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A Law of the Iterated Logarithm for Martingales*

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1. Statement of the Theorem

In this note, techniques of Chow and Teicher [2], Heyde [3] and Stout [5] will be employed to prove the following law of the iterated logarithm for identically distributed martingale difference sequences. ($I(A)$ will denote the indicator function of an event A).

Theorem. *Let $X_1, X_2, \dots$ be identically distributed random variables (rv) with $E(X_1^2)=1$. Suppose $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots$ are sigma-fields such that X_n is $\mathscr{F}_n$-measurable and $E(X_n|\mathscr{F}_{n-1})=0$ almost surely (a.s.) for each $n \geqq 1$.*

Let $\{a_n, n \geqq 1\}$ be a real sequence satisfying

(i) $A_n \equiv \sum_{j=1}^{n} a_j^2 \to \infty$ *as* $n \to \infty$,

(ii) $n a_n^2 = O(A_n)$ *as* $n \to \infty$, *and*

(iii) $\lim_{n\to\infty} A_n^{-1} \sum_{j=1}^{n} a_j^2 E(X_j^2|\mathscr{F}_{j-1}) = 1$ *a.s.*

For $m \geqq 1$, $n \geqq 1$, define the rv

$$Y_{m,n} \equiv X_n I(|X_n| \leqq m) + m(I(X_n > m) - I(X_n < -m))$$

and

$$Y \equiv \liminf_{m\to\infty} \liminf_{n\to\infty} A_n^{-1} \sum_{j=1}^{n} a_j^2 E(Y_{m,j}^2|\mathscr{F}_{j-1}).$$

If (iv) $Y>0$ *a.s. then*

$$Y^{1/2} \leqq \limsup_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_j}{(2A_n \log\log A_n)^{1/2}} \leqq 1 \quad \text{a.s.} \tag{1}$$

In particular, if (v) $Y=1$ *a.s., then*

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$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_j}{(2A_n \log\log A_n)^{1/2}} = 1 \quad \text{a.s.} \tag{2}$$

Chow and Teicher [2] showed that (2) holds if $X_1, X_2, \ldots$ are independent, identically distributed rv with $E(X_1)=0$ and $E(X_1^2)=1$ and if the a_n's satisfy (i) and (ii). Their theorem is implied by our result since, taking $\mathscr{F}_n$ to be the sigma-field generated by $X_1, \ldots, X_n$, (iii) and (v) are obvious in this case. (See also the work of Teicher [6].)

Our result also contains Heyde's theorem [3] (and, hence, Stout's theorem [5]), but this fact is by no means obvious. Heyde showed that

$$\limsup_{n\to\infty} (2n \log\log n)^{-1/2} \sum_{j=1}^{n} X_j = 1 \quad \text{a.s.}$$

for every stationary martingale difference sequence satisfying

$$E(X_1^2)=1 \quad \text{and} \quad \lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} E(X_j^2 | \mathscr{F}_{j-1}) = 1 \quad \text{a.s.}$$

Taking $a_n = 1$ for $n \geqq 1$, (i) and (ii) are trivial, whereas (iii) is precisely Heyde's second hypothesis. The validity of (v) can be established by reproducing much of the proof of Heyde's result (cf. expression (14) of [3]); it must be stressed that the derivation of (v) in this case is neither short nor simple.

2. Proof of the Theorem

Choose a positive real sequence $d_n \to \infty$ exactly as in [2] and let

$$b_n = n^{1/2} (d_n \log\log A_n)^{-1/2}.$$

Now introduce Stout's truncation scheme [5]: for $n \geqq 1$, let

$$Y_n \equiv X_n I(|X_n| \leqq b_n) + b_n (I(X_n > b_n) - I(X_n < -b_n)),$$

$$Z_n \equiv Y_n - E(Y_n | \mathscr{F}_{n-1}),$$

$$Z_{m,n} = Y_{m,n} - E(Y_{m,n} | \mathscr{F}_{n-1}) \quad \text{for } m \geqq 1,$$

$$Y_n^* \equiv X_n - Y_n \quad \text{and} \quad Z_n^* = Y_n^* - E(Y_n^* | \mathscr{F}_{n-1}).$$

Since (ii) holds and the X_n's are identically distributed, the arguments in expression (8) of [2] yield

$$\sum_{n=1}^{\infty} \frac{|a_n| E(|X_n| I(|X_n| > b_n))}{(A_n \log\log A_n)^{1/2}} < \infty.$$

But $|Y_n^*| \leqq |X_n| I(|X_n| > b_n)$ so

$$\sum_{n=1}^{\infty} \frac{|a_n| E|Y_n^*|}{(A_n \log\log A_n)^{1/2}} < \infty. \tag{3}$$

Since $E|Z_n^*| \leqq 2E|Y_n^*|$, (3) and (i) imply (using Kronecker's lemma as in [2]),

$$\sum_{j=1}^{n} a_j Z_j^*/(A_n \log\log A_n)^{1/2} \to 0 \text{ a.s.} \quad \text{as } n \to \infty. \tag{4}$$

Following [3], note that $E(Y_n|\mathscr{F}_{n-1}) = -E(Y_n^*|\mathscr{F}_{n-1})$ and $|Y_n| \leqq b_n$ so, for all $n \geqq 1$,

$$E(E^2(Y_n|\mathscr{F}_{n-1})) \leqq b_n E|Y_n^*|. \tag{5}$$

Recalling that $d_n \to \infty$ and using (ii),

$$A_n^{-1} a_n^2 b_n = O(|a_n|(A_n \log\log A_n)^{-1/2}). \tag{6}$$

(3), (5) and (6) show that

$$\sum_{n=1}^{\infty} A_n^{-1} a_n^2 E(E^2(Y_n|\mathscr{F}_{n-1})) < \infty.$$

Hence $\sum_{n=1}^{\infty} A_n^{-1} a_n^2 E^2(Y_n|\mathscr{F}_{n-1}) < \infty$ a.s.

