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On the Balayage of Green Functions on Finely Open Sets

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Introduction

Let E be a locally compact space with countable base and let $((W_\alpha), (\hat{W}_\alpha), m)$ be a triple with the following three properties (henceforth referred to as case B_3):

a) $(W_\alpha)_{\alpha \geqq 0}$ is a Hunt resolvent on E which maps bounded measurable functions with compact support into bounded continuous functions (by "measurable" we always understand "Borel measurable");

b) $(\hat{W}_\alpha)_{\alpha \geqq 0}$ is a Hunt co-resolvent on E which maps bounded measurable functions with compact support into bounded continuous functions;

c) m is a Radon measure such that (W_α) and $(\hat{W}_\alpha)$ are in duality with respect to m.

Then it is well known that there exists a Green function $g: E \times E \to \overline{\mathbb{R}}_+$. Moreover, g has the property that for every $y \in E$ the excessive function $g(\ ,y): x \to g(x,y)$ is invariant under balayage onto neighborhoods of y.

This note arose from the question of characterizing those situations where

(1) $g(\cdot, y)$ is invariant under balayage onto *fine* neighborhoods of y for every $y \in E$.

The answer is that (1) holds if and only if every finely open set is not co-thin at each of it's points, or equivalently, iff every co-finely open set is not thin at each of it's points. Moreover, (1) implies the strong domination principle $(\bar{D})$.

Let $(E, \mathscr{H})$ be a harmonic space in the sense of Bauer [1] such that the constant 1 is superharmonic. If a Green function g exists for $(E, \mathscr{H})$, then we can always associate a triple $((W_\alpha), (\hat{W}_\alpha), m)$ which satisfies the assumptions of B_3 (c.f. Taylor [13]). As a by-product we obtain the following extension of Chapter 33 in [5]: If there exists an adjoint harmonic space for $(E, \mathscr{H})$, then $\hat{g}: (x, y) \to g(y, x)$ defines a Green function for this adjoint harmonic space.

As application we prove the existence of a strong harmonic space which satisfies the domination principle (D) but not the strong domination principle $(\bar{D})$.

In an appendix we show that for Brelot spaces under the assumption of the existence of a Green function several properties are equivalent with $(\bar{D})$, e.g. the above mentioned property (1), or the fine boundary minimum principle. We do not know if (D) implies these properties in this case.

The second author takes the opportunity offered by this note to correct an error in his previous papers [9] and [10] (see the remark at the end of the appendix).

1. Preparations

Under the hypothesis B_3 each of the triples $((W_\alpha), (\hat{W}_\alpha), m)$ and $((\hat{W}_\alpha), (W_\alpha), m)$ satisfies the Kunita-Watanabe hypothesis. Consequently, the potential theory as developed in [11] can be applied by symmetry either to the Hunt process X of (W_α) or to the Hunt co-process $\hat{X}$ of $(\hat{W}_\alpha)$.

Let us recall some notions and results of [11]:

1.1. *Definition.* (a) An excessive (resp. co-excessive) function h is called *harmonic* (resp. *co-harmonic*) iff h is m-almost surely finite and

$$\hat{R}_h^{\complement K} = h \text{(resp. } \hat{R}_h^{*\,\complement K} = h)$$

for every compact subset K of E.

(b) An excessive (resp. co-excessive) function p is called a *potential* (resp. *co-potential*) if p is m-almost surely finite and if 0 is the only harmonic (resp. co-harmonic) minorant of p.

(c) Let $g: E \times E \to \overline{\mathbb{R}}_+$ be the Green function of the process X, which exists according to $T4$ in Chapter II of [11]. For every positive Radon measure μ on E, if the excessive function p_μ (which is defined by $p_\mu(x) = \int g(x, y)\, d\mu(y)$ for $x \in E$) is m-almost surely finite, then p_μ is called the *Green potential of* μ. *Co-Green-potentials* are defined in the obvious same way with respect to the *co-Green function* $\hat{g}: (x, y) \to g(y, x)$ for $\hat{X}$.

