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A Generalization of a Formula of Steiner

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Summary. Combinatorial formulae for the numbers of cells of subdivision, of varying dimensions, when hyperplanes in general position intersect an open convex set of $\mathbb{R}^d$, are derived. They coincide with basic formulae for the probabilities of combinations of events. The dual result is stated.

Suppose $\mathscr{H}$ is a set of N hyperplanes $H_1, \ldots, H_N$ in $\mathbb{R}^d$ (N, d arbitrary positive integers), which are in (mutual) *general position*, inasmuch as the intersection of any m of them has dimension at most $d-m$ ($m=2, \ldots, d+1$). In fact, it follows that this intersection has dimension either -1 (no intersection) or $d-m$. If this dimension is always $d-m$, we say $\mathscr{H}$ has *restricted* general position. In the 19th century, Schläfli (1950) showed that, if $\mathscr{H}$ has restricted general position, the remainder of $\mathbb{R}^d$ is thereby partitioned into

$$\binom{N}{0} + \ldots + \binom{N}{d}$$

disjoint open sets (in an obvious way); in fact, Steiner (1826) had earlier derived this formula for $d \leqq 3$. Buck (1943) extended the formula to the total number of k-dimensional 'cells' formed ($k=0, \ldots, d$). We now generalize Buck's result in another direction, first taken for $d \leqq 3$ by Santaló (1976, Chap. 16, Note 7).

Henceforth $\mathscr{H}$ is supposed to have full general position. Let K be an open convex set in $\mathbb{R}^d$. Should a hyperplane H intersect K, then (ignoring $K \cap H$) it partitions K into two disjoint nonvoid open convex sets, in an obvious way. Thus the combined effect of $\mathscr{H}$ is to partition (the remainder of) K into a finite number $v_d \geqq 1$ of open convex sets. Let α_{d-k} be the number of distinct $(d-k)$-subsets of $\mathscr{H}$ which intersect, in a k-flat, which itself intersects K (0-flat $=$point); denote these flats by $F_{k,i}$ ($i=1, \ldots, \alpha_{d-k}$; $k=0, \ldots, d-1$). We make the convention that $\alpha_0 = 1$. Consider a k-dimensional open convex set $K_{k,i} = K \cap F_{k,i}$ ($k \neq 0$). Those of the 'remaining' $N-d+k$ members of $\mathscr{H}$ which

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intersect $F_{k,i}$ do so in k-dimensional hyperplanes having general position within $F_{k,i}$. Thus the set-up $(K; H_1, \ldots, H_N)$ in $\mathbb{R}^d$ is reproduced, with lower dimensionalities, in each such $F_{k,i}$. Suppose $K_{k,i}$ is accordingly partitioned into $v_{k,i}$ disjoint open k-dimensional convex regions, and define

$$v_k = \sum_{i=1}^{\alpha_{d-k}} v_{k,i} \qquad (k=1, \ldots, d-1),$$

$$v_0 = \alpha_d,$$

with v_d previously defined. The v_k are the total numbers of k-dimensional cells in the decomposition of K by $\mathscr{H}$. Their values are determined in terms of those of the α_n as follows.

Theorem A.

$$v_k = \sum_{n=d-k}^{d} \binom{n}{d-k} \alpha_n \qquad (k=0, \ldots, d).$$

The proof is by induction on both d and N. Let $\{d, N\}$ represent the proposition of Theorem A, and assume both $\{d-1, N-1\}$ and $\{d, N-1\}$ are true. With the superfixes $'$ and $*$ relating respectively to $(K; H_1, \ldots, H_{N-1})$ and $(K \cap H_N; H_1 \cap H_N, \ldots, H_{N-1} \cap H_N)$, we have

$$\alpha_n = \alpha'_n + \alpha^*_{n-1} \qquad (n=1, \ldots, d).$$

In obtaining $(K; H_1, \ldots, H_N)$ by adding H_N to $(K; H_1, \ldots, H_{N-1})$, new k-dimensional regions may be formed both in H_N itself, and by partition of already existing such regions by H_N. It follows that

$$v_k = v'_k + v^*_k + v^*_{k-1} \qquad (k=1, \ldots, d).$$

Thus, by the inductive hypotheses,

$$\begin{aligned} v_k &= \sum_{n=d-k}^{d} \binom{n}{d-k} \alpha'_n + \sum_{n=d-k-1}^{d-1} \binom{n}{d-k-1} \alpha^*_n + \sum_{n=d-k}^{d-1} \binom{n}{d-k} \alpha^*_n \\ &= \sum_{n=d-k}^{d} \binom{n}{d-k} \alpha'_n + \sum_{n=d-k-1}^{d-1} \binom{n+1}{d-k} \alpha^*_n \\ &= \sum_{n=d-k}^{d} \binom{n}{d-k} (\alpha'_n + \alpha^*_{n-1}) \\ &= \sum_{n=d-k}^{d} \binom{n}{d-k} \alpha_n, \end{aligned}$$

