Z. Wahrscheinlichkeitstheorie verw. Gebiete
58, 299–308 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

A Class of Two-Type Point Processes

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Summary. A construction is given of a class of two-type point processes with Poisson marginals but possibly negative correlation between points of different types. Examples of the construction are given. The correlation structure of the processes is determined, and criteria obtained for the processes to be stationary, ergodic and mixing.

1. Introduction

In this paper we study a method for constructing point processes in which each of the points has one of two labels attached to it. Such a process is called a *bivariate* or *two-type* point process; because of the ambiguity inherent in the term 'bivariate' we shall always refer to two-type processes.

The carrier space for our bivariate point processes will be $\mathbb{R}^d$, $d \geqq 1$, but the construction is not dependent on this choice and works equally well on any suitable topological space. The class of two-type point processes introduced has the property that the marginal process of points of either type is a Poisson process, while there is interaction between points of different types. In particular we shall concentrate on processes where there is inhibition between the different types of points, though the formal construction also allows positive correlation between points.

There are three main reasons for considering such processes. The first is to provide a class of models for observed two-type processes with marginal Poisson structure but interaction between different types. The second is to provide a class of alternatives for tests of interaction in two-type processes. Finally, the investigation may be regarded as a thought experiment which demonstrates that the Poisson process is a plausible model for a single-type point process on $\mathbb{R}^d$ even if there is a second, possibly unobserved, Poisson process negatively correlated with the first.

* Research supported by Social Science Research Council Spatial Data Project

0044-3719/81/0058/0299/$02.00

Section 2 below contains an informal discussion of the class of processes introduced, together with some examples. This is followed by a formal definition, after which the correlation properties of the processes are discussed. Finally, results on stationarity, ergodicity and mixing properties of the processes are obtained.

There are many examples of two-type point processes in the literature, but very few of them exhibit both marginal randomness and negative spatial correlation between points of different types. One exception is quoted by Cox and Lewis (1972); this is based on a renewal process and so is restricted to the line. Griffiths, Milne and Wood (1979) provide two more examples, in each of which only a finite number of points is possible, so that the marginals cannot be homogeneous Poisson processes. In an attempt to induce as much negative correlation as possible into our process, we shall make use of minimally correlated Poisson random variables as investigated by Griffiths, Milne and Wood (1979). Some remarks about the analysis of two-type point processes are made by Hanisch and Stoyan (1979).

2. Informal Description and Examples

In this section we shall describe informally a general method of defining a two-type process whose marginals are homogeneous Poisson processes negatively correlated with one another. Some examples of our construction will also be given. The definition will be extended and made precise in Sect. 3 below.

For $a, b \geqq 0$ and $\gamma \in \mathbb{R}$, a random vector (M, N) will be said to have a $\mathrm{Poi}(a, b; \gamma)$ distribution if and only if $\mathrm{Cov}(M, N) = \gamma$ and the marginal distributions of M and N are respectively Poisson (a) and Poisson (b). For given a and b, such bivariate integer distributions exist for a range of values of γ and are not in general unique for any particular γ. Let $\gamma_{\min}(a, b)$ be the smallest γ for which $\mathrm{Poi}(a, b; \gamma)$ distributions exist. Write $\mathrm{Poi}_*(a, b)$ for the $\mathrm{Poi}(a, b; \gamma_{\min}(a, b))$ distribution, which is unique; see Griffiths, Milne and Wood (1979). Simulating $\mathrm{Poi}_*(a, b)$ random vectors is easy; if X is uniform $(0, 1)$ and F_c^{-1} is the inverse of the Poisson (c) distribution function, then $(F_a^{-1}(X), F_b^{-1}(1 - X))$ has the $\mathrm{Poi}_*(a, b)$ distribution.

The construction of the two-type process proceeds as follows. Suppose the required marginal intensities are α and β. Possibly using some random mechanism, construct a countable set of integrable functions f_i such that $\sum f_i(x) = 1$ for all x in $\mathbb{R}^d$ and $\int f_i < \infty$ for all i. For each i, generate a pair (M_i^∞, N_i^∞) from a $\mathrm{Poi}_*(\alpha \int f_i, \beta \int f_i)$ distribution, and generate M_i^∞ points of type 1 and N_i^∞ points of type 2 independently from the probability density function proportional to f_i. The required process is the union over i of all the collections of points thus generated. It will be shown in Sects. 3 and 4 that this construction does indeed satisfy the required conditions.