The following result holds because of the strong regularity conditions imposed by B_3:

1.2. **Theorem.** *Assume the case B_3. Then for every potential p on E there exists a unique Radon measure μ such that*

$$p(x) = \int g(x, y)\, d\mu(y) \qquad \textit{for all } x \in E.$$

Moreover, every Green potential is a potential[1].

Proof. The first assertion is proved in $T9$, $T1$ in Chapter III of [11]. By $T12$ and $D3$ in Chapter III of [11] it only remains to show that $g(\ ,y)$ is not harmonic for every $y \in E$. Hence assume $g(\ ,y)$ is harmonic for some $y \in E$ and let K be a compact neighborhood of y. In particular, $\mathscr{C}K$ is not co-thin at y. On the other hand the harmonicity of $g(\ ,y)$ implies $\hat{R}_{g(\ ,y)}^{\complement K} = g(\ ,y)$, hence by duality

$$\hat{R}_{g(x,\ )}^{*\,\complement K}(y) = g(x, y) \qquad \text{for every } x \in X,$$

hence $\hat{R}_u^{*\,\complement K}(y) = u(y)$ for every co-excessive function u. This contradicts the fact that K is co-thin at y.

2. Balayage of the Green Function on Finely Open Sets

In this chapter we will always assume the case B_3. Denote by $\mathbf{0}$ (respectively $\mathbf{0}^*$) the class of finely (respectively co-finely) open subsets of E. g denotes the Green function for the process X.

2.1. **Theorem.** *The following properties are equivalent:*

(1) $\hat{R}_{g(\ ,y)}^{0}(x) = g(x, y)$ *for all* $(y, x) \in 0 \times E$, $0 \in \mathbf{0}$.

(2) $0 \subset b^*(0)$ *for all* $0 \in \mathbf{0}$, *where* $b^*(A)$ *denotes the set of points which are co-regular for a subset A of E.*

[1] By symmetry this result remains true if we replace "$g(x, y)$", "potential", "Green potential" by "$\hat{g}(x, y)$", "co-potential", "co-Green potential" respectively.

(3) $0_1 \subset b(0_1)$ *for all* $0_1 \in \mathbf{0}^*$, *where* $b(A)$ *denotes the set of points which are regular for a subset* A *of* E.

(4) $\hat{R}^{*0_1}_{g(x,\ )}(y) = g(x, y)$ *for all* $(x, y) \in 0_1 \times E$, $0_1 \in \mathbf{0}^*$.

Proof. By a symmetry argument it is enough to prove (2)$\Rightarrow$(3)$\Leftrightarrow$(4).

(2)$\Rightarrow$(3): Assume the contrary, i.e. there exists $0_1 \in \mathbf{0}^*$ and $y \in 0_1 \setminus b(0_1)$. Since 0_1 is thin at y we conclude that $V := \{y\} \cup \complement 0_1$ is a fine neighborhood of y. Let 0 be the fine interior of V and denote by W the fine open set $0 \setminus \{y\}$. Then we conclude from $W \subset \complement 0_1$

$$y \in 0_1 \subset X \setminus b^*(\complement 0_1) \subset X \setminus b^*(W).$$

On the other hand we know that 0_1 is thin at $y \in 0_1$, hence $\{y\}$ is totally thin. This implies by (2)

$$y \in 0 \subset b^*(0) = b^*(F \cup \{y\}) = b^*(W) \cup b^*(\{y\}) = b^*(W),$$

i.e. a contradiction.

(3)$\Rightarrow$(4): By duality we have $\hat{R}^{*0_1}_{g(x,\ )}(y) = \hat{R}^{0_1}_{g(\ ,y)}(x)$ for all $(x, y) \in E \times E$, $0_1 \in \mathbf{0}^*$. Consequently, for $x \in b(0_1)$ the property (4) holds. This implies the wanted implication.

(4)$\Rightarrow$(3): Let $0_1 \in \mathbf{0}^*$. If (4) holds, then we have by duality $\hat{R}^{0_1}_{g(\ ,y)}(x) = g(x, y)$ for all $y \in E$, $x \in 0_1$. Hence we conclude for every $x \in 0_1$ that 0_1 is not thin at x, i.e. $0_1 \subset b(0_1)$.

