, \infty)$, or n-monotone on $(0, \infty)$ with $f(0+)$ finite, for some $n \geqq 2$, then*

$$s^k f^{(k)}(s) \to 0 \qquad \text{as } s \downarrow 0,\ 1 \leqq k \leqq n-1. \tag{2.27}$$

Proof. In either case, $f(0+)$ exists and is finite. Thus $f(s) - f(s/2) \to 0$ as $s \downarrow 0$, and (2.21) follows from (2.25) by induction. □

Proof of Theorem 2.2. In view of the remark preceding Lemma 2.3, we may take (2.13) to be established for all bounded n-monotone functions f on $[0, \infty)$. The proof for unbounded f requires the removal from f of its (possible) linear part and then a truncation argument.

Suppose first that $f(s) = -cs$, $s \geqq 0$, where $c > 0$. Inequality (2.8) can be expressed as $\sum_1^n \mathscr{E} Z_i^2 \leqq \sum_1^n \mathscr{E} Z_i'^2$, which must hold since (2.12) implies Z_i^2 is stochastically smaller than $Z_i'^2$, $i = 1, \dots, n$.

Now suppose f is any unbounded n-monotone function on $[0, \infty)$, so that $f(\infty) = -\infty$. Since $f^{(1)}$ is nonpositive and nondecreasing, the finite nonpositive limit $f^{(1)}(\infty)$ exists. We shall assume, without loss of generality, that $f^{(1)}(\infty) = 0$. For otherwise, we may express f as the sum of two n-monotone functions, $f = f_1 + f_2$ where $f_1^{(1)}(\infty) = 0$ and $f_2(s) = f^{(1)}(\infty) \cdot s$, $s \geqq 0$, and treat the parts independently. We shall truncate f as follows: Define h on $(0, \infty)$ by (2.22) and $h(0) = (n-1) f(0)$. For $x > 0$, let $h_x(v) = h(v \wedge x)$, $v \geqq 0$, and define f_x by (see (2.16))

$$f_x(s) = \int_1^\infty \frac{(u-1)^{n-2}}{u^n} h_x(us)\, du, \qquad s \geqq 0. \tag{2.28}$$

Since h is nonincreasing on $(0, \infty)$ (see (2.23)) and $h(0+)=(n-1)f(0+)\leqq(n-1)f(0)=h(0)$ (cf. Lemma 2.5), it follows that h is nonincreasing on $[0, \infty)$ and, consequently, h_x is a bounded nonincreasing function on $[0, \infty)$ for every $x>0$. This implies that (2.13) holds for each function f_x, and it only remains to show, if possible, that $f_x \downarrow f$ as $x\to\infty$ (so that (2.13) follows for f itself by the monotone convergence theorem). Since h_x is nonincreasing in x, so is f_x (apparent from (2.28)), and thus it is only necessary to show the pointwise convergence of f_x to f.

From (2.28), we have $f_x(0)=(n-1)h(0)=f(0)$ for all $x>0$. Thus we may focus our attention exclusively on points $s>0$. For such points, it is more convenient to use the following variant of (2.28) (see (2.18)):

$$f_x(s)=s\int_s^\infty \frac{(v-s)^{n-2}}{v^n} h_x(v)\,dv, \qquad s>0.$$

For $x>s>0$, we have (using (2.24))

$$\begin{aligned}
f_x(s)&=s\int_s^x \frac{(v-s)^{n-2}}{v^n} h(v)\,dv+s\int_s^\infty \frac{(v-s)^{n-2}}{v^n}\,dv\cdot h(x)\\
&=s\int_s^x \frac{(v-s)^{n-2}}{v^n}\left[\frac{(-1)^{n-1}}{(n-2)!}\int_s^v u^{n-1}\,df^{(n-1)}(u)\right]dv\\
&\quad+\frac{h(x)}{(n-1)}\left[1-\left(1-\frac{s}{x}\right)^{n-1}\right]+\frac{h(s)}{n-1}\left(1-\frac{s}{x}\right)^{n-1}\\
&=\frac{(-1)^{n-1}}{(n-1)!}\int_s^x u^{n-1}\left[\left(1-\frac{s}{x}\right)^{n-1}-\left(1-\frac{s}{u}\right)^{n-1}\right]df^{(n-1)}(u)\\
&\quad+\frac{h(x)}{n-1}\left[1-\left(1-\frac{s}{x}\right)^{n-1}\right]+\frac{h(s)}{n-1}\left(1-\frac{s}{x}\right)^{n-1}\\
&=-\frac{(-1)^{n-1}}{(n-1)!}\int_s^x (u-s)^{n-1}\,df^{(n-1)}(u)+\frac{h(x)}{n-1}.
\end{aligned}$$

By repeatedly integrating by parts (much as in the proof of Lemma 2.3), we obtain

$$f_x(s)=f(s)+\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}f^{(k)}(x)\{x^k-(x-s)^k\}, \qquad x>s>0.$$

Since, as $x\to\infty$, $f^{(1)}(x)\to f^{(1)}(\infty)=0$ (assumed without loss of generality), it follows from Lemma 2.4 that the sum converges to zero as $x\to\infty$. Thus $f_x(s)\to f(s)$ as $x\to\infty$, which completes the proof. □

Theorem 2.2 can be extended to n-monotone functions on $(0, \infty)$, to allow for functions which are unbounded at zero as well as at infinity.

