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Continuity of Stochastic Processes with Values in the Dual of a Nuclear Space

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Summary. Let E' be the dual of a nuclear Fréchet space E (of which the Schwartz space $\mathscr{S}'(\mathbb{R}^d)$ of tempered distributions is one). We give a simple sufficient condition for the strong continuity of a weakly continuous E'-valued stochastic process and as applications examine the Kolmogorov-Hahn-Delporte criteria.

1. Introduction and Results

Let $\mathscr{S}(\mathbb{R}^d)$ be the nuclear space of all rapidly decreasing functions on $\mathbb{R}^d$ and $\mathscr{S}'(\mathbb{R}^d)$ the topological dual space of $\mathscr{S}(\mathbb{R}^d)$. A. Martin-Löf [7] gave a class of strongly continuous $\mathscr{S}'(\mathbb{R}^d)$-valued Gaussian processes and K. Itô [5] proved the strong continuity of a weakly continuous $\mathscr{S}'(\mathbb{R}^d)$-valued additive process. Let E' be the topological dual space of a nuclear Fréchet space E. Then, motivated by [5], the author proved the strong continuity of a weakly continuous E'-valued Gaussian process [8] and proved the existence of a version which is right continuous with left limits in the strong topology for an E'-valued martingale associated with a right continuous and increasing family of σ-fields [9].

The aim of this paper is to give a simple sufficient condition for the strong continuity of a weakly continuous E'-valued stochastic process and as applications to examine the Kolmogorov-Hahn-Delporte criteria.

Before stating results we give some notations. Let $\|\cdot\|_1 \leqq \|\cdot\|_2 \leqq \dots \leqq \|\cdot\|_p \leqq \dots$ be an increasing sequence of Hilbertian semi-norms defining the topology of E, E_p the completion of E by $\|\cdot\|_p$, E'_p the topological dual space of E_p and $\|\cdot\|_{-p}$ the dual norm of E'_p. Let $\langle x, \xi\rangle$ be the canonical bilinear form on $E' \times E$ and $\langle x, \xi\rangle_p$ the canonical bilinear form on $E'_p \times E_p$. We assume all stochastic processes in this paper are defined on a complete probability space, so if necessary we denote by $(\Omega, \mathscr{F}, P)$ the probability space.

Theorem 1. *Let E be a nuclear Fréchet space and $\mathbf{X} = \{X_t; t \in [0,1]\}$ an E'-valued stochastic process. If for each ξ in E the real stochastic process X_ξ*

$=\{\langle X_t,\xi\rangle; t\in[0,1]\}$ *has a continuous version and there exist positive number* δ *and a countable dense subset* Q *of* $[0,1]$ *which are independent of* ξ *such that*

$$E[\sup_{t\in Q}|\langle X_t,\xi\rangle|^\delta]<\infty, \tag{1.1}$$

where $E[\cdot]$ *is the mathematical expectation, then there exists a natural number* p *such that* $\mathbf{X}$ *has a* $\|\cdot\|_{-p}$*-continuous version.*

As applications of Theorem 1, we shall examine the criteria given by J. Delporte [1] and M.G. Hahn [3] as follows.

Theorem 2. *Let* E *be a nuclear Fréchet space and* $\mathbf{X}=\{X_t; t\in[0,1]\}$ *an* E'*-valued stochastic process. Suppose that for each* ξ *in* E *the real process* $\{\langle X_t,\xi\rangle; t\in[0,1]\}$ *is stochastically continuous and there exist a number* $\alpha_\xi>0$, *a non-negative function* ϕ_ξ *and a sequence of divisions* $\{\Delta_n(\xi)\}$ *of* $[0,1]$ *such that*

