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A Remark on the Behaviour at Infinity of the Potential Kernel

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Let $(\mu_t)_{t>0}$ be a vaguely continuous convolution semigroup of probability measures on a locally compact abelian group G, and suppose that $(\mu_t)_{t>0}$ is transient, i.e. that the integral $\int_0^\infty \mu_t\,dt$ converges vaguely. The positive measure

$$\kappa = \int_0^\infty \mu_t\,dt$$

is then called the *potential kernel* for $(\mu_t)_{t>0}$. The purpose of the note is to give some results on the behaviour at infinity of the symmetrized measure $\frac{1}{2}(\kappa+\check{\kappa})$, using the theory of Fourier transformation of positive definite measures.

A (complex) measure μ on G is said to be *positive definite* if

$$\langle \mu, f*\tilde{f}\rangle \geqq 0 \quad \text{for all } f\in C_c(G),$$

where $C_c(G)$ is the set of continuous complex functions on G with compact support, and where $\tilde{f}$ for $f\in C_c(G)$ is the function on G defined by $\tilde{f}(x)=\overline{f(-x)}$ for $x\in G$.

Let ω_G be a fixed Haar measure on G and let $\omega_{\hat{G}}$ be the "dual" Haar measure on the dual group $\hat{G}$. Given a positive definite measure μ on G there exists a uniquely determined positive measure σ on the dual group $\hat{G}$ such that (cf. [1])

$$\mu*f*\tilde{f}(x)=\int_{\hat{G}}(x,\gamma)\,|\hat{f}(\gamma)|^2\,d\sigma(\gamma) \quad \text{for } f\in C_c(G) \text{ and } x\in G,$$

where $\hat{f}(\gamma)$ denotes the value at the character $\gamma\in\hat{G}$ of the Fourier transform $\hat{f}$ of $f\in C_c(G)$. The measure σ on $\hat{G}$ is called the *Fourier transform* of μ and is also denoted $\mathscr{F}\mu$.

The convolution semigroup $(\mu_t)_{t>0}$ on G is determined in the following way

$$\hat{\mu}_t(\gamma)=\exp(-t\psi(\gamma)) \quad \text{for } t>0 \text{ and } \gamma\in\hat{G},$$

where $\psi\colon \hat{G}\to\mathbb{C}$ is the *continuous, negative definite function* on $\hat{G}$ associated with $(\mu_t)_{t>0}$ (cf. [2]). The closed subgroup H of G generated by $\bigcup_{t>0}\operatorname{supp}(\mu_t)$ is the annihilator of the closed subgroup $\{\gamma\in\hat{G}\,|\,\psi(\gamma)=0\}$ of $\hat{G}$.

It is shown in [1] that the symmetrization $\nu=\frac{1}{2}(\kappa+\check{\kappa})$ of the potential kernel κ is a positive definite measure on G and that the function $\operatorname{Re}\psi^{-1}$ is locally integrable on $\hat{G}$. Furthermore, denoting by ω_H a fixed Haar measure on H, the Fourier transform of ν has the representation

$$\mathscr{F}\nu=\beta\mathscr{F}\omega_H+(\operatorname{Re}\psi^{-1})\,\omega_{\hat{G}}, \tag{1}$$

where β is a non-negative constant. The measure ω_H, considered as a measure on G, is positive definite, and $\mathscr{F}\omega_H$ is a Haar measure on the subgroup $H^\perp = \{\gamma\in\hat{G} \mid \psi(\gamma)=0\}$ of $\hat{G}$.

It follows by (1) that

$$\nu * f * \tilde{f}(x) = \beta\,\omega_H * f * \tilde{f}(x) + \int_{\hat{G}} (x,\gamma) |\hat{f}(\gamma)|^2 \operatorname{Re} \psi^{-1}(\gamma)\, d\omega_{\hat{G}}(\gamma) \tag{2}$$

for $f\in C_c(G)$ and $x\in G$, and the function

$$\hat{G} \ni \gamma \mapsto |\hat{f}(\gamma)|^2 \operatorname{Re} \psi^{-1}(\gamma)$$

is in particular integrable on $\hat{G}$.

The constant β in (1) can be characterized in the following intrinsic way.

Theorem 1. *The set of numbers $\alpha\geqq 0$, such that the measure $\nu-\alpha\omega_H$ is positive definite, is the closed interval $[0,\beta]$, i.e. β is the largest number $\alpha\geqq 0$ such that the measure $\nu-\alpha\omega_H$ is positive definite.*

The proof is based on the following result.

Lemma 2. *Suppose that G is non-compact and let φ be a real-valued, continuous positive definite function on G. Then*

$$\limsup_{x\to\infty} \varphi(x) \geqq 0,$$

i.e. for every $\varepsilon>0$ and every compact set $K\subseteqq G$ there exists $x\in G\setminus K$ such that $\varphi(x)\geqq -\varepsilon$.