We now give some examples. Example 3 illustrates how the f_i can be chosen randomly; in Examples 1 and 3 the f_i are uniform on certain regions while this is not the case in Example 2.

Example 1. (The chessboard process in $\mathbb{R}^d$.) For each cell of the integer lattice generate a pair (M, N) from the $\mathrm{Poi}_*(\alpha, \beta)$ distribution, and place M type 1 points and N type 2 points uniformly within that cell.

Example 2. Letting ϕ be the standard normal density function, define a finite constant k by

$$k = 2\sum_{i=1}^{\infty} \phi(i - \tfrac{1}{2}).$$

Now let

$$\eta(t) = \begin{cases} 1 - k^{-1} \sum_{i=1}^{\infty} \{\phi(t+i) + \phi(t-i)\} & -\tfrac{1}{2} < t \leqq \tfrac{1}{2} \\ k^{-1}\phi(t) & \text{otherwise} \end{cases}$$

so that η is everywhere positive and $\sum_{-\infty}^{\infty} \eta(t+i) = 1$ for all t. Let $f_i(x) = \eta(x+i)$ for each integer i, $-\infty < i < \infty$, and proceed as in the definition of the construction.

Example 3. (The random Dirichlet cell process in the plane.) Generate a Poisson process ξ in the plane, and construct the Dirichlet tessellation of ξ; see Rogers (1964) for definitions. This gives a random partition of the plane into cells. For each cell C, generate (M, N) from the $\mathrm{Poi}_*(\alpha A, \beta A)$ distribution, where A is the area of C. Place M type 1 points and N type 2 points uniformly in C.

The idea of Example 3 can clearly be extended to any random partition of the plane, for example to the Delaunay triangulation (see Rogers, 1964) of a Poisson process, or to a lattice of random position, orientation and/or size. The only restriction is that all the cells must be of finite area.

The results of Sect. 5 below will show that the random Dirichlet cell process is stationary under rigid motions, ergodic and mixing. In addition it is

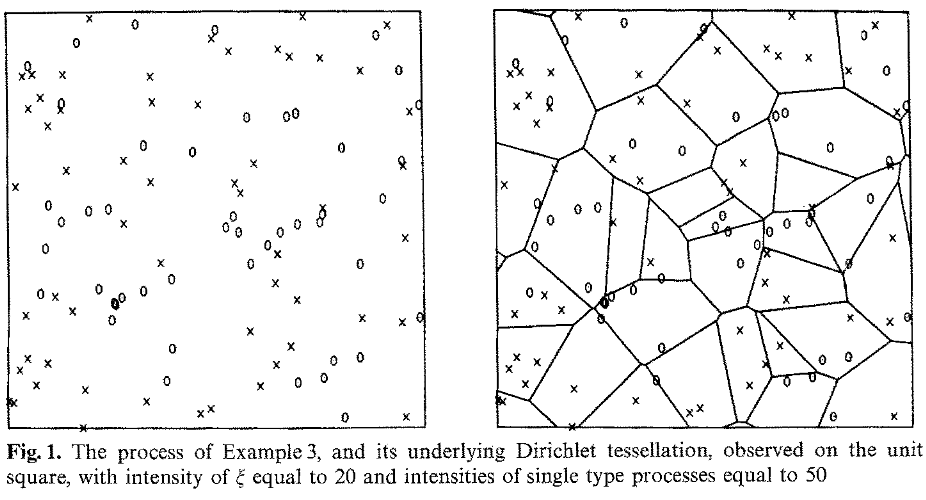

Fig. 1. The process of Example 3, and its underlying Dirichlet tessellation, observed on the unit square, with intensity of ξ equal to 20 and intensities of single type processes equal to 50

easily simulated using the algorithm of Green and Sibson (1978) to construct the tessellation. There are three parameters to be chosen, the intensity of the underlying process ξ as well as the intensities of the two single type processes; part of a sample realisation is given in Fig. 1. To avoid edge effects, the tessellation was simulated on a square which was large enough to ensure that the realisation would not be changed by simulating more of the tessellation. In contrast with many other models for spatial point processes, realisations can be obtained quickly; the example given took about $2\frac{1}{2}$ seconds to obtain on a Honeywell Series 60 Level 68/DPS machine. By comparison, the same computer took over 16 seconds to produce a single realisation of a single-type hard core point process on the unit square, using the program of Ripley (1979) with $n=100$ and $r=0.05$. Lotwick (1981) has succeeded in fitting a random Dirichlet cell process to some data on the positions of veins and arteries in the cerebral cortex.