2.2. **Proposition.** *Assume that one of the equivalent properties of* (2.1) *holds. Then the following statements are true:*

i) *Let* p *be the Green potential of a measure with compact support* K *such that* p *is finite on* K. *Then* $p = R^K_p$.

ii) *The strong domination principle* $(\bar{D})$ *holds, i.e. every finite potential is a countable sum of continuous potentials.*

iii) *The corresponding "co-statements" hold.*

Proof. a) Let

$$p(x) = \int g(x, y)\, d\mu(y) \qquad (x \in E)$$

be a Green potential of the measure μ with compact support K such that p is finite. Then we have

$$R^K_p(x) = \inf\{R^0_p(x) \colon K \subset 0,\ 0 \in \mathbf{0}\}.$$

By assumption we have $R^0_{g(\ ,y)} = g(\ ,y)$ for every $y \in 0$, hence we obtain by integration $\hat{R}^0_p = R^0_p = p$. Consequently we have $p = R^K_p = \hat{R}^K_p$.

b) As a consequence of (a) we conclude the maximum principle for Green potentials, i.e. if

$$p(x) := \int g(x, y)\, d\mu(y) \qquad (x \in E)$$

defines a potential and if $S(\mu)$ is compact, then we have

$$\sup\{p(x) \colon x \in S(\mu)\} = \sup\{p(x) \colon x \in X\},$$

since positive constants are excessive.

By Proposition (7.2) in [2] this implies the strong continuity principle for Green potentials, i.e. the Green potential of a measure with compact support C

is continuous on E provided it is finite and continuous on C. Now (ii) is an easy consequence from the fact that Green potentials are lower semi-continuous and from Lusin's theorem.

c) The "co-results" follow by symmetry.

2.3. **Consequences for Harmonic Spaces.** a) Let $(E, \mathscr{H})$ be a strong harmonic space in the sense of Bauer [1] such that a Green function in the sense of [13] exists (necessary and sufficient conditions for the existence of a Green function can be found in [6]). Assume that positive constants are superharmonic.

By the results of Taylor [13] we can conclude that there exists a nice process associated with $(E, \mathscr{H})$ which has a dual in the sense of Kunita-Watanabe such that we are in the case B_3.

Hence all the above results apply here; e.g. if $0 \subset \underline{b}^*(0)$ for every finely open set 0, then the strong domination principle $(\bar{D})$ holds for $(E, \mathscr{H})$. This extends some of the results which have been obtained by the authors in [7] and [8].

b) In particular, if the co-excessive functions are the hyperharmonic functions of some harmonic space $(E, \mathscr{K})$ (e.g. if the adjoint harmonic space in the sense of Mme. Hervé [5] exists) then by 1.2 the hypothesis of proportionality of extreme potentials with respect to $\mathscr{K}$ is satisfied.

Consequently, $(E, \mathscr{H})$ is an adjoint harmonic space for $(E, \mathscr{K})$ (this extends the results of Mme. Hervé in [5], Chapter 33).

c) The result of 2.2 has an analogue for general harmonic spaces, where a Green function does not necessarily exist. Let $\mathscr{H}^*$ be a sheaf of hyperharmonic functions on E such that $(E, \mathscr{H}^*)$ is a $\mathfrak{P}$-harmonic space in the sense of Constantinescu-Cornea [4]. Assume that Doob's convergence axiom holds.

Proposition [2]. *Assume that for every extreme potential p and for every fine neighborhood V of the superharmonic support of p we have $R_p^V = p$. Then $(\bar{D})$ holds. In particular, $(E, \mathscr{H}^*)$ is a Brelot space for which semipolar sets are polar.*

Proof. The second assertion follows from Chapter 9.2 in [4]. To prove $(\bar{D})$ just adapt the proof of 2.2 to the integral representation for the positive superharmonic functions (which is guaranteed by Chapter 11.5 in [4]) and use Proposition 11.4.12 b) of [4].