i.e. the truth of $\{d-1, N-1\}$ and $\{d, N-1\}$ implies that of $\{d, N\}$. But $\{1, N\}$ $(N=1, 2, \ldots)$ and $\{d, 0\}$ $(d=1, 2, \ldots)$ are both trivially true (this explains why $\alpha_0 = 1$). Hence $\{d, N\}$ $(d=1, 2, \ldots; N=0, 1, \ldots)$ is true.

This result was derived in the author's unpublished doctoral thesis (1961) under the condition of restricted general position. Rolf Schneider pointed out its validity under full general position, and utilizes the result in the adjacent paper (1982).

Remarkably, the parameter N does not appear in these formulae. In fact, they are identical to basic formulae in the theory of probabilities of combinations of events (Feller, 1957, Chapter IV, Relation (5.3)). Translating to our notation, in that theory there are d events, α_i is the probability that exactly i of the d events occur, and v_{d-i} is the sum, with $\binom{d}{i}$ terms, of the probabilities that specified i-subsets of events occur. Thus we have, by Feller's Relation (3.1),

Corollary 1.

$$\alpha_n = \sum_{k=0}^{d-n} (-1)^{d-n-k} \binom{d-k}{n} v_k \qquad (n=0, \ldots, d).$$

Moreover, our α's and v's satisfy Bonferroni's Inequalities (Feller, 1957, Chap. IV, Sect. 5(c)). Note, however, that in Feller's theory (i) $\{\alpha_0, \ldots, \alpha_d\}$ is a probability mass function, and (ii) there is no equivalent of our parameter N. Aside from the common formulae, the link between Feller's and our models does not seem clearcut, and is perhaps best illustrated when N is chosen equal to d, and each hyperplane corresponds to an event.

Corollary 2. *The sums*

$$\sum_{i=0}^{d} \alpha_i = v_d, \qquad \sum_{j=0}^{d} v_j = \sum_{i=0}^{d} 2^i \alpha_i.$$

Corollary 3. *When* $K = \mathbb{R}^d$, $\alpha_n = \binom{N}{n}$ $(n=0, \ldots, d)$ *(maximal) and*

$$v_k = \binom{N}{d-k} \sum_{n=d-k}^{d} \binom{N-d+k}{N-n} \qquad (k=0, \ldots, d).$$

Corollary 3 was obtained by Buck (1943, Corollary 2 to Theorem 1), apart from his not reducing the expression to simplest form!

Harding (1967, Theorem 1) considered the case of $N+1$ points in 'pointwise' general position in $\mathbb{R}^d$ (i.e., no $i+2$ points lie in an i-flat $(i=0, \ldots, d-1)$), and showed that the total number of distinct (combinatorial) ways in which they may be partitioned into two disjoint subsets by a separating hyperplane is v_d. Watson (1969) demonstrated the duality existing between Schläfli's and Harding's models. Surprisingly, he omitted to mention that Harding's Theorem 2, relating to the corresponding number of partitions by a hyperplane constrained to contain a given j-flat, corresponds under duality to the intersection of a given $(d-j-1)$-flat with N hyperplanes in restricted general position in $\mathbb{R}^d$. A related interesting application of duality has been made by Ziezold (1970). Extending duality to our Theorem A we obtain

Theorem B. *Suppose* K *is a compact convex set and* $\mathscr{X}$ *is a set of* N *points in pointwise general position in* $\mathbb{R}^d$. *Let* $\alpha_0 = 1$, *and* α_i *be the number of distinct* i-*subsets of* $\mathscr{X}$ *whose containing* $(i-1)$-*flat does not intersect* K $(i=1, \ldots, d)$. *Then the total number of distinct (combinatorial) ways in which a hyperplane containing exactly* $j+1$ *points of* $\mathscr{X}$ *and not intersecting* K *separates the* $(N-j)$-*set* *comprising K and the remaining $N-j-1$ points of $\mathscr{X}$ into two subsets is v_{d-j-1} (defined in Theorem A)* $(j+1=0, \ldots, d)$.

Harding's theorems correspond to the 'limit' of Theorem B in which $\mathscr{X}$ is fixed and K 'shrinks' to become the $(N+1)$st point; in particular, his Theorem 1 corresponds to the case $j=-1$. Theorem B may alternatively be proved by the inductive proof dual to that of Theorem A, i.e. the generalizations of Harding's proofs.

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