Corollary 2.2. *If, in addition to* (2.12), $P(Z_1=0,\dots,Z_n=0)=0$, *then* ($P(Z'_1=0,\dots,Z'_n=0)=0$ *and*) (2.13) *holds for each n-monotone function f on* $(0, \infty)$ *for which the expectations in* (2.13) *are defined.*

Proof. Let f be n-monotone on $(0, \infty)$ with $f(0+)=\infty$ and $f(\infty)=-\infty$. (Functions f with smaller ranges can be handled similarly or more easily.) Let

s_0 be the zero of f, $f(s_0)=0$, and for each $k>(2/s_0)$ define $f_k(s)=f\left(s+\frac{1}{k}\right)$, $s\geqq 0$. Then each f_k is n-monotone on $[0,\infty)$, and by Theorem 2.2, $\mathscr{E} f_k(\|Z\|^2)\geqq \mathscr{E} f_k(\|Z'\|^2)$. Also as $k\uparrow\infty$, $f_k\uparrow f$ on $(0,\infty)$. More precisely, $f_k^+\uparrow f^+$ and by monotone convergence $\mathscr{E} f_k^+(\|Z\|^2)\uparrow \mathscr{E} f^+(\|Z\|^2)$. Also for $s\geqq s_0-\frac{1}{k}$, since $0\leqq -f^{(1)}\downarrow$, we have

$$0\leqq f_k^-(s)-f^-(s)\leqq -f\left(s+\frac{1}{k}\right)+f(s)=-\int_s^{s+\frac{1}{k}} f^{(1)}(u)\,du$$
$$\leqq -f^{(1)}\left(s_0-\frac{1}{k}\right)\frac{1}{k}\leqq \frac{1}{k}\left|f^{(1)}\left(\frac{s_0}{2}\right)\right|,$$

and thus $0\leqq f_k^-(s)-f^-(s)\leqq \frac{1}{k}\left|f^{(1)}\left(\frac{s_0}{2}\right)\right|$, $s\geqq 0$. It follows that $\mathscr{E} f_k^-(\|Z\|^2)$ and $\mathscr{E} f^-(\|Z\|^2)$ are finite or infinite together and thus $\mathscr{E} f_k^-(\|Z\|^2)\downarrow \mathscr{E} f^-(\|Z\|^2)$ (by dominated convergence if they are finite or trivially if they are infinite). Since $\mathscr{E} f(\|Z\|^2)$ is defined by $\mathscr{E} f^+(\|Z\|^2)-\mathscr{E} f^-(\|Z\|^2)$ iff at least one of the two terms if finite, (2.13) follows. □

Some examples of an ∞-monotone function, to which Corollary 2.2 is applicable, are: $s^\alpha(\alpha<0)$, $-\log s$.

We have already shown that the 2-monotone functions provide the appropriate class for the result of Theorem 2.1. The following example shows that 3-monotone functions provide the appropriate class for the result of Theorem 2.2 for $n=3$ (by constructing, for a 2-monotone function which is not 3-monotone, 3-dimensional random vectors Z and Z' which satisfy (2.12) and for which (2.13) fails), and we anticipate that similar examples would show the same for $n>3$.

Example. Suppose f is 2-monotone but not 3-monotone on $[0,\infty)$. Then for some a and b, $a\geqq 0$, $b>0$, one has

$$f(a+3b)-3f(a+2b)+3f(a+b)-f(a)>0. \tag{2.29}$$

(Implicitly we are saying that functions f which are 2-monotone and satisfy the converse of (2.29) for all a and b are 3-monotone, which can be verified.) Let $3\alpha^2=a$ and $2\alpha^2+\beta^2=a+b$ be used to define α and β $(0\leqq\alpha<\beta)$, and let Z and Z' be three-dimensional random vectors whose distributions are described in Table 1. From the last two columns of Table 1, it is apparent that condition (2.12) holds. Now for $R^2=ZZ^t$ and $R'^2=Z'Z'^t$, we have

$$\mathscr{E} f(R^2)=\tfrac{11}{81}f(a)+\tfrac{12}{81}f(a+b)+\tfrac{24}{81}f(a+2b)+\tfrac{34}{81}f(a+3b),$$
$$\mathscr{E} f(R'^2)=\tfrac{9}{81}f(a)+\tfrac{18}{81}f(a+b)+\tfrac{18}{81}f(a+2b)+\tfrac{36}{81}f(a+3b).$$

From (2.29), it follows that $\mathscr{E} f(R'^2)>\mathscr{E} f(R^2)$. Consequently, the assumption of 3-monotonicity in Theorem 2.2 when Z and Z' are three-dimensional is essential; it is impossible to consider a larger class of functions. □

Table 1

z	P(Z=z)	P(Z′=z)	P(Z≤z)	P(Z′≤z)
(α,α,α)	$\frac{11}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{9}{81}$
(α,α,β)	$\frac{4}{81}$	$\frac{6}{81}$	$\frac{15}{81}$	$\frac{15}{81}$
(α,β,α)	$\frac{4}{81}$	$\frac{6}{81}$	$\frac{15}{81}$	$\frac{15}{81}$
(β,α,α)	$\frac{4}{81}$	$\frac{6}{81}$	$\frac{15}{81}$	$\frac{15}{81}$
(α,β,β)	$\frac{8}{81}$	$\frac{6}{81}$	$\frac{27}{81}$	$\frac{27}{81}$
(β,α,β)	$\frac{8}{81}$	$\frac{6}{81}$	$\frac{27}{81}$	$\frac{27}{81}$
(β,β,α)	$\frac{8}{81}$	$\frac{6}{81}$	$\frac{27}{81}$	$\frac{27}{81}$
(β,β,β)	$\frac{34}{81}$	$\frac{36}{81}$	1	1

We have thus shown in Theorem 2.2 that when $n \geqq 2$, (C1) $\Rightarrow$ (C2) with $\mathscr{A}$ the class of all closed symmetric rectangles in $\mathbb{R}^n$ and

$$\mathscr{F}=\{k(x_1,\ldots,x_n)=f(x_1^2+\ldots+x_n^2),\ f\colon[0,\infty)\to\mathbb{R}^1\ n\text{-monotone}\}.$$

As in the case of Theorem 2.2, using the separation approach, we obtain (C1) $\Rightarrow$ (C2) $\Leftrightarrow$ (C3) with $\mathscr{G}=\{(l,k)\colon l(x_1,\ldots,x_n)=g(x_1^2+\ldots+x_n^2),\ k(x_1,\ldots,x_n)=f(x_1^2+\ldots+x_n^2),\ g\geqq h\geqq f,\ h\colon[0,\infty)\to\mathbb{R}^1\ n\text{-monotone}\}$, but we have not succeeded in obtaining a direct description of the class $\mathscr{G}$.