3. Applications

We consider the example of the Brelot harmonic space $(X, \mathscr{H}_\mu)$ which was given by Constantinescu and Cornea in [3]: Let $r_0, r_1 \in \mathbb{R}$, $0 < r_0 < r_1$, $X_0 = \{z \in \mathbb{C}: r_0 < |z| < r_1\}$ and let X be the topological space obtained from $\{z \in \mathbb{C}: |z| < r_1\}$ by identification of the points in $\{z \in \mathbb{C}: |z| \leqq r_0\}$.

Let μ be a probability measure on $C_{r_0} = \{z \in \mathbb{C}: |z| = r_0\}$. For an open subset U of X and a function h defined on U we say $h \in \mathscr{H}_\mu(U)$ iff h is continuous on U, $h_{|U \cap X_0}$ satisfies the Laplace equation, and if $\{x_0\} = X \setminus X_0$ is contained in U, then

$$\int_{C_{r_0}} \frac{\partial h}{\partial n}(\xi)\, d\mu(\xi) = 0$$

[2] This proposition has been announced in [7].

$\left(\text{here } \frac{\partial h}{\partial n}(\xi) = \lim_{\substack{r \to 1 \\ r > 1}} \frac{h(r\,\xi) - h(x)}{r-1} \; (\xi \in C_{r_0}) \text{ exists and defines a continuous function on } C_{r_0}\right)$.

We denote by σ the surface area measure on C_{r_0}. The following results have been obtained in [3]:

3.1. *Properties of* $(X, \mathscr{H}_\mu)$.

a) $(X, \mathscr{H}_\mu)$ is a Brelot harmonic space.

b) Every semipolar set is polar, $\{x_0\}$ is not polar.

c) If $\mu = f\sigma$ with f continuous such that f is σ-a.e. strictly positive on C_{r_0}, then $(X, \mathscr{H}_\mu)$ has the property (D), i.e. every locally bounded potential is a countable sum of continuous potentials.

d) If $\mu = f\sigma$ with f continuous such that f is σ-a.e. strictly positive on C_{r_0}, and if $\{z \in \mathbb{C}\colon f(z) = 0\}$ has at least two elements, then there are two nonproportional potentials with superharmonic support $\{x_0\}$.

Repeating the proof of Theorem 1.3 in [3] with minor change we obtain

3.2. **Proposition.** *If $\mu = f\sigma$ where f is continuous and strictly positive everywhere on C_{r_0}, then $(X, \mathscr{H}_\mu)$ has the property $(\bar{D})$.*

Since constants are harmonic for $(X, \mathscr{H}_\mu)$ we obtain immediately from the definition of $(X, \mathscr{H}_\mu)$:

3.3. **Lemma.** *The function p_0, defined by $p_0(x_0) = \log \frac{r_1}{r_0}$,*

$$p_0(z) = \log\left(\frac{r_1}{|z|}\right) \qquad (z \in X_0)$$

is a potential for $(X, \mathscr{H}_\mu)$ with $S(p_0) = \{x_0\}$ (where $S(p_0)$ denotes the superharmonic support of p_0).

3.4. **Proposition.** *The following properties are equivalent for $(X, \mathscr{H}_\mu)$:*

(a) *$(\bar{D})$ holds.*

(b) *Every potential p with $S(p) = \{x_0\}$ is proportional to p_0.*

(c) *The hypothesis of proportionality holds.*

Proof. (a)$\Rightarrow$(b): Let p be a potential with $S(p) = \{x_0\}$. Then $\alpha = p(x_0)$ is finite since $\{x_0\}$ is not polar. By $(\bar{D})$ we conclude from the finiteness of p

$$p = \hat{R}_p^{\{x_0\}} = \hat{R}_\alpha^{\{x_0\}} = \alpha\, \hat{R}_1^{\{x_0\}},$$

i.e. (b) holds.