3. The Formal Construction

In order to make precise the description of our process, and to facilitate the investigation of its properties, we now give a definition using the formalism of Kallenberg (1976). For notational simplicity, we restrict attention to two-type processes on $\mathbb{R}^d$ $(d \geqq 1)$ where the intensity measures for points of either type are identical; all the results are easily extended to the more general case of unequal intensities. Note that the marginals will not be restricted to homogeneous processes.

In the notation of Sect. 2 above, the idea of the formalism is to regard the random set $\{f_i\}$ as a point process on the set of finite measures on $\mathbb{R}^d$, in other words as an integer valued random measure on a set of measures. In the definitions below, the f_i are the densities of the measures Λ_i, and choice of the random directing measure Λ corresponds to the random choice of sequence f_i.

Throughout, let B be an arbitrary bounded Borel subset of $\mathbb{R}^d$ and let $\mathcal{M}_0$ be the collection of finite measures on $\mathbb{R}^d$ excluding the zero measure. Given any suitable set C, let $\mathcal{N}(C)$ be the set of locally finite integer valued measures on C, so that random elements of $\mathcal{N}(C)$ correspond to point processes on C. For convenience write λ in $\mathcal{N}(\mathcal{M}_0)$ as (λ_i) and write $\sum \lambda_i(B)$ instead of the formally correct but less transparent $\int v(B)\, \lambda(dv)$.

Let μ be a fixed measure on $\mathbb{R}^d$ which will be the intensity measure of each of the single-type processes. A *directing point process* $\Lambda = (\Lambda_i)$ will be a random element of $\mathcal{N}(\mathcal{M}_0)$ such that, for all B,

$$\sum_i \Lambda_i(B) = \mu(B). \tag{3.1}$$

We assume given a set $\{\psi(.;\alpha), \alpha \geqq 0\}$ of probability generating functions of marginally Poisson(α) bivariate random vectors. It is the correlation structure of these vectors which leads to the correlation between the processes; to obtain positively correlated processes choose ψ so that the components of these vectors are positively correlated. Conditional on Λ, for each i let (M_i^∞, N_i^∞) be a random vector distributed, independently for each i, according to $\psi(.; \Lambda_i(\mathbb{R}^d))$. Then distribute M_i^∞ points of type 1 and N_i^∞ points of type 2 independently with probability law $\Lambda_i/\Lambda_i(\mathbb{R}^d)$. Let $\mathbf{N}=(M,N)$ be the random element of $\mathcal{N}(\mathbb{R}^d \times \{1,2\}$ defined by

$$\mathbf{N}(B \times \{1\}) = \sum_i M_i(B),$$

$$\mathbf{N}(B \times \{2\}) = \sum_i N_i(B).$$

We shall call $\mathbf{N}$ a bivariate Poisson process directed by Λ. If the $\psi(.; \alpha)$ are all probability generating functions of Poi_* random vectors then we shall call $\mathbf{N}$ a bivariate $\mathrm{Poisson}_*$ process. In any case, the single-type processes are Poisson processes with the required intensity because they are Cox processes directed by $\sum \Lambda_i = \mu$. Of course, this construction is a special case of the cluster process construction of Matthes, Kersten and Mecke (1978). In our case the cluster field, χ, is on $\mathcal{M}_0$ and H is the distribution of Λ. The condition (3.1) ensures that $H(\lambda \notin M'_\chi)=0$, in their terminology.

The case where the single-type intensities are different requires the directing point process to be a random element of $\mathcal{N}(\mathcal{M}_0 \times \mathcal{M}_0)$ and the probability generating functions ψ to be defined for pairs (α_1, α_2) in $\mathbb{R}^+ \times \mathbb{R}^+$; if the intensities are merely multiples of one another it suffices to restrict attention to pairs (ν, π) in $\mathcal{M}_0 \times \mathcal{M}_0$ and (α_1, α_2) in $\mathbb{R}^+ \times \mathbb{R}^+$ which are corresponding multiples of one another.