3. Normal and Elliptically Contoured Distributions

In this section, we refer to the notation of Sect. 1 and consider two-dimensional random vectors $Z=(X,Y)$ and $Z'=(X',Y')$ with common marginal distributions and with bivariate distribution functions H and H', and we take $\mathscr{A}=\{(-\infty,x]\times(-\infty,y],\ x,y\in\mathbb{R}^1\}$. Then (C1) $\Leftrightarrow$ $H\geqq H'$, and it is shown in [3, 16] that (C1) $\Leftrightarrow$ (C2) with $\mathscr{F}$ *the class of all quasi-monotone functions* (cf. (1.4)) for which the expectations in (C2) are defined and which satisfy certain minor regularity conditions (see Theorem 1 in [3]). The separation approach yields (1.3) with $\mathscr{G}_{\mathscr{F}}$ defined by (1.1) as the class of all pairs of functions l,k which can be separated by a quasi-monotone function m: $l=m+f$, $k=m-g$, where $f,g\geqq 0$. While $(l,k)\in\mathscr{G}_{\mathscr{F}}$ implies (1.5), the converse is not generally true. *There exist functions* l,k *satisfying* (1.5) *which are sufficiently close that no quasi-monotone function can exist between them. Also,* (C1) *in general does not imply* (C3) *with*

$$\mathscr{G}=\{(l,k)\colon \text{(1.5) is satisfied, and the expectations appearing in (C3) make sense}\}. \tag{3.1}$$

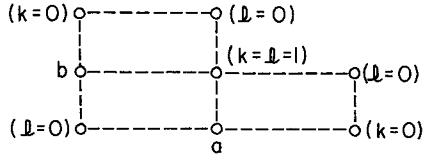

Fig. 1

Proof. Define k as -10, say, and l as 10 at all points in the plane except for those points in an array as shown in Fig. 1 for which an explicit definition is given. Then (1.5) is satisfied, and it is easily checked that it is impossible to define a quasi-monotone function m at the points a and b satisfying $l \geqq m \geqq k$. Now let the distribution of Z (respectively Z') assign mass 1/3 to each of the points in Figure 1 at which $l=0$ (respectively, k is explicitly defined). It is easily checked that (C1) holds, but $\mathscr{E}l(Z) < \mathscr{E}k(Z')$. □

We now show that in certain special cases (C1) $\Leftrightarrow$ (C3) with $\mathscr{G}$ defined in (3.1). This is accomplished by using certain surrogate random variables which satisfy the properties described in the following condition.

(C0) There exists a four-dimensional random vector (X_1, X_2, Y_1, Y_2) whose values are in the set $F = \{(x_1, x_2, y_1, y_2)\colon (x_2 - x_1)(y_2 - y_1) \geqq 0\}$ and whose bivariate marginals $F_{11}, F_{12}, F_{21}, F_{22}$ for (X_1, Y_1), (X_1, Y_2), (X_2, Y_1), (X_2, Y_2) respectively satisfy $F_{11} + F_{22} = 2H$ and $F_{12} + F_{21} = 2H'$, where H and H' are the distribution functions of Z and Z' respectively.

When (1.5) holds, condition (C0) implies

$$l(X_1, Y_1) + l(X_2, Y_2) \geqq k(X_1, Y_2) + k(Y_2, X_1) \quad \text{a.s.,} \tag{3.2}$$

which, upon taking expectations (assuming they are defined), yields (C3). Hence (C0) $\Rightarrow$ (C3), and the latter can be established by constructing surrogate random variables satisfying (C0), i.e. by using the surrogate approach. We begin with the normal case.

Theorem 3.1. *Suppose Z and Z' are bivariate normal random variables with common means and variances and with correlation coefficients ρ and ρ'. Then*

$$\rho \geqq \rho' \Leftrightarrow H \geqq H' \Leftrightarrow \text{(C0)} \Leftrightarrow \text{(C2)} \Leftrightarrow \text{(C3)}. \tag{3.3}$$

Proof. Assume $\rho \geqq \rho'$ and let (S, T, U) be normally distributed with a zero mean vector and the covariance matrix

$$\begin{pmatrix} 1 & \rho & (\rho'-\rho) \\ \rho & 1 & (\rho'-\rho) \\ (\rho'-\rho) & (\rho'-\rho) & 2(\rho-\rho') \end{pmatrix}.$$

Define

$$X_1 = \mu_x + \sigma_x S, \qquad Y_1 = \mu_y + \sigma_y T, \qquad X_2 = \mu_x + \sigma_x(S+U), \qquad Y_2 = \mu_y + \sigma_y(T+U),$$

where (μ_x, μ_y) is the common mean vector, and σ_x^2 and σ_y^2 are the common variances of H and H'. It is easily checked that (X_1, Y_1) and (X_2, Y_2) have distribution function H and that (X_1, Y_2) and (X_2, Y_1) have distribution function H'. Moreover, $(X_2 - X_1)(Y_2 - Y_1) = \sigma_x \sigma_y U^2 \geqq 0$. Thus condition (C0) is satisfied. It follows that $\rho \geqq \rho' \Rightarrow$ (C0), and since (C0) $\Rightarrow$ (C3) $\Rightarrow$ (C2) $\Rightarrow$ (C1) $\Leftrightarrow H \geqq H'$, the result follows. □

The implication $\rho \geqq \rho' \Rightarrow H \geqq H'$ is a special case of an n-dimensional result due to Slepian [12]. The equivalence $H \geqq H' \Leftrightarrow$ (C2) is a slight improvement in a special case of Theorem 1 in [3], in that no regularity conditions are imposed beyond existence of expectations. The equivalences in Theorem 3.1 involving (C0) and (C3) are novel results.