$\phi_\xi(h)$ *is non-decreasing in* h *for sufficiently small* h,
$\phi_\xi(h)\to 0$ *as* $h\to 0$,

$$\Delta_n(\xi)=\{t_j^n;\ t_0^n=0<t_1^n<\dots<t_{2^n}^n=1\},$$
$$t_{2k}^{n+1}=t_k^n \quad \text{for } k=0,1,\dots,2^n, \tag{1.2}$$

$$\bigcup_{n=0}^{\infty}\Delta_n(\xi) \quad \text{is a dense subset of } [0,1], \tag{1.3}$$

$$\sum_{n=0}^{\infty}[\phi_\xi(1/2^{n+2})]^{-1}E[\sup_{\Delta_n(\xi)}|\langle X_{t_j^n},\xi\rangle-\langle X_{t_{j-1}^n},\xi\rangle|^{\alpha_\xi}]^{\lambda_\xi}<\infty, \tag{1.4}$$

where $\lambda_\xi=\min\{1,1/\alpha_\xi\}$.

If $\alpha=\inf_\xi\alpha_\xi>0$, *then there exists a natural number* p *such that* $\mathbf{X}$ *has a* $\|\cdot\|_{-p}$*-continuous version.*

Theorem 3. *Let* E *be a nuclear Fréchet space,* $\mathbf{X}=\{X_t; t\in[0,\infty)\}$ *an* E'*-valued stochastic process and for each* ξ *in* E *let* $V_t(\xi)$ *be a non-negative, non-decreasing and continuous function of* t. *Suppose that for each* $T>0$ *and each* ξ *in* E *there exist a number* $\alpha_{\xi,T}>0$ *and a non-negative function* $\Psi_{\xi,T}$ *which is non-decreasing in a neighborhood of* 0 *such that*

$$E[|\langle X_t,\xi\rangle-\langle X_s,\xi\rangle|^{\alpha_{\xi,T}}]\leqq\Psi_{\xi,T}(V_t(\xi)-V_s(\xi)),\quad 0\leqq s\leqq t\leqq T, \tag{1.5}$$

$$\int_0 y^{-(1+\lambda_{\xi,T})}\,\Psi_{\xi,T}(y)^{\lambda_{\xi,T}}\,dy<\infty,\ \text{where } \lambda_{\xi,T}=\min\{1,1/\alpha_{\xi,T}\}. \tag{1.6}$$

If $\alpha_T=\inf_\xi\alpha_{\xi,T}>0$, *then* $\mathbf{X}$ *has a strongly continuous version.*

As a corollary of Theorem 3 we have a generalized Kolmogorov's criterion.

Corollary. *Let* $\mathbf{X}=\{X_t; t\in[0,\infty)\}$ *be an* $\mathscr{S}'(\mathbb{R}^d)$*-valued stochastic process such that for each* $T>0$ *and each* φ *in* $\mathscr{S}(\mathbb{R}^d)$,

$$E[|X_t(\varphi)-X_s(\varphi)|^{\alpha_{\varphi,T}}]\leqq\gamma_{\varphi,T}(V_t(\varphi)-V_s(\varphi))^{\beta_{\varphi,T}+1},\quad 0\leqq s\leqq t\leqq T,$$

where $\alpha_{\varphi,T}$, $\beta_{\varphi,T}$ and $\gamma_{\varphi,T}$ are positive numbers. If $\alpha_T = \inf_{\varphi} \alpha_{\varphi,T} > 0$, then **X** *has a strongly continuous version.*

An important application of the corollary is

Example 4. (Theorem 4.1 of [5]). Let $\mathbf{X} = \{X_t; t \in [0, \infty)\}$ be an $\mathscr{S}'(\mathbb{R}^d)$-valued Gaussian process with independent increments such that $X_0 = 0$. Let $m(t, \varphi)$ and $n(t, \varphi)$ be mean and variance of $X_t(\varphi)$. Suppose that $m(t, \varphi)$ and $n(t, \varphi)$ are continuous real functions of t. Then there always exists a strongly continuous curve m_t in $\mathscr{S}'(\mathbb{R}^d)$ such that $m_t(\varphi) = m(t, \varphi)$. Set up $Y_t = X_t - m_t$, then

$$E[|Y_t(\varphi) - Y_s(\varphi)|^4] = 3(n(t, \varphi) - n(s, \varphi))^2.$$

By the corollary, **X** has a strongly continuous version.