4. Correlations

In this section a formula is obtained for the covariance of $M(B)$ and $N(B')$ in the bivariate Poisson process introduced in the last section. The minimality properties of the covariance are also discussed. We shall need some notation. Define a subset A_* of $\mathbb{R}^+ \times \mathbb{R}^+$ by

$$A_* = \{(u,v): e^{-u} + e^{-v} \geqq 1\}.$$

From Griffiths, Milne and Wood (1979) it follows that A_* is precisely the set of (u,v) with this property: if $(M,N) \sim \mathrm{Poi}_*(u,v)$ then $MN=0$ with probability one. Also, it is immediate that (u,u) is in A_* if and only if $u \leqq \log_e 2$.

Lemma 1. *Suppose that, for i in some subset of $\mathbb{N}$, (M_i, N_i) are independent $\mathrm{Poi}_*(\mu_i, \nu_i)$ random vectors with $\sum \mu_i = \mu < \infty$ and $\sum \nu_i = \nu < \infty$. Then*

$$\mathrm{Cov}(\textstyle\sum M_i, \sum N_i) \geqq \gamma_{\min}(\mu, \nu) \tag{4.1}$$

with equality if and only if either $\mu\nu = 0$ or, for some i, $\mu_i = \mu$ and $\nu_i = \nu$. If (X_i), (Y_i) are independent respectively Bernoulli (p) and Bernoulli (q) sequences (p, q in $[0,1]$) and (M,N) is some $\mathrm{Poi}(\mu, \nu; \gamma_0)$ random vector, then

$$\begin{aligned}\mathrm{Cov}\left(\sum_1^M X_i, \sum_1^N Y_i\right) &= p\,q\,\gamma_0 \\ &\geqq \gamma_{\min}(p\,\mu, q\,\nu).\end{aligned} \tag{4.2}$$

Suppose that (M, N) *is* $\mathrm{Poi}_*(\mu, \nu)$. *In that case, equality occurs in* (4.2) *if and only if* $(\mu, \nu) \in A_*$ *or* $pq = 0$ *or* 1.

Proof. Inequality (4.1) follows immediately from the fact that $(\sum M_i, \sum N_i)$ has a $\mathrm{Poi}(\mu, \nu; \sum \gamma_{\min}(\mu_i, \nu_i))$ distribution. The criteria for equality are precisely those which ensure that the support of $(\sum M_i, \sum N_j)$ is that of a minimally correlated bivariate Poisson distribution; see Griffiths, Milne and Wood (1979). The proof of the second part of the lemma is similar to that of the first.

We can now state and prove the main result of this section.

Theorem 1. *If* (M, N) *is a bivariate Poisson process directed by* Λ, *then for all sets* B *and* B'

$$\mathrm{Cov}\{M(B), N(B')\} = E(\sum P_i Q_i \Gamma_i) \tag{4.3}$$

where

$$\Gamma_i = \mathrm{Cov}(M_i^\infty, N_i^\infty | \Lambda)$$
$$P_i = \Lambda_i(B)/\Lambda_i(\mathbb{R}^d)$$
$$Q_i = \Lambda_i(B')/\Lambda_i(\mathbb{R}^d).$$

Suppose that (M, N) *is bivariate* $\mathrm{Poisson}_*$ *directed by* Λ *and that* $\mu(B)\,\mu(B') > 0$. *Then*

$$E(\sum P_i Q_i \Gamma_i) = \gamma_{\min}\{\mu(B), \mu(B')\} \tag{4.4}$$

if and only if either

(i) $\mu(B \triangle B') = 0$ *and, with probability* 1, *there exists* i *such that*

$$\Lambda_i(B) = \Lambda_i(B') = \Lambda_i(\mathbb{R}^d) = \mu(B)$$

or

(ii) $\mu(B \cup B') \leqq \log_e 2$ *and, with probability* 1, *there exists* i *such that*

$$\Lambda_i(\mathbb{R}^d) \leqq \log_e(2) \quad \textit{and} \quad \Lambda_j(B \cup B') = 0, \quad \textit{for } j \neq i.$$

Proof. The proof of (4.3) follows easily from the definitions. For the second part, condition on Λ, and note that, for all k, $M_k(B)$ and $N_k(B')$ are then random sums of Bernoulli random variables; apply the conditions of (4.2) to conclude that, for all k,

$$P_k Q_k \Gamma_k \geqq \gamma_{\min}\{\Lambda_k(B), \Lambda_k(B')\} \tag{4.5}$$

with equality if and only if either $\Lambda_k(\mathbb{R}^d) \leqq \log_e 2$ or $\Lambda_k(B)\Lambda_k(B') = 0$ or $\Lambda_k(B) = \Lambda_k(B') = \Lambda_k(\mathbb{R}^d)$, in which case $\Lambda_k(B \triangle B') = 0$.