Theorem 3.1 can be extended to higher dimensions and from normal to elliptically contoured distributions. If Z is an n-dimensional random (row) vector and, for some n-(row)-vector μ and some $n \times n$ nonnegative definite matrix Σ, the characteristic function $\phi_{Z-\mu}(s)$ of $Z - \mu$ is a function of the quadratic form $s \Sigma s^t$, $\phi_{Z-\mu}(s) = \phi(s \Sigma s^t)$, we say that Z has an elliptically contoured distribution with parameters μ, Σ and ϕ, and we write $Z \sim EC_n(\mu, \Sigma, \phi)$. When $\phi(u) = \exp(-u/2)$, $EC_n(\mu, \Sigma, \phi)$ is the normal distribution $N_n(\mu, \Sigma)$. The class of admissible fuctions ϕ depends on the rank k of Σ, $r(\Sigma) = k$, and consists of all functions of the form

$$\phi(u) = \int_{[0,\infty)} \Omega_k(r^2 u)\, dF(r), \qquad u \geqq 0,$$

for some distribution function F on $[0, \infty)$, where $\Omega_k(\|s\|^2)$, $s \in \mathbb{R}^k$, is the characteristic function of the uniform distribution on the surface of the unit sphere of $\mathbb{R}^k$. This follows from a theorem of Schoenberg [11] and is discussed in [2] where the following useful stochastic representation is also introduced. Let $\Sigma = A^t A$ be a rank factorization of Σ, i.e. A is $k \times n$ and $r(\Sigma) = k = r(A)$. Then Z has the stochastic representation

$$Z \stackrel{d}{=} \mu + R U^{(k)} A,$$

where the equality is in distribution, R is a nonnegative random variable (with distribution F), $U^{(k)}$ is a k-dimensional random vector uniformly distributed on the surface of the unit sphere in $\mathbb{R}^k$, and R and $U^{(k)}$ are independent.

Theorem 3.2. *Suppose that* $Z \sim EC_2(\mu, \Sigma, \phi)$ *and* $Z' \sim EC_2(\mu, \Sigma', \phi)$, *where*

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}, \qquad \Sigma' = \begin{pmatrix} \sigma_1^2 & \rho' \sigma_1 \sigma_2 \\ \rho' \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}.$$

Then (3.3) *is satisfied.*

Proof. As in the proof of Theorem 3.1, it suffices to show $\rho \geqq \rho' \Rightarrow$ (C0). If $\rho = 1$ and $\rho' = -1$, we have $Z \stackrel{d}{=} \mu + R U^{(1)} A_1$, $Z' \stackrel{d}{=} \mu + R U^{(1)} A_1'$, where $A_1 = (\sigma_1, \sigma_2)$, $A_1' = (\sigma_1, -\sigma_2)$. Since R and $U^{(1)}$ are independent, in order to show $\mathscr{E} l(Z) \geqq \mathscr{E} k(Z')$, it suffices to show

$$\mathscr{E} l(\mu + r U^{(1)} A_1) \geqq \mathscr{E} k(\mu + r U^{(1)} A_1'), \qquad r \geqq 0. \tag{3.4}$$

Since $k(\cdot)$ and $l(\cdot)$ satisfy (1.5), so do $k(\mu+r\cdot)$ and $l(\mu+r\cdot)$ for every μ and $r\geqq 0$. Thus it suffices to show (3.4) when $\mu=0$ and $r=1$, i.e. it suffices to show $\mathscr{E}\, l(U^{(1)}A_1)\geqq \mathscr{E}\, k(U^{(1)}A_1')$, which is written as

$$\tfrac{1}{2}l(-\sigma_1,-\sigma_2)+\tfrac{1}{2}l(\sigma_1,\sigma_2)\geqq \tfrac{1}{2}k(-\sigma_1,\sigma_2)+\tfrac{1}{2}k(\sigma_1,-\sigma_2)$$

and follows from (1.5).

Now assume that at least one of ρ,ρ' differs from 1 in absolute value. Putting

$$A=\begin{pmatrix}\sigma_1\cos\alpha & \sigma_2\sin\alpha\\ \sigma_1\sin\alpha & \sigma_2\cos\alpha\end{pmatrix},\qquad A'=\begin{pmatrix}\sigma_1\cos\alpha' & \sigma_2\sin\alpha'\\ \sigma_1\sin\alpha' & \sigma_2\sin\alpha'\end{pmatrix},$$

where α and α', $-\pi/4\leqq\alpha'\leqq\alpha\leqq\pi/4$, are defined by $\rho=\sin 2\alpha$ and $\rho'=\sin 2\alpha'$, we have $\Sigma=A^tA$ and $\Sigma'=A'^tA'$. When $-1<\rho'<\rho<1$, then both Σ,Σ' are full rank and $r(A)=2=r(A')$, so that $\Sigma=A^tA$, $\Sigma'=A'^tA'$ are rank factorizations of Σ,Σ'. It then follows that

$$Z\overset{d}{=}\mu+RU^{(2)}A,\qquad Z'\overset{d}{=}\mu+RU^{(2)}A'. \tag{3.5}$$

When one, but not both, of ρ,ρ' equals 1 in absolute value, say $-1<\rho'<\rho=1$, then $Z'\overset{d}{=}\mu+RU^{(2)}A'$, and $\mathscr{E}\exp[is(Z'-\mu)^t]=\phi(s\Sigma's^t)=\int_{[0,\infty)}\Omega_2(r^2s\Sigma's^t)\,dF(r)$, where F is a distribution of R. Since

$$\mathscr{E}\exp[is(Z-\mu)^t]=\phi(s\Sigma s^t)=\int_{[0,\infty)}\Omega_2(r^2s\Sigma s^t)\,dF(r),$$

it is easily checked that $Z\overset{d}{=}\mu+RU^{(2)}A$. Hence (3.5) holds, provided at least one of ρ,ρ' differs from 1 in absolute value. Because of the independence of $R, U^{(2)}$, arguing as before, it suffices to show that $\mathscr{E}\, l(U^{(2)}A)\geqq\mathscr{E}\, k(U^{(2)}A')$. This will be done by defining a random vector (X_1,X_2,Y_1,Y_2) which satisfies condition (C0); (X_1,Y_2) and (X_2,Y_2) will be distributed as $U^{(2)}A$, (X_1,Y_2) and (X_2,Y_1) will be distributed as $U^{(2)}A'$, and the product $(X_2-X_1)(Y_2-Y_1)$ will be nonnegative.