The proofs of Theorems 1, 2 and 3 are given in Sect. 2. Section 3 is devoted to the special cases where $E[|\langle X_t, \xi\rangle - \langle X_s, \xi\rangle|^{\alpha_T}] \leqq \gamma_{\xi,T} \Psi_T(t, s)$ for every ξ in E.

2. Proofs

Proof of Theorem 1. We first prove a variation of the Banach-Steinhaus theorem.

Lemma 2.1. *Under the condition* (1.1) *there exist a natural number q and a positive number K which are independent of ξ such that*

$$E[\sup_{t \in Q} |\langle X_t, \xi\rangle|^\delta] \leqq K \|\xi\|_q^\delta \quad \textit{for every } \xi \textit{ in } E. \tag{2.1}$$

Proof. Set up

$$V(\xi) = E[\sup_{t \in Q} |\langle X_t, \xi\rangle|^\delta].$$

By the argument similar to the proof of Lemma 1 of [8], $V(\xi)$ is a lower semi-continuous function on E. Put $E_n = (\xi \in E; V(\xi) \leqq n\}$. Since $V(\xi) < \infty$, $\bigcup_{n=1}^{\infty} E_n = E$. Each E_n is a closed subset of E because $V(\xi)$ is a lower semi-continuous function of ξ. Since E is a complete metrizable space, by the Baire category theorem, (page 62 of [4]), there exist a natural number n_0, an element $\xi_0 \in E$, a positive number ρ and a natural number q such that

$$\xi_0 + \{\xi \in E; \|\xi\|_q \leqq \rho\} \subset E_{n_0}.$$

Hence if $\|\zeta\|_q \leqq \rho$, $\xi_0 - \zeta \in E_{n_0}$. Since $V(\xi)$ is symmetric, so $-\xi_0 + \zeta \in E_{n_0}$. Of course $\xi_0 + \zeta \in E_{n_0}$.

On the other hand if $0 < \delta < 1$, for every $\xi, \eta \in E$ we have

$$V(\xi + \eta) \leqq V(\xi) + V(\eta).$$

Therefore we obtain

$$\begin{aligned} v(2\zeta) &= 2^{\delta} V(\zeta) \\ &= V(-\xi_0+\zeta)+(\xi_0+\zeta)) \\ &\leqq V(-\xi_0+\zeta)+V(\xi_0+\zeta) \\ &\leqq 2n_0. \end{aligned}$$

Thus

$$V(\zeta) \leqq 2^{1-\delta} n_0.$$

Put $K = 2^{1-\delta} n_0/\rho^{\delta}$, then we have the desired inequality.

If $\delta \geqq 1$, $V(\xi)$ is a convex function of ξ, so that the proof is completed similarly.

Without loss of generality we may assume $\delta \leqq 1$. Since X_ξ has a continuous version, by Lemma 2.1 we have

$$\sup_{0 \leqq t \leqq 1} E[|\langle X_t, \xi \rangle|^{\delta}] \leqq K \|\xi\|_q^{\delta} \quad \text{for every } \xi \text{ in } E. \tag{2.2}$$