By the first part of Lemma 1,

$$\sum_k \gamma_{\min}\{\Lambda_k(B), \Lambda_k(B')\} \geqq \gamma_{\min}\{\mu(B), \mu(B')\} \tag{4.6}$$

with equality if and only if, for some i, $\Lambda_i(B) = \mu(B)$ and $\Lambda_i(B') = \mu(B')$, so that $P_j = Q_j = \Lambda_j(B \cup B') = 0$ for $j \neq i$. Now (4.4) will be true if and only if there is almost sure equality in (4.5) and (4.6) for all k; a careful but elementary argument completes the proof of the theorem.

A first conclusion from Theorem 1 is that $M(B)$ and $N(B')$ are strictly negatively correlated in a bivariate Poisson$_*$ process whenever $\sum \Lambda_i(B)\Lambda_i(B')>0$ with positive probability. In Examples 2 and 3 of Sect. 2, this is the case for any non-trivial sets B and B', while in Example 1 the correlation will be zero if no lattice cell intersects both B and B'.

The theorem gives a complete characterisation of those sets B and B' for which minimal correlation will be achieved. In terms of the examples given, it is clear that minimal correlation will never be achieved in the random Dirichlet cell process, because for non-trivial sets B there is always finite probability that B will intersect two or more cells of the tessellation, thus violating the conditions. The same remark holds for any of the processes given with randomly chosen directing measure Λ. For the chessboard process with fixed unit lattice, minimal correlation will be achieved if B and B' both essentially coincide with the same lattice cell, and alternatively if the intensity of the marginal processes are at most $\log_e 2$ and B and B' are both essentially contained in the same lattice cell.

Theorem 1 may be extended to the case where the marginal intensities are different; as the exact statement and proof of the last part are more complicated the extensions are left to the reader.

5. Stationarity and Ergodicity Properties

Definitions of stationarity, ergodicity and mixing for two-type point processes may be found in Matthes, Kerstan and Mecke (1978), Chap. 6. Informally, mixing means that the process on one part of $\mathbb{R}^d$ is nearly independent of a distant part of the process, while the weaker property of ergodicity ensures that estimators of parameters obtained by taking averages over areas will enjoy strong consistency as the areas increase in size. We shall also require stationarity, ergodicity and mixing for the directing processes, which take values in $\mathcal{N}(\mathcal{M}_0)$. Define the translation operator, T_y: $\mathcal{M}_0 \to \mathcal{M}_0$, $y \in \mathbb{R}^d$, by

$$T_y \lambda(B) = \lambda(B+y).$$

We shall use the same symbol, T_y, for the translation operator on any space. Thus we define T_y: $\mathcal{N}(\mathcal{M}_0) \to \mathcal{N}(\mathcal{M}_0)$ by

$$T_y\{\sum_i \delta_{\lambda_i}\} = \sum_i \delta_{T_y \lambda_i}.$$

The definitions of stationarity and ergodicity for a point process on $\mathcal{M}_0$ are now exactly analogous to those for a point process on $\mathbb{R}^d \times \{1, 2\}$.

The following result enables properties of our two-type point process to be deduced from properties of the directing point process. The mixing part of the Theorem is analogous to Theorem 6 of Westcott (1971), but the present proof seems simpler for our case. Moreover an adaptation of Westcott's argument would not be possible without auxiliary results.

Theorem 2. *Suppose* $\mathbf{N}=(N, M)$ *is a bivariate Poisson process directed by* Λ. *If* Λ *is stationary then* $\mathbf{N}$ *is stationary. If in addition* Λ *is ergodic or mixing, then* $\mathbf{N}$ *is also ergodic or mixing, respectively.*

We first remark that the proof of the stationarity assertion of Theorem 2 extends to stationarity, with suitable definitions, under any transformation group. Thus, for example, the distribution of $\mathbf{N}$ is invariant under an orthogonal change of coordinates if the distribution of Λ is similarly invariant. Next, we note that the converse statements to those in Theorem 2 are false, since, if $\mathbf{N}$ is constructed with ψ always corresponding to independent Poisson random variables then M and N are independent and the structure of Λ is irrelevant to the structure of $\mathbf{N}$. Finally, we remark that only notational difficulties are encountered in extending Theorem 2 to the case where the marginal intensities are different.