(b)$\Rightarrow$(c): Let $x \in X_0$ and let q be an extreme potential for $(X_0, \mathscr{H}_{\mu|X_0})$ with $S(q) = \{x\}$. q is up to a multiplicative factor uniquely determined. By Satz 5.3.6 of Bauer [1] there exists a unique potential p for $(X, \mathscr{H}_\mu)$ such that $S(p) = \{x\}$ and $p_{|X_0} = q + h$ for some harmonic function h on X_0. Since every potential v for $(X, \mathscr{H}_\mu)$ with $S(v) = \{x\}$ arises in this way from a potential for $(X_0, \mathscr{H}_{\mu|X_0})$ we conclude that p is up to a constant factor the unique potential v for $(X, \mathscr{H}_\mu)$ with $S(v) = \{x\}$.

(c)⇒(a): We have to show that every finite potential is a countable sum of continuous potentials. Hence let p be a finite potential. Then $p=\sum_{n\geq 0} p_n$, where $S(q_0)\subset\{x_0\}$ and $S(q_n)\subset K_n$ for a compact subset $K_n\subset X_0$ such that $q_{n|K_n}$ is continuous ($n\in\mathbb{N}$). Since $(\bar{D})$ holds for Laplace equation we conclude that q_n is continuous for $n\in N$. By the assumption of (c) we know that q_0 is proportional to p_0, hence q_0 is also continuous.

3.5. *Consequence.* From 3.4 and 3.1.c, 3.1.d follows the existence of a harmonic space $(X, \mathcal{H}_\mu)$ which satisfies (D) but not $(\bar{D})$.

To develop an adjoint theory in the sense of Mme. Hervé [5] we have to prove the existence of a base of completely determining sets (and the hypothesis of proportionality).

Remember that an open subset V of a harmonic space is called *completely determining* iff $\hat{R}_p^{\complement V}=p$ for every potential p on X such that $S(p)\subset \complement V$.

3.6. *Remarks.* a) V is completely determining iff $\hat{R}_p^{\complement V}=p$ for every extreme potential p such that $S(p)$ is contained in the boundary of V.

b) It is well known that for the harmonic space which is associated with the Laplace equation in a relatively compact open subset Ω of $\mathbb{R}^2$ the sets $V_1\cap V_2$ are completely determining if V_1, V_2 are balls in $\mathbb{R}^2$ such that $\bar{V}_1, \bar{V}_2\subset\Omega$.

c) To prove the following lemma we use the following result of Sieveking ([12], p. 21):

Let p be an extreme potential. Then $\{E\subset X\colon \hat{R}_p^{\complement E}\neq p\}$ is a filter of subsets of X. Obviously, this filter contains every neighborhood of $S(p)$.

3.7. **Lemma.** *The harmonic space $(X, \mathcal{H}_\mu)$ has a base of completely determining sets.*

Proof. Consider the class $\mathcal{V}$ of subsets V of X such that V is an open ball in $\mathbb{C}$ with $\bar{V}\subset X_0$ or

$$V=\{x_0\}\cup\left\{z\in\mathbb{C}\colon r_0<|z|<r_0+\frac{1}{n}\right\} \quad \text{with } r_0+\frac{1}{n}<r_1 \quad (n\in\mathbb{N}).$$

Every $V\in\mathcal{V}$ is completely determining: If $\bar{V}\subset X_0$ then we obtain this by restriction to some open neighborhood W of $\bar{V}$ with $\bar{W}\subset X_0$ by 3.6.b; in the case $x_0\in V$, assume that for some $x\in V$ and some y in the boundary of V we have $\hat{R}_{p_y}^{\complement V}(x)\neq p_y(x)$, where p_y is an extreme potential with $S(p_y)=\{y\}$. Let W be an open ball in $\mathbb{R}^2$ such that $y\in W$ and $\bar{W}\subset X_0$. By 3.6.c we conclude

$$\hat{R}_{p_y}^{\complement (V\cap W)}(x_1)\neq p_y(x_1)$$

for some $x_1\in V\cap W$. But this contradicts the fact that $V\cap W$ is completely determining for $(X_0, \mathcal{H}_{\mu|X_0})$ (c.f. 3.6.b).