Proof. Suppose on the contrary that there exist an $\varepsilon>0$ and a compact set $K\subseteqq G$ such that

$$\varphi(x) < -\varepsilon \qquad \text{for } x\in G\setminus K.$$

Putting $a=\int_K |\varphi(x)|\, d\omega_G(x)$, we may choose a compact set $L\subseteqq G$ such that $K\subseteqq L$ and $\omega_G(L\setminus K) > 2a\varepsilon^{-1}$. Since G is amenable, we may choose (cf. [3], p. 61) $g\in C_c^+(G)$ such that $0\leqq g*\tilde{g}\leqq 1$ and $g*\tilde{g}\geqq \frac{1}{2}$ on L, and it follows that

$$\begin{aligned} 0 &\leqq \int \varphi(x) g*\tilde{g}(x)\, d\omega_G(x) = \int_K \varphi(x) g*\tilde{g}(x)\, d\omega_G(x) + \int_{G\setminus K} \varphi(x) g*\tilde{g}(x)\, d\omega_G(x) \\ &\leqq a - \varepsilon \int_{L\setminus K} g*\tilde{g}(x)\, d\omega_G(x) < a - a = 0, \end{aligned}$$

which is a contradiction.

Proof of Theorem 1. For $f\in C_c^+(G)$ such that $\langle \omega_H, f*\tilde{f}\rangle > 0$ and $\alpha\geqq 0$ such that $\nu-\alpha\omega_H$ is positive definite we have

$$0 \leqq \langle \nu-\alpha\omega_H, f*\tilde{f}\rangle = \langle \nu, f*\tilde{f}\rangle - \alpha \langle \omega_H, f*\tilde{f}\rangle,$$

which shows that the set

$$I = \{\alpha\geqq 0 \mid \nu-\alpha\omega_H \text{ is positive definite}\}$$

is bounded. With $\alpha_0=\sup I$ it is easy to see that $I=[0,\alpha_0]$.

Replacing f by $\tilde{f}$ and putting $x=0$ in (2) we get

$$\langle \nu-\beta\omega_H, f*\tilde{f}\rangle = \int_{\hat{G}} |\hat{f}(-\gamma)|^2 \operatorname{Re}\psi^{-1}(\gamma)\, d\omega_{\hat{G}}(\gamma) \geqq 0,$$

which implies that $\nu-\beta\omega_H$ is positive definite, and hence $\beta \leqq \alpha_0$.

Writing $\sigma=\nu-\alpha_0\omega_H$ we get from (2) that

$$\sigma*f*\tilde{f}(x) = -(\alpha_0-\beta)\omega_H*f*\tilde{f}(x) + \int_{\hat{G}} (x,\gamma)|\hat{f}(\gamma)|^2 \operatorname{Re}\psi^{-1}(\gamma)\, d\omega_{\hat{G}}(\gamma) \tag{3}$$

for all $f\in C_c(G)$ and $x\in G$.

We now apply (3) to a $f\in C_c^+(G)$ for which $\langle\omega_H, f*\tilde{f}\rangle=1$. Since the second member on the righthand side of (3) tends to zero at infinity by the Riemann-Lebesgue lemma, there exists for every given $\varepsilon>0$ a compact set $K\subseteqq G$ such that

$$\left|\int (x,\gamma)|\hat{f}(\gamma)|^2 \operatorname{Re}\psi^{-1}(\gamma)\, d\omega_{\hat{G}}(\gamma)\right| < \varepsilon/2 \qquad \text{for } x\in G\setminus K.$$

By Lemma 2, applied to the restriction of $\sigma*f*\tilde{f}$ to the subgroup H of G, (H is easily seen to be non-compact, since $(\mu_t)_{t>0}$ is transient), there exists $x_0\in H\setminus K$ such that $\sigma*f*\tilde{f}(x_0)\geqq -\varepsilon/2$, and since $\omega_H*f*\tilde{f}(x_0)=\langle\omega_H, f*\tilde{f}\rangle=1$, it follows that

$$\beta-\alpha_0 = \sigma*f*\tilde{f}(x_0) - \int_{\hat{G}} (x_0,\gamma)|\hat{f}(\gamma)|^2 \operatorname{Re}\psi^{-1}(\gamma)\, d\omega_{\hat{G}}(\gamma) \geqq -\varepsilon,$$

and hence $\beta\geqq\alpha_0$.

A (complex) measure μ on G is said to *tend to zero at infinity*, if the function $\mu*f$ tends to zero at infinity for all $f\in C_c(G)$.

Let $C_0(G)$ denote the set of continuous complex functions on G tending to zero at infinity, which is a Banach space under the uniform norm. If the measure μ tends to zero at infinity, the linear mapping $f\mapsto\mu*f$ of $C_c(G)$ into $C_0(G)$, is continuous when $C_c(G)$ is equipped with the usual inductive limit topology, on account of the closed graph theorem.

Lemma 3. *Let μ be a measure on G which tends to zero at infinity. For every compact subset $A\subseteqq C_c(G)$ and every $\varepsilon>0$ there exists a compact subset $K\subseteqq G$ such that*

$$|\mu*f(x)|\leqq\varepsilon$$

for all $f\in A$ and all $x\in G\setminus K$.