To provide examples of the application of Theorem 2, consider first the process obtained as in Example 1 of Sect. 2, but with the position of the lattice chosen uniformly at random for each realisation. The resulting distribution of Λ is clearly stationary and also ergodic, since every realisation is a translation of the integer lattice. Therefore the resulting point process $\mathbf{N}$ is stationary and ergodic. However this example is not mixing, which is a serious practical deficiency. To see this, note that the position of the lattice may be recovered from a realisation of $\mathbf{N}$ by looking at empirical correlations of numbers of points of the two types. Further the position of the lattice is invariant under integer shifts.

The process Λ in the Dirichlet tessellation example is translation isomorphic, in the sense of Billingsley (1965) page 53, to the point process ξ which generates it. Hence, stationarity, ergodicity and mixing for Λ are equivalent to the same concepts for ξ. In particular, if ξ is a homogeneous Poisson process, then $\mathbf{N}$ is stationary and mixing. Of course, the same isomorphism holds for Delaunay triangulations.

Proof of Theorem 2. We first note that the distribution of $\mathbf{N}$ is determined by that of Λ. A proof of this intuitively obvious fact follows from this formula for the Laplace transform (Kallenberg (1976)), $L_{\mathbf{N}}$, of $\mathbf{N}$:

$$L_{\mathbf{N}}(f)=L_{\Lambda}(H^f),$$

where L_Λ is the Laplace transform of the point process Λ and for μ in $\mathscr{M}$, putting $\alpha=\mu(\mathbb{R}^d)$,

$$H^f(\mu)=-\log\psi\left(\int e^{-f_1}\,d\mu/\alpha, \int e^{-f_2}\,d\mu/\alpha; \alpha\right).$$

It is clear that, for all y, $T_y\mathbf{N}$ is a bivariate Poisson process directed by $T_y\Lambda$. Hence, if $T_y\Lambda$ and Λ are identically distributed, so are $T_y\mathbf{N}$ and $\mathbf{N}$, completing the proof of the stationarity part.

Now suppose that Λ is stationary and ergodic, and let Y be an invariant subset of $\mathscr{N}(\mathbb{R}^d\times\{1,2\})$. Let h be the indicator of Y. Now $E\{h(\mathbf{N})|\Lambda=\lambda\}$ is easily shown to be an invariant function of λ. Hence, the ergodicity of Λ implies that $E(h(\mathbf{N})|\Lambda)$ is a.s. constant. It follows that

$$\operatorname{Var}(h(\mathbf{N}))=E\{\operatorname{Var}(h(\mathbf{N})|\Lambda)\}. \tag{5.1}$$

Let $\varepsilon>0$. For any Borel set A of $\mathbb{R}^d$, let $\mathscr{F}(A)$ be the σ-field generated by $\{\mathbf{N}(B)\colon B\subseteq A\}$. Now $[\mathbf{N}\in Y]$ is in the σ-field generated by $\bigcup\mathscr{F}(A)$, where A ranges over rectangles. Hence, for a large enough rectangle A, it is possible to find a subset Z of $\mathscr{N}(\mathbb{R}^d\times\{1,2\})$ for which $[\mathbf{N}\in Z]$ is in $\mathscr{F}(A)$ and such that $P(\mathbf{N}\in Z\bigtriangleup Y)$ is arbitrarily small. Further, for any y in $\mathbb{R}^d$, the last probability equals $P(\mathbf{N}\in T_y Z\bigtriangleup Y)$, since $\mathbf{N}$ is stationary and Y is invariant. Thus we can choose A and Z so that for λ outside a subset E_1 of $\mathscr{N}(\mathscr{M}_0)$ with

$$P(\Lambda\in E_1)<\varepsilon$$

and for T_y a fixed translation, we have

$$\begin{aligned} &P(\mathbf{N}\in Z\bigtriangleup Y|\Lambda=\lambda)<\varepsilon,\\ \text{and}\quad &\\ &P(\mathbf{N}\in T_y Z\bigtriangleup Y|\Lambda=\lambda)<\varepsilon. \end{aligned} \tag{5.2}$$