3.8. **Theorem.** *Assume that $(X, \mathcal{H}_\mu)$ has a Green function g*[3]*. Then there exists an adjoint harmonic space $(X, \hat{\mathcal{H}}_\mu)$ in the sense of Mme. Hervé. Moreover, $(X, \mathcal{H}_\mu)$ and $(X, \hat{\mathcal{H}}_\mu)$ have the property $(\bar{D})$, and $\hat{g}\colon (x, y)\to g(y, x)$ defines a Green function for $(X, \hat{\mathcal{H}}_\mu)$.*

[3] By 3.4 this is equivalent with $(\bar{D})$. The assumption is satisfied if $\mu=f\sigma$ such that f is a continuous strictly positive function on C_{r_0}.

Proof. a) By 3.7 we know that $(X, \mathscr{H}_\mu)$ has a base of completely determining sets. Hence the assumption of the existence of a Green function implies the existence of an adjoint harmonic space $(X, \hat{\mathscr{H}}_\mu)$ in the sense of Mme. Hervé [5].

b) To prove the second part of the statement, it suffices according to 2.1 and 2.3 to show the following:

Every $y \in X$ has a base $\mathscr{V}$ of fine neighborhoods such that $\hat{R}^V_{g(\,,y)} = g(\,,y)$ for every $V \in \mathscr{V}$.

Now, if $y = x_0$, then $g(\,,y)$ is proportional with the continuous potential p_0. Consequently we have for every fine neighborhood V of y

$$g(\,,y) = \hat{R}^{\{x_0\}}_{g(\,,y)} \leqq \hat{R}^V_{g(\,,y)} \leqq g(\,,y),$$

i.e. $g(\,,y) = \hat{R}^V_{g(\,,y)}$.

Next, if $y \in X_0$, then let V be any fine neighborhood of y such that $\overline{V} \subset X_0$, and $\overline{V}$ is compact. To prove $\hat{R}^V_{g(\,,y)} = g(\,,y)$ it is by (3.6.c) enough to prove $\hat{R}^{\complement V}_{g(\,,y)} \neq g(\,,y)$.

Since the swept measure $\varepsilon_x^{\complement V}$ $(x \in \overline{V})$ is the same for both of the harmonic spaces $(X, \mathscr{H}_\mu)$ and $(X_0, \mathscr{H}_{\mu|X_0})$ we conclude $\hat{R}^{\complement V}_{g(\,,y)} \neq g(\,,y)$ from the self-adjointness of $(X_0, \mathscr{H}_{\mu|X_0})$ and théorème 32.5.2 in [5].

Appendix

The harmonic space studied in § 3 gives an example of a harmonic space which satisfies the domination principle (axiom D) but not the strong domination principle (axiom $\bar{D}$). But in this example there exist two nonproportional potentials with superharmonic support $\{x_0\}$, i.e. no Green function exists. This suggests the following natural question:

Let $(E, \mathscr{H})$ be a harmonic space in the sense of [1] for which a Green function exists. Does axiom (D) imply axiom $(\bar{D})$?

We did not succeed to find an answer of this question. In the following we give several equivalent statements of axiom $(\bar{D})$.

Theorem. *Let $(E, \mathscr{H})$ be a harmonic space with a Green function g such that axiom (D) is satisfied. Assume that 1 is hyperharmonic. Then the following properties are equivalent:*

(a) *The fine boundary minimum principle: Let f be a finely hyperharmonic function on a finely open set U in E such that $f \geqq -p|_U$ for some potential p on E. If fine $\liminf_{x \to z} f(x) \geqq 0$ for every z in the fine boundary $\partial_f U$ of U, then $f \geqq 0$.*

(b) *For every finely open set $0 \subset E$ and every $y \in 0$ we have*

$$R^0_{g(\,,y)}(x) = g(x,y) \qquad \textit{for every } x \in E. \tag{1}$$

(Remark that this is one of the equivalent statements of 2.1.*)*

(c) *The maximum principle for Green potential: Let p be the Green potential of a measure μ with compact support K. Then*

$$\sup\{p(x) \colon x \in K\} = \sup\{p(x) \colon x \in E\}.$$

(d) *The strong continuity principle for Green potentials: Let p be the Green potential of a measure μ with compact support K. If the restriction of p to K is finite and continuous, then p is continuous on E.*

(e) *The strong domination principle ($\bar{D}$).*

(f) *Let μ be a measure with compact polar support K. If the Green potential p, defined by*

$$p(x)=\int g(x,y)\,d\mu(y) \quad (x\in E),$$

is finite on K, then

$$\text{co-fine}\limsup_{x\to y} p(x)<+\infty \quad \textit{for all } y\in K.$$

(g) *For every Green potential p of a measure μ on E, μ does not charge polar subsets of* $\{x\in E\colon p(x)<\infty\}$.