Proof. By the above remarks the set $\{\mu*f \mid f\in A\}$ is compact in $C_0(G)$, so for every $\varepsilon>0$ there exist finitely many functions $f_1,\dots,f_n\in A$ such that for every $f\in A$ there exists an index $i\in\{1,\dots,n\}$ such that $\|\mu*f-\mu*f_i\|_\infty<\varepsilon$, and the conclusion of the Lemma follows immediately.

Lemma 4. *Let μ be a measure on G which is concentrated on a closed subgroup H of G. Then μ (considered as a measure on H) tends to zero at infinity on H if and only if μ tends to zero at infinity on G.*

Proof. If μ tends to zero at infinity on G, then μ tends to zero at infinity on H as a simple consequence of the Tietze extension theorem.

Suppose conversely that μ tends to zero at infinity on H and let $f \in C_c(G)$ and $\varepsilon > 0$ be given. Denoting by K the compact support of f we consider the set

$$A = \{(\tau_{-x} f)|_H \mid x \in K\}$$

of restrictions to H of the functions $\tau_{-x} f$, $x \in K$, where $\tau_{-x} f(y) = f(x+y)$ for $y \in G$. The set A is easily seen to be compact in $C_c(H)$, so by Lemma 3 there exists a compact subset L of H such that

$$\left|\int f(x+h-y)\, d\mu(y)\right| \leqq \varepsilon \qquad \text{for } x \in K,\, h \in H \setminus L.$$

This implies that $|\mu * f(z)| \leqq \varepsilon$ for all $z \in G \setminus (L+K)$. In fact, for $z \in G \setminus (H+K)$ we even have $\mu * f(z) = 0$, and for $z \in (H+K) \setminus (L+K)$, writing $z = h + x$, with $h \in H \setminus L$ and $x \in K$, we find

$$|\mu * f(z)| = \left|\int f(x+h-y)\, d\mu(y)\right| \leqq \varepsilon.$$

Proposition 5. *The measure $\nu - \beta\,\omega_H$ (with β as in Theorem 1) tends to zero at infinity.*

Proof. By Lemma 4 we may suppose that $H = G$. For every compact subset $K \subseteqq G$ the function

$$x \mapsto \nu(x+K), \qquad x \in G$$

is bounded, because there exists $f \in C_c^+(G)$ such that $f * \tilde{f}(x) \geqq 1$ for all $x \in K$, and then

$$\nu(x+K) \leqq \nu * f * \tilde{f}(x) \qquad \text{for all } x \in G,$$

which gives the assertion, since $\nu * f * \tilde{f}$ is a positive definite function, in particular bounded.

For every $f \in C_c^+(G)$ the function $\nu * f$ is uniformly continuous. To see this let $\varepsilon > 0$ be given and let V_0 be a compact symmetric neighbourhood of 0. Putting

$$a = \sup_{x \in G} \nu(x + V_0 - \operatorname{supp}(f)),$$

there exists a neighbourhood V of 0 such that $V \subseteqq V_0$ and

$$|f(x) - f(y)| \leqq \varepsilon / a \qquad \text{for } x, y \in G \text{ with } x - y \in V,$$

and then

$$|\nu * f(x) - \nu * f(y)| \leqq \int |f(x-z) - f(y-z)|\, d\nu(z) \leqq \varepsilon \qquad \text{for } x, y \in G \text{ with } x - y \in V.$$

It follows that there exists for $\varepsilon > 0$ a neighbourhood V of 0 such that

$$|\nu * f(x) - \nu * f * g(x)| \leqq \varepsilon \qquad \text{for } x \in G \tag{4}$$

for all $g \in C_c^+(G)$ satisfying $\operatorname{supp}(g) \subseteqq V$ and $\int g\, d\omega_G = 1$.

By the Riemann-Lebesgue lemma and formula (2), the function $(\nu - \beta\,\omega_G) * h * \tilde{h}$ tends to zero at infinity for every $h \in C_c(G)$, and by polarization we get that $(\nu - \beta\,\omega_G) * f * g$ tends to zero at infinity for all $f, g \in C_c(G)$. With $g \in C_c^+(G)$ choosen such that (4) holds, we find that

$$\begin{aligned} |(\nu - \beta\,\omega_G) * f(x)| &\leqq |\nu * f(x) - \nu * f * g(x)| + |(\nu - \beta\,\omega_G) * f * g(x)| \\ &\leqq \varepsilon + |(\nu - \beta\,\omega_G) * f * g(x)|, \end{aligned}$$

which shows that the measure $\nu - \beta\,\omega_G$ tends to zero at infinity.

Remarks

1) The constant β can also be characterized as the uniquely determined non-negative number α such that the measure $\nu - \alpha\,\omega_H$ tends to zero at infinity.

2) In the case $H = G$ we have for all $f \in C_c(G)$:

$$\lim_{x \to \infty} \nu * f(x) = \beta \langle \omega_G, f \rangle.$$

3) If the convolution semigroup $(\mu_t)_{t>0}$ consists of symmetric measures, then $\beta = 0$ (cf. [1]).

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