Since E is a nuclear Fréchet space, there exists an integer $p > q$ such that

$$\sum_{j=1}^{\infty} \|\xi_j\|_q^{\delta} < \infty \tag{2.3}$$

for a C.O.N.S. (Complete Orthonormal System) $\{\xi_j\}$ of E_p (Theorem 3 of [6]). By (2.2) and Sazonov-Minlos' theorem (Theorem 3 of Chap. IV of [2]), we have

$$P(\|X_t\|_{-p} < \infty) = 1 \quad \text{for all } 0 \leqq t \leqq 1, \tag{2.4}$$

so that we can extend (2.1) to

$$E[\sup_{t \in Q} |\langle X_t, \xi \rangle_p|^{\delta}] \leqq K \|\xi\|_q^{\delta} \quad \text{for every } \xi \text{ in } E_p \tag{2.5}$$

and also we have

$$P\left(\|X_t\|_{-p}^2 = \sum_{j=1}^{\infty} \langle X_t, \xi_j \rangle_p^2,\ t \in Q\right) = 1. \tag{2.6}$$

It follows from (2.3), (2.5), (2.6) and $\delta \leqq 1$ that

$$\begin{aligned} E[\sup_{t \in Q} \|X_t\|_{-p}^{\delta}] &\leqq \sum_{j=1}^{\infty} E[\sup_{t \in Q} |\langle X_t, \xi_j \rangle_p|^{\delta}] \\ &\leqq K \sum_{j=1}^{\infty} \|\xi_j\|_q^{\delta} < \infty. \end{aligned} \tag{2.7}$$

Therefore

$$\sup_{t \in Q} \|X_t\|_{-p} < \infty \quad \text{(a.s.P)} \tag{2.8}$$

and

$$\sum_{j=1}^{\infty} \sup_{t \in Q} |\langle X_t, \xi_j \rangle_p|^2 < \infty \quad \text{(a.s.P)}. \tag{2.9}$$

For each ξ in E denote by $X(t, \xi, \cdot)$ the continuous version of X_ξ. Suppose that $\xi \in E_p$, $\eta_n \in E$, $n=1, 2, \ldots$ and $\|\xi - \eta_n\|_p \to 0$ as $n \to \infty$. Then by (2.8) there exists a subset Λ of Ω with probability one such that for any $\omega \in \Lambda$, $\{X(t, \eta_n, \omega)\}$ is a Cauchy sequence in $C[0, 1]$ which is a Banach space of continuous real functions on $[0, 1]$. Set up

$$X_p(t, \xi, \omega) = \text{the limit of } \{X(t, \eta_n, \omega)\} \text{ if } \omega \in \Lambda,$$
$$= 0 \quad \text{if } \omega \notin \Lambda.$$

Then $X_p(t, \xi, \cdot)$ is a continuous version of $\{\langle X_t, \xi\rangle_p; t \in [0, 1]\}$. It follows from (2, 9) that we can construct a $\|\cdot\|_{-p}$-continuous process $\{Z_t; t \in [0, 1]\}$ as follows:

$$Z_t(\omega) = \sum_{j=1}^{\infty} X_p(t, \xi_j, \omega)\, \hat{\xi}_j,$$

where $\{\hat{\xi}_j\}$ is the biorthonormal basis of $\{\xi_j\}$ and the right hand limit means $\|\cdot\|_{-p}$-convergence. What $\{Z_t; t \in [0, 1]\}$ becomes a version of $\{X_t; t \in [0, 1]\}$ follows from (2.4). This completes the proof.

Note. Although the above proof is similar with that in [8], we must mention that we need the inequality (2.3) for our purposes, which was remarked by Professor S. Koshi, to whom the author wishes to express his hearty thanks.

Before going to the proof of Theorem 2, we give a slight modification of Theorem 5.2.2.C of J. Delporte [1].