The random vector $U^{(2)}$ can be taken to be $(\sin\theta,\cos\theta)$, where θ is uniformly distributed on any interval of length 2π. Then

$$U^{(2)}A=(\sigma_1\sin(\theta+\alpha),\sigma_2\cos(\theta-\alpha)),\qquad U^{(2)}A'=(\sigma_1\sin(\theta+\alpha'),\sigma_2\cos(\theta-\alpha')).$$

Let

$$S=\sin(\theta-\alpha),\qquad T=\cos(\theta-\alpha),\qquad V=\sin(\alpha'-\theta),$$

and define

$$X_1=\sigma_1 S,\qquad X_2=\sigma_1(S+2\cos(\alpha+\alpha')V),$$
$$Y_1=\sigma_2 T,\qquad Y_2=\sigma_2(T+2\sin(\alpha-\alpha')V).$$

Since $2\sin(\alpha-\alpha')\cos(\alpha+\alpha')=\sin 2\alpha-\sin 2\alpha'=\rho-\rho'\geqq 0$, it follows that $(X_2-X_1)(Y_2-Y_1)\geqq 0$. By further trigonometric manipulations, one obtains

$$S+2\cos(\alpha+\alpha')V=\sin((2\alpha'-\theta)+\alpha),\qquad T+2\sin(\alpha-\alpha')V=\cos((2\alpha'-\theta)-\alpha).$$

Since $2\alpha' - \theta$ is uniformly distributed on an interval of length 2π, (X_2, Y_2) as well as (X_1, Y_1) is distributed as $U^{(2)}A$. Similarly, one finds that (X_1, Y_2) and (X_2, Y_1) are distributed as $U^{(2)}A'$. This completes the proof. □

Remark. The success of this proof depends implicitly on an interesting geometric fact about any two ellipses which are inscribed in the same rectangle. Each point on one of the ellipses is the vertex of a rectangle whose opposite vertex is on the same ellipse and whose adjacent vertices are on the other ellipse. (In fact, there are two such rectangles.) Whether this is a known fact of projective geometry is unknown to the authors. (In the present context, the two ellipses are the ranges of the random vectors $U^{(2)}A$ and $U^{(2)}A'$.)

Similar comments to those following the proof of Theorem 3.1 are applicable to Theorem 3.2. A higher dimensional version of Theorem 3.2 can be proven by reducing the dimension to 2 through conditioning.

Theorem 3.3. *Suppose* $Z \sim EC_n(\mu, \Sigma, \phi)$ *and* $Z' \sim EC_n(\mu, \Sigma', \phi)$, *where* $\Sigma = (\sigma_{ij})$, $\Sigma' = (\sigma'_{ij})$, *and* $\sigma'_{ii} = \sigma_{ii}$ *for* $1 \leqq i \leqq n$. *Then*

$$\{\sigma_{ij} \geqq \sigma'_{ij},\ i \neq j\} \Leftrightarrow H \geqq H' \Leftrightarrow (\mathrm{C1}) \Leftrightarrow (\mathrm{C2}) \Leftrightarrow (\mathrm{C3}), \tag{3.6}$$

where $\mathscr{A} = \{(-\infty, z],\ z \in \mathbb{R}^n\}$, $\mathscr{G}$ *is the class of all pairs* (l, k) *of functions of n variables for which the expectations appearing in* (C3) *are defined and which satisfy* (1.5) *as functions of any two of the n variables for all fixed values of the remaining* $n-2$ *variables, and* $\mathscr{F} = \mathscr{F}_{\mathscr{G}}$.

Proof. The main task (and the only one we shall address) is that of showing the first condition in (3.6) implies the last. According to the argument in the first paragraph of the proof of Theorem 5.1 in [4], it suffices to prove it in the case where $\sigma_{12} > \sigma'_{12}$ and $\sigma_{ij} = \sigma'_{ij}$ for all $(i,j) \neq (1,2), (2,1)$. Write

$$Z = (Z_1, Z_2), \qquad \mu = (\mu_1, \mu_2), \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where Z_1, μ_1 are two-dimensional and Σ_{11} is 2×2. It is easily seen via the characteristic function that

$$(Y_1, Y_2) = (Z_1 - \mu_1, Z_2 - \mu_2) \begin{pmatrix} I & 0 \\ -\Sigma_{22}^{+}\Sigma_{21} & I \end{pmatrix} \sim EC_n\left(0, \begin{pmatrix} \Sigma^* & 0 \\ 0 & \Sigma_{22} \end{pmatrix}, \phi\right),$$

where Σ_{22}^{+} is the self-adjoint generalized inverse of Σ_{22} and $\Sigma^* = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{+}\Sigma_{21}$. Now let $RU^{(n)} = R(U_1, U_2) \sim EC_n(0, I, \phi)$, where U_1 is two-dimensional. Then

$$(Y_1, Y_2) \stackrel{d}{=} R(U_1, U_2) \begin{pmatrix} \Sigma^* & 0 \\ 0 & \Sigma_{22} \end{pmatrix}^{\frac{1}{2}} = R(U_1 \Sigma^{*\frac{1}{2}}, U_2 \Sigma_{22}^{\frac{1}{2}})$$

and

$$Z = (Z_1, Z_2) \stackrel{d}{=} (\mu_1 + RU_2\Sigma_{22}^{\frac{1}{2}}\Sigma_{22}^{+}\Sigma_{12} + RU_1\Sigma^{*\frac{1}{2}},\ \mu_2 + RU_2\Sigma_{22}^{\frac{1}{2}}).$$