The translation T_y is chosen together with a number j and a subset E_2 of $\mathscr{N}(\mathscr{M}_0)$ so that the following hold: $T_y A\cap A=\emptyset$,

$$P(\Lambda\in E_2)<\varepsilon$$

and, for λ outside E_2, both

$$\begin{aligned} &\sum_{j+1}^{\infty}\lambda(A)<\varepsilon,\\ \text{and}\quad &\\ &\sum_{1}^{j}\lambda(T_y A)<\varepsilon. \end{aligned} \tag{5.3}$$

A measure λ is now fixed in $(E_1\cup E_2)^c$. Let $\mathbf{N}'$ be a bivariate Poisson process directed by λ. Having fixed x in $(A\cup T_y A)^c$, we construct a process $\mathbf{N}^*$ from $\mathbf{N}'$ by moving to x all the points (of *both* types) in $T_y A$ drawn from $\lambda_1,\ldots,\lambda_j$ and all the points in A drawn from $\lambda_{j+1},\lambda_{j+2},\ldots$. From (5.3) it follows that for any subset S of $\mathscr{N}(\mathbb{R}^d\times\{1,2\})$

$$\begin{aligned}|P(\mathbf{N}'\in S)-P(\mathbf{N}^*\in S)| &\leqq P(\mathbf{N}'\neq\mathbf{N}^*)\\ &\leqq 2(1-e^{-2\varepsilon})\end{aligned}$$

since the number of points of each type that are moved has a Poisson distribution with mean at most 2ε. But, by construction, $\mathbf{N}^*$ on A is independent of $\mathbf{N}^*$ on $T_y A$. Thus,

$$|P(\mathbf{N}'\in Z\cap T_y Z)-P(\mathbf{N}'\in Z)\,P(\mathbf{N}'\in T_y Z)|<6(1-2e^{-2\varepsilon}).$$

Using (5.2) it is now straightforward to see that

$$\operatorname{Var}(h(\mathbf{N}'))<4\varepsilon+6(1-e^{-2\varepsilon}).$$

Since λ was arbitrary in $(E_1 \cup E_2)^c$, we may use (5.1) to bound $\mathrm{Var}(h(\mathbf{N}))$ by the right side of (5.4) plus 2ε, and ergodicity of $\mathbf{N}$ follows.

For the last part of the theorem suppose Λ is stationary and mixing. By the argument of Billingsley (1965) Theorem 1.2, it suffices to show that for each rectangle A and each pair of sets Z_1 and Z_2 with $[\mathbf{N}\in Z_1]$ and $[\mathbf{N}\in Z_2]$ in $\mathscr{F}(A)$ we have

$$P(\mathbf{N}\in Z_1 \cap T_y Z_2) \to P(\mathbf{N}\in Z_1)\, P(\mathbf{N}\in Z_2) \quad \text{as } \|y\| \to \infty. \tag{5.5}$$

By an argument similar to (but simpler than) that used before, we may establish that, if $\|y\|$ is sufficiently large,

$$E|P(\mathbf{N}\in Z_1 \cap T_y Z_2|\Lambda) - P(\mathbf{N}\in Z_1|\Lambda)\, P(\mathbf{N}\in T_y Z_2|\Lambda)| < 6(1-e^{-2\varepsilon}) + \varepsilon. \tag{5.6}$$

It follows from the mixing property of Λ that, for *bounded* measurable functions u and v defined on $\mathscr{N}(\mathscr{M}_0)$,

$$E\{u(\Lambda)\, v(T_y \Lambda)\} \to E\{u(\Lambda)\}\, E\{v(\Lambda)\} \quad \text{as } \|y\| \to \infty.$$

Hence, using F_Λ for the distribution of Λ,

$$\begin{aligned} &\int P(\mathbf{N}\in Z_1|\Lambda=\lambda)\, P(\mathbf{N}\in T_y Z_2|\Lambda=\lambda)\, F_\Lambda(d\lambda) \\ &\qquad = \int P(\mathbf{N}\in Z_1|\Lambda=\lambda)\, P(\mathbf{N}\in Z_2|\Lambda=T_{-y}\lambda)\, F_\Lambda(d\lambda) \\ &\qquad \to P(\mathbf{N}\in Z_1)\, P(\mathbf{N}\in Z_2). \end{aligned} \tag{5.7}$$

Combining (5.6) with (5.7) and the arbitrariness of ε, we obtain (5.5), as required.

Acknowledgements. We are very grateful to Robin Sibson, Adrian Bowyer and Wyn Lotwick for assistance with the computing, and to the first named for the use and adaptation of the TILE 004 package.

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Received August 15, 1981