Proposition 5. *Let $\{X_t; t \in [0, 1]\}$ be a real stochastically continuous process. Suppose that there exist a number $\beta > 0$, a non-negative function ϕ and a sequence of divisions $\{\Delta_n\}$ of $[0, 1]$ such that*

$\phi(h)$ is non-decreasing in h for sufficiently small h,
$\phi(h) \to 0$ as $h \to 0$,

$$\Delta_n = \{t_j^n;\, t_0^n = 0 < t_1^n < \ldots < t_{2^n}^n = 1\}$$
$$t_{2k}^{n+1} = t_k^n \quad \textit{for } k = 0, 1, \ldots, 2^n, \tag{2.10}$$

$$\bigcup_{n=0}^{\infty} \Delta_n \quad \textit{is a dense subset of } [0, 1], \tag{2.11}$$

$$\sum_{n=0}^{\infty} [\phi(1/2^{n+2})]^{-1} E[\sup_{\Delta^n} |X_{t_j^n} - X_{t_{j-1}^n}|^\beta]^\lambda < \infty, \tag{2.12}$$

where $\lambda = \min\{1, 1/\beta\}$.
Then $\{X_t; t \in [0, 1]\}$ has a continuous version.

Proof. Set up $D = \bigcup_{n=1}^{\infty} \Delta_n$, $A_n = \sup_{\Delta^n} |X_{t_j^n} - X_{t_{j-1}^n}|$ and $h_n = \min_{\Delta_n} |t_j^n - t_{j-1}^n|$. Then by (2.10) and (2.11) obviously $h_n > 0$, $h_n > h_{n+1}$ and $\lim_{n \to \infty} h_n = 0$. Let m be a sufficiently large natural number such that ϕ is non-decreasing over $(0, 1/2^{m+2}]$. Then by (2.10), (2.12) and an estimation similar to the proof of the theorem of J. Delporte [1], for almost all $\omega \in \Omega$ we have

$$|X_t(\omega)-X_s(\omega)|\leqq\phi(1/2^{m+2})^{\mu}A(\omega)\qquad\text{for any } s,t\in D, |s-t|\leqq h_m,$$

where

$$\mu=\max\{1,1/\beta\}\quad\text{and}\quad A(\omega)=3\left(\sum_{n=0}^{\infty}\phi(1/2^{n+2})^{-\mu}A_n(\omega)\right)\in L^{\beta}(\Omega,\mathscr{F},P).$$

Therefore for almost all $\omega\in\Omega$, $X_t(\omega)$ is uniformly continuous on a dense subset D of $[0,1]$, so that the proof is completed by the assumption of the stochastic continuity (see Theorem 2.1 of [3]).

Proof of Theorem 2. Let Q be a countable dense subset of $[0,1]$. Set up $Y_t=X_t-X_0$, $Q_\xi=\bigcup_{n=0}^{\infty}\Delta_n(\xi)$, $\delta_\xi=\min\{1,\alpha_\xi\}$ and $\varepsilon=\min\{1,\alpha\}$. Then by making use of the inequalities (1.4), $\alpha_\xi\geqq\alpha$ and

$$\sup_{t\in Q_\xi}|\langle Y_t,\xi\rangle|^{\delta_\xi}\leqq\sum_{n=0}^{\infty}\sup_{\Delta_n(\xi)}|\langle Y_{t_j^n},\xi\rangle-\langle Y_{t_{j-1}^n},\xi\rangle|^{\delta_\xi}\quad\text{(a.s.P)},$$

we have

$$E[\sup_{t\in Q_\xi}|\langle Y_t,\xi\rangle|^{\varepsilon}]^{1/\varepsilon}\leqq\left(\sum_{n=0}^{\infty}E[\sup_{\Delta_n(\xi)}|\langle Y_{t_j^n},\xi\rangle-\langle Y_{t_{j-1}^n},\xi\rangle|^{\alpha_\xi}]^{\lambda_\xi}\right)^{\mu_\xi}<\infty,$$
$$\text{where}\quad \mu_\xi=\max\{1,1/\alpha_\xi\}. \tag{2.13}$$