The proof of the equivalence will proceed in the following way:

$$\begin{array}{c}(a)\Rightarrow(b)\Rightarrow(c)\Rightarrow(d)\\ \nwarrow \qquad\qquad \swarrow\\ (g)\Leftarrow(f)\Leftarrow(e)\end{array}$$

(a)$\Rightarrow$(b): Let 0 be finely open and let $y\in 0$. Let 0_1 be a finely open set with fine closure $\bar{0}_1^f\subset 0$ and $y\in 0_1$. Define $U=E\setminus\bar{0}_1^f$. U is finely open. Moreover, the function f which is defined by

$$f(x)=R^0_{g(\,,y)}(x)-g(x,y) \quad (x\in U)$$

is finely hyperharmonic on U and satisfies $f\geqq -g(\,,y)|_U$. Furthermore we have fine $\lim_{x\to y} f(x)=0$ for every $z\in\partial_f U$.

Consequently we obtain $f\geqq 0$ from (a), i.e. $R^0_{g(\,,y)}=g(\,,y)$.

(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(e): see the proof of 2.2. Remark that the proof of (c)$\Rightarrow$(d) (which was given by Proposition 7.2 of Blumenthal-Getoor in [2]) is a standard analytical proof.

(e)$\Rightarrow$(f): Since p is finite, we can represent p as the sum of a sequence of continuous potentials. Since a continuous potential with compact polar support vanishes on E, we conclude $p=0$, i.e. (f).

(f)$\Rightarrow$(g): Let K be a compact polar subset of $[p<\infty]$ and let μ_1 be the restriction of μ on K. We have to show that $p_1(x):=\int g(x,y)\,d\mu_1(y)=0$ for every $x\in E$.

Obviously p_1 is a potential which is finite on K. Let $\tilde{p}$ be the function defined by $\tilde{p}:=p_1$ on $E\setminus K$ and $\tilde{p}(x):=\text{co-fine}\limsup_{y\to x} p_1(y)$ for $x\in K$.

By (f) we know that $\tilde{p}$ is finite. Moreover, $\tilde{p}$ is co-finely upper semi-continuous on E, hence we conclude from Lemma 5 in [10]

$$R^K_{\tilde{p}}=\inf\{R^0_{\tilde{p}}\colon 0\supset K,\ 0 \text{ is co-finely open}\}.$$

Since K is polar we obtain

$$0=\hat{R}^K_{\tilde{p}}=\widehat{\inf}\{R^0_{\tilde{p}}\colon 0\supset K,\ 0 \text{ is co-finely open}\}.$$

On the other hand we have for every co-finely open set $0 \supset K$ and every $x \in E$

$$\hat{R}_{\bar{p}}^{0}(x) = \hat{R}_{p_1}^{0}(x) = \int_K \hat{R}_{g(\,,y)}^{0}(x)\, d\mu_1(y)$$
$$= \int_K \hat{R}_{g(x,\,)}^{*0}(y)\, d\mu_1(y) = \int_K g(x,y)\, d\mu_1(y) = p_1(x).$$

Consequently, p_1 vanishes everywhere except on a polar set; p_1 being a potential, this exceptional set is empty.

(g)⇒(a): This is exactly the proof of the fine boundary minimum principle given in [9].

Remark. In the papers [9] and [10] the second author used the statement (g) to prove some generalizations of the fine boundary minimum principle. Hence, one of the seven equivalent statements of the last theorem should be added to the hypothesis called B_2 in [9] and [10] in order to give a correct proof of the fine boundary minimum principles of these papers. This of course is superfluous if it turns out that the answer to the question of the beginning of this section is yes.

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