Applying Kronecker's lemma, we get

$$A_n^{-1} \sum_{j=1}^{n} a_j^2 E^2(Y_j|\mathscr{F}_{j-1}) \to 0 \text{ a.s. as } n \to \infty. \tag{7}$$

For $n \geqq 1$, define the *rv*

$$u_n^2 = \sum_{j=1}^{n} a_j^2 E(X_j^2|\mathscr{F}_{j-1}), \qquad s_n^2 = \sum_{j=1}^{n} a_j^2 E(Z_j^2|\mathscr{F}_{j-1}),$$

and

$$K_n = 2|a_n| b_n s_n^{-1} (\log\log s_n^2)^{1/2};$$

note that K_n is $\mathscr{F}_{n-1}$-measurable.

Fix $m \geqq 1$. If $b_n \geqq m$ then

$$E(Y_{m,n}^2|\mathscr{F}_{n-1}) \leqq E(Y_n^2|\mathscr{F}_{n-1}) \leqq E(X_n^2|\mathscr{F}_{n-1}) \quad \text{a.s.} \tag{8}$$

and

$$E(Z_{m,n}^2|\mathscr{F}_{n-1}) \leqq E(Z_n^2|\mathscr{F}_{n-1}) \leqq E(X_n^2|\mathscr{F}_{n-1}) \quad \text{a.s.}; \tag{9}$$

(8) is evident, whereas (9) follows from corollary 1 of [1] (as noted by Stout [5]).

For each $m \geqq 1$, we have, in view of (8),

$$\liminf_{n\to\infty} A_n^{-1} \sum_{j=1}^{n} a_j^2 E(Y_{m,j}^2|\mathscr{F}_{j-1}) \leqq \liminf_{n\to\infty} A_n^{-1} \sum_{j=1}^{n} a_j^2 E(Y_j^2|\mathscr{F}_{j-1}) \quad \text{a.s.}$$

Letting $m \to \infty$ and applying (iv) and (7), then,

$$\liminf_{n\to\infty} A_n^{-1} s_n^2 \geqq Y > 0 \quad \text{a.s.} \tag{10}$$

Furthermore, (iii) and (9) imply that

$$\limsup_{n\to\infty} A_n^{-1} s_n^2 \leqq 1 \quad \text{a.s.} \tag{11}$$

Note that $E(Y_j^2)=E(X_1^2 I(|X_1|\leqq b_j))+b_j^2 I(|X_1|>b_j)\to E(X_1^2)=1$. By the Toeplitz lemma and the fact that $E(Y_n^2|\mathscr{F}_{n-1})\leqq E(X_n^2|\mathscr{F}_{n-1})$ a.s. for all $n\geqq 1$,

$$A_n^{-1} E\left|u_n^2-\sum_{j=1}^n a_j^2 E(Y_j^2|\mathscr{F}_{j-1})\right|=A_n^{-1}\left(E(u_n^2)-\sum_{j=1}^n a_j^2 E(Y_j^2)\right)$$

$$=1-A_n^{-1}\sum_{j=1}^n a_j^2 E(Y_j^2)\to 0.$$

Hence

$$\left(u_n^2-\sum_{j=1}^n a_j^2 E(Y_j^2|\mathscr{F}_{j-1})\right)\Big/A_n\to 0 \quad \text{in probability.}$$

In light of (iii) and (7), $s_n^2/A_n\to 1$ in probability. This fact and (11) ensure that $\limsup_{n\to\infty} s_n^2/A_n^2=1$ a.s. whence

$$s_n^2\to\infty \quad \text{a.s.} \tag{12}$$

Moreover, using (ii), there exists $C>0$ such that

$$K_n=\frac{2|a_n|n^{1/2}(\log\log s_n^2)^{1/2}}{(d_n s_n^2\log\log A_n)^{1/2}}\leqq C d_n^{-1/2}(A_n^{1/2}/s_n)(\log\log s_n^2/\log\log A_n)^{1/2},$$

so, taking (10) and (11) into account,

$$K_n\to 0 \text{ a.s.} \quad \text{as } n\to\infty. \tag{13}$$

But $(\log\log s_n^2)^{1/2}\dfrac{|a_n||Z_n|}{s_n}\leqq K_n$ a.s. so, because (12) and (13) hold and $(Z_n, \mathscr{F}_n, n\geqq 1)$ is a martingale difference sequence, theorems 1 and 2 of Stout [4] yield

$$\limsup_{n\to\infty}\frac{\sum_{j=1}^n a_j Z_j}{(2s_n^2\log\log s_n^2)^{1/2}}=1 \quad \text{a.s.} \tag{14}$$

It is clear from (10) and (11) that

$$\lim_{n\to\infty}(\log\log s_n^2/\log\log A_n)=1 \quad \text{a.s.} \tag{15}$$

(1) now follows readily from (4), (10), (11), (14) and (15). If (v) holds, (2) is immediate from (1). Q.E.D.

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