Since $U^{(n)}$ is uniformly distributed on the surface of the n-dimensional unit sphere, $(U_1 | U_2 = u_2) \stackrel{d}{=} [1 - u_2 u_2^t]_+^{\frac{1}{2}} U^{(2)}$, where $[a]_+ = \max(a, 0)$. (See, for instance, Lemma 2 in [2].) Since R and V are independent, it follows that for all $r \geqq 0$ and u_2,

$$((Z_1, Z_2)|R=r, U_2=u_2) \stackrel{d}{=} (\mu^* + r^* U^{(2)} \Sigma^{*\frac{1}{2}}, \mu_2 + r u_2 \Sigma_{22}^{\frac{1}{2}}), \tag{3.7}$$

where

$$\mu^* = \mu_1 + r u_2 \Sigma_{22}^{\frac{1}{2}} \Sigma_{22}^{+} \Sigma_{12}, \qquad r^* = r[1 - u_2 u_2^t]_+^{\frac{1}{2}}, \qquad \Sigma^* = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{+} \Sigma_{21}.$$

Since $\Sigma' = \begin{pmatrix} \Sigma'_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, $Z' = (Z'_1, Z'_2)$ satisfies (3.7) with $\mu^{*\prime} = \mu^*$, $r^{*\prime} = r^*$, and $\Sigma^{*\prime} = \Sigma'_{11} - \Sigma_{12} \Sigma_{22}^{+} \Sigma_{21}$. In order to verify $\mathscr{E} l(Z) \geqq \mathscr{E} k(Z')$, it thus suffices to show that for all $r \geqq 0$ and u_2,

$$\mathscr{E} l(\mu^* + r^* U^{(2)} \Sigma^{*\frac{1}{2}}, \mu_2 + r u_2 \Sigma_{22}^{\frac{1}{2}}) \geqq \mathscr{E} k(\mu^* + r^* U^{(2)} \Sigma^{*\prime\frac{1}{2}}, \mu_2 + r u_2 \Sigma_{22}^{\frac{1}{2}}),$$

and this follows from Theorem 3.2 □

The implication $\{\sigma_{ij} \geqq \sigma'_{ij}\} \Rightarrow H \geqq H'$ is a well-known result due to Slepian [12] for normal distributions, and for elliptically contoured distributions, it has been obtained by Das Gupta et al. [4] under the assumptions that the matrices Σ, Σ' are invertible and that densities exist.

The approach used in proving Theorem 3.3 can be used to extend the theorem to random vectors Z and Z' with more general distributions than elliptically contoured. For instance, the theorem holds for random variables with distributions $Z \stackrel{d}{=} \mu + U^{(k)} A R$ and $Z' \stackrel{d}{=} \mu + U^{(k)} A' R$, where R is any random matrix with nonnegative components, which is independent of $U^{(k)}$.

4. More on the Surrogate Approach

It is possible to characterize the bivariate distribution functions H and H' which satisfy condition (C0). (Here, we freely refer to the notation of Sects. 1 and 3.) This is accomplished by a straightforward generalization of the proof given by Sudakov [15] of a theorem by Strassen [14]. Cast in our context, this slight extension of Strassen's theorem says that

$$(\mathrm{C0}) \Leftrightarrow (\mathrm{C1'}) \Leftrightarrow (\mathrm{C3}),$$

where $\mathscr{G}$ is defined by (3.1) and

(C1′) $P(Z \in A) \geqq P(Z' \in A')$, $\quad (A, A') \in \mathscr{B} = \{(A, A'): 1_A, 1_{A'} \text{ satisfy } (1.5)\}$

(with $l = 1_A$ and $k = 1_{A'}$).

Regarding (C1′), it should be noted that $(A, A') \in \mathscr{B}$ if and only if $A \supset A'$, and for all $x_1 < x_2$, $y_1 < y_2$, and $a = (x_1, y_1)$, $b = (x_2, y_1)$, $c = (x_2, y_2)$, $d = (x_1, y_2)$, the number of points of $\{a, c\}$ in A is larger than or equal to the number of points $\{b, d\}$ in A'. By choosing various such pairs of sets, one finds that

$$(\mathrm{C1'}) \Rightarrow \{H, H' \text{ have common marginals and } H \geqq H'\}.$$

The converse is also true in the case of normal or elliptically contoured distribution functions H and H', as shown in Theorems 3.1 and 3.2, while the case of more general distributions requires further investigation.

The equivalence of (C0), (C1′), and (C3) is a special case of a more general situation, which provides new ways of obtaining inequalities of the type (C3) and of showing the existence of joint distributions with fixed support and with certain marginals fixed. To illustrate the power and novelty of this approach, let us consider a few examples, which have obvious $2n$-dimensional analogues.

Let F be a closed subset of $\mathbb{R}^4$ and consider an inequality between functions k and l of the following type (simpler than (1.5))

(CF) $l(x_1, y_1) \geqq k(x_2, y_2), \quad (x_1, x_2, y_1, y_2) \in F,$

and the following conditions which depend on F.

(C0F) There exists a random vector (X_1, X_2, Y_1, Y_2) whose values are in the set F and which is such that the bivariate marginal distribution functions of (X_1, Y_1) and (X_2, Y_2) are H and H' respectively.

(C1F) $P(Z \in A) \geqq P(Z' \in A'), \quad (A, A') \in \mathscr{B} = \{(A, A') \colon 1_A, 1_{A'} \text{ satisfy (CF)}\}.$

(C3F) $\mathscr{E} l(Z) \geqq \mathscr{E} k(Z'),$

$(l, k) \in \mathscr{F}_2 = \{(l, k) \colon l, k \text{ satisfy (CF), and the expectations are defined}\}.$

By Strassen's theorem, (C0F) $\Leftrightarrow$ (C1F), and as before, (C0F) $\Rightarrow$ (C3F) $\Rightarrow$ (C1F). Thus

$$\text{(C0F)} \Leftrightarrow \text{(C1F)} \Leftrightarrow \text{(C3F)}.$$

Example 4.1. When $F = \{x_1 \geqq x_2,\ y_1 \geqq y_2\}$, (CF) becomes $l(x_1, y_1) \geqq k(x_2, y_2)$, $x_1 \geqq x_2$, $y_1 \geqq y_2$; (C1F) is equivalent to $\{H(I) \geqq H'(I)$ for all increasing sets $I\}$; and the result includes Theorem 1(i), (iv), and (vi) of Kamae, Krengel and O'Brien [5]. (Here and below, the distribution *functions* H and H' are treated as if they were distributions.) Of course, any one or both of the inequalities in the definition of F could be reversed with corresponding results.

Example 4.2. When $F = \{|x_1| \geqq |x_2| \text{ or } |y_1| \geqq |y_2|\}$, then (C1F) becomes equivalent to

$$H(A) \leqq H'(A), \quad \text{for all rectangles, } A = [-a, a] \times [-b, b], \tag{4.1}$$

and the corresponding conditions (C0F), (C3F), and (4.1) are equivalent. When H and H' are absolutely continuous elliptically contoured distributions $EC(0, \Sigma, \phi)$ and $EC_2(0, \Sigma', \phi)$, where Σ, Σ' are as in Theorem 3.2 with $|\rho| \leqq |\rho'|$, then (4.1) is Theorem 2.1 of [4], and thus (C0F) and (C3F) hold, both new results.

Example 4.3. When $F = \{(x_1, y_1) \overset{m}{\geqq} (x_2, y_2)\}$ where for two-dimensional vectors, $(x_1, y_1) \overset{m}{\geqq} (x_2, y_2)$ means $\max(x_1, y_1) \geqq \max(x_2, y_2)$ and $x_1 + y_1 = x_2 + y_2$, then (C1F) becomes equivalent to

$$H(A) \geqq H'(A) \quad \text{for all measurable Schur-convex sets } A \tag{4.2}$$

(A is Schur-convex if $z \in A$ and $z' \overset{m}{\geqq} z$ imply $z' \in A$), and the corresponding conditions (C0F), (C3F), and (4.2) are equivalent. This is Theorem 2.2 in [8].

Example 4.4. Let S be a (nonempty) convex, compact and symmetric subset of $\mathbb{R}^2$, and $\|\cdot\|_S$ the norm on $\mathbb{R}^2$ for which S is the unit sphere, i.e. $\|(x,y)\|_S = \lambda \Leftrightarrow (x,y) \in \lambda S$ and $\lambda \geqq 0$ (see pp. 111–112 of Stein and Weiss [13]). When $F = \{\|(x_1,y_1)\|_S \leqq \|(x_2,y_2)\|_S\}$, then (C1F) is equivalent to

$$H(\lambda S) \geqq H'(\lambda S) \quad \text{for all } \lambda \geqq 0, \tag{4.3}$$

and the corresponding conditions (C0F), (C3F), and (4.3) are equivalent. As special cases, when S is an ellipse, $S = \{x^2/a^2 + y^2/b^2 \leqq 1\}$, then $\|(x,y)\|_S^2 = x^2/a^2 + y^2/b^2$; when S is diamond-shaped, i.e. $S = \{|x/a + y/b| \leqq 1,\ |x/a - y/b| \leqq 1\}$, then $\|(x,y)\|_S = |x/a + y/b|$ if $xy \geqq 0$, $= |x/a - y/b|$ if $xy \leqq 0$; and when S is a rectangle, $S = [-a,a] \times [-b,b]$, then $\|(x,y)\|_S = \max\{|x|/a, |y|/b\}$. (Notice that, unlike (4.1) which is satisfied for all rectangles, when S is a ractangle, (4.3) is satisfied only for those rectangles which are homothetic to S.) When H and H' are absolutely continuous $EC_2(0,\Sigma,\phi)$ and $EC_2(0,\Sigma',\phi)$ distributions and $\Sigma' - \Sigma$ is nonnegative definite, then Theorem 3.3 of [4] implies (4.3), and therefore (C0F) and (C3F) hold. For the special case where S is a square and Σ, Σ' are as in Theorem 3.2 with $\sigma_1 = \sigma_2$, we now give a simple proof of (C0F) through a construction similar to that used to prove Theorem 3.2. Thus, for S a square, we establish (C3F) (as well as (4.3)) without using Strassen's theorem (and also for not necessarily absolutely continuous elliptically contoured distributions). Even though we are assuming for simplicity of the construction that the common variances of Σ, Σ' are equal, no doubt a similar but somewhat more involved construction would be feasible when the variances are not equal. (An analogous construction may even be feasible for the stronger result corresponding to Example 4.2.)

Theorem 4.1. *Suppose that* $Z \sim EC_2(0,\Sigma,\phi)$ *and* $Z' \sim EC_2(0,\Sigma',\phi)$, *where*

$$\Sigma = \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}, \quad \Sigma' = \begin{pmatrix} \sigma^2 & \rho'\sigma^2 \\ \rho'\sigma^2 & \sigma^2 \end{pmatrix}.$$

Then $|\rho| \geqq |\rho'| \Rightarrow$ (C0F) $\Rightarrow$ (C3F), *with S a symmetric square* $[-a,a] \times [-a,a]$.

Proof. As in the proof of Theorem 3.2, it suffices to show $\mathscr{E} l(U^{(2)}A) \geqq \mathscr{E} k(U'^{(2)}A')$ (here $\mu = 0$). This will be done by defining a random vector (X_1, X_2, Y_1, Y_2) which satisfies condition (C0F); i.e. $(X_1, Y_1) \overset{d}{=} U^{(2)}A$, $(X_2, Y_2) \overset{d}{=} U'^{(2)}A'$, and $\max(|X_1|, |Y_1|) \leqq \max(|X_2|, |Y_2|)$.

We can take $U^{(2)} = (\cos\theta, \sin\theta)$ and $U'^{(2)} = (\cos\theta', \sin\theta')$, where θ and θ' are uniformly distributed on intervals of length 2π, and

$$(X_1, Y_1) = \sigma(\cos(\theta - \alpha), \sin(\theta + \alpha)), \quad (X_2, Y_2) = \sigma(\cos(\theta' - \alpha'), \sin(\theta' + \alpha')),$$

where $\sin 2\alpha = \rho$, $\sin 2\alpha' = \rho'$, $-\pi/4 \leqq \alpha$, $\alpha' \leqq \pi/4$. We will now determine the joint distribution of (θ, θ') so that the marginals will be uniform on intervals of length 2π and

$$\max\{|\cos(\theta - \alpha)|, |\sin(\theta + \alpha)|\} \leqq \max\{|\cos(\theta' - \alpha')|, |\sin(\theta' + \alpha')|\}. \tag{4.4}$$

One can easily check that

$$
\begin{aligned}
|\cos(\theta'-\alpha')| \geqq |\sin(\theta'+\alpha')| &\Leftrightarrow \cos 2\theta' \geqq 0\\
|\cos(\theta-\alpha)| \leqq |\cos(\theta'-\alpha')| &\Leftrightarrow \sin(\theta+\theta'-\gamma)\sin(\theta-\theta'-\beta) \geqq 0\\
|\sin(\theta+\alpha)| \leqq |\cos(\theta'-\alpha')| &\Leftrightarrow \cos(\theta+\theta'+\beta)\cos(\theta-\theta'+\gamma) \geqq 0\\
|\cos(\theta-\alpha)| \leqq |\sin(\theta'+\alpha')| &\Leftrightarrow \cos(\theta+\theta'-\beta)\cos(\theta-\theta'-\gamma) \leqq 0\\
|\sin(\theta+\alpha)| \leqq |\sin(\theta'+\alpha')| &\Leftrightarrow \sin(\theta+\theta'+\gamma)\sin(\theta-\theta'+\beta) \leqq 0,
\end{aligned}
$$

where $\beta=\alpha-\alpha'$, $\gamma=\alpha+\alpha'$ $(0\leqq\beta\leqq\pi/4,\ 0\leqq\gamma\leqq\pi/2)$. Thus (4.4) is equivalent to

$$
\left\{\begin{matrix} \cos 2\theta' \geqq 0 \\ \sin(\theta+\theta'-\gamma)\sin(\theta-\theta'-\beta) \geqq 0 \\ \cos(\theta+\theta'+\beta)\cos(\theta-\theta'+\gamma) \geqq 0 \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} \cos 2\theta' \leqq 0 \\ \cos(\theta+\theta'-\beta)\cos(\theta-\theta'-\gamma) \leqq 0 \\ \sin(\theta+\theta'+\gamma)\sin(\theta-\theta'+\beta) \leqq 0 \end{matrix}\right\}. \qquad (4.5)
$$

The two sets of inequalities in (4.5) determine the set within which the support of the joint distribution of (θ,θ') must lie.

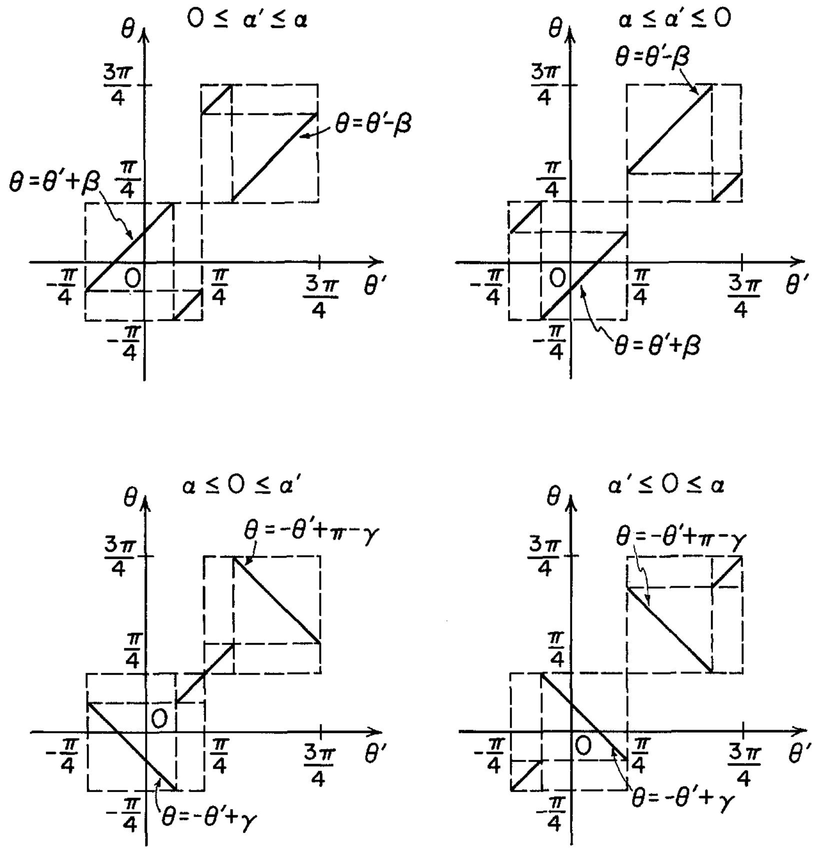

Fig. 2

Let us first consider the case $0 \leqq \rho' \leqq \rho$, i.e. $0 \leqq \alpha' \leqq \alpha \leqq \pi/4$. When $\alpha = \alpha'$, i.e. $\beta = 0$, we can take $\theta = \theta'$. In the general case, $\beta > 0$, we can take θ to be the function of θ' graphed in the upper right section of Fig. 2. (The graph is plotted only for $-\pi/4 \leqq \theta' < 3\pi/4$. For $3\pi/4 \leqq \theta' < 7\pi/4$, the graph is obtained by shifting the plotted graph by (π, π). This applies to the other cases as well.) Since the relationship between θ and θ' in $[-\pi/4, 7\pi/4)$ is one-to-one and piecewise linear with slope 1, if θ' is uniformly distributed on $[-\pi/4, 7\pi/4)$, then so is θ. Moreover, the pairs (θ, θ'), appearing in the graph, satisfy conditions (4.5).

The remaining cases are treated similarly, and Figure 2 shows the graph of $\theta = f(\theta')$, which achieves the desired properties. □

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