By the assumption of Theorem 1 and Proposition 5, $\{\langle Y_t,\xi\rangle;t\in[0,1]\}$ has a continuous version for each ξ in E, so that we denote it by $Y(t,\xi,\cdot)$. It follows from (1.3) that

$$\sup_{t\in Q_\xi}|\langle Y_t(\omega),\xi\rangle|^{\varepsilon}=\sup_{t\in Q_\xi}|Y(t,\xi,\omega)|^{\varepsilon}=\sup_{0\leqq t\leqq 1}|Y(t,\xi,\omega)|^{\varepsilon}\quad\text{(a.s.P)},$$

so that by (2.13) we have

$$E[\sup_{t\in Q}|\langle Y_t,\xi\rangle|^{\varepsilon}]\leqq E[\sup_{0\leqq t\leqq 1}|Y(t,\xi,\omega)|^{\varepsilon}]$$
$$=E[\sup_{t\in Q_\xi}|\langle Y_t,\xi\rangle|^{\varepsilon}]<\infty.$$

Thus Theorem 2 is a corollary of Theorem 1.

As an application of Theorem 2 we have

Proposition 6. *Let* $\mathbf{X}=\{X_t;t\in[0,\infty)\}$ *be an E'-valued stochastic process. Suppose that for each $T>0$ and each ξ in E there exist a number $\alpha_{\xi,T}>0$, a non-negative function $\Psi_{\xi,T}$, a non-negative function $\phi_{\xi,T}$ and a sequence of divisions $\{\Delta_n(\xi,T)\}$ of $[0,T]$ such that*

$$\Psi_{\xi,T}(h)\to 0\qquad as\ h\to 0,$$
$$E[|\langle X_t,\xi\rangle-\langle X_s,\xi\rangle|^{\alpha_{\xi,T}}]\leqq\Psi_{\xi,T}(V_t(\xi)-V_s(\xi)),\qquad 0\leqq s\leqq t\leqq T, \tag{2.14}$$

$\phi_{\xi,T}(h)$ is non-decreasing in h for sufficiently small h,

$$\phi_{\xi,T}(h) \to 0 \quad \textit{as } h \to 0,$$
$$\Delta_n(\xi, T) = \{t_j^n; t_0^n = 0 < t_1^n < \ldots < t_{2^n}^n = T\},$$
$$t_{2k}^{n+1} = t_k^n \quad \textit{for } k = 0, 1, \ldots, 2^n, \tag{2.15}$$

$$\bigcup_{n=0}^{\infty} \Delta_n(\xi, T) \textit{ is a dense subset of } [0, T], \tag{2.16}$$

$$\sum_{n=0}^{\infty} [\phi_{\xi,T}(1/2^{n+2})]^{-1} \Big(\sum_{\Delta_n(\xi,T)} \Psi_{\xi,T}(V_{t_j^n}(\xi) - V_{t_{j-1}^n}(\xi))\Big)^{\lambda_{\xi,T}} < \infty, \tag{2.17}$$

where $\lambda_{\xi,T} = \min\{1, 1/\alpha_{\xi,T}\}$.

If $\alpha_T = \inf_{\xi} \alpha_{\xi,T} > 0$, *then* $\mathbf{X}$ *has a strongly continuous version.*

Remark. If $\mathbf{X}$ is an E'-valued Gaussian process with mean 0 there exists a number $C_{\xi,T} = C(\alpha_{\xi,T}) > 0$ such that

$$C_{\xi,T} E[|\langle X_t, \xi\rangle - \langle X_s, \xi\rangle|^2]^{\alpha_{\xi,T}/2} \leqq E[|\langle X_t, \xi\rangle - \langle X_s, \xi\rangle|^{\alpha_{\xi,T}}],$$

so that if $\alpha_{\xi,T} < 1$, from the conditions (2.14) and (2.17) we can derive the conditions such that $\alpha_{\xi,T} = 2$ and $\Psi_{\xi,T}$ and $\phi_{\xi,T}$ are changed for another $\Psi'_{\xi,T}$ and $\phi'_{\xi,T}$ in (2.14) and (2.17). In this case we may consider the condition of the conclusion of Proposition 6 is satisfied automatically.

Proof. By (2.14) and (2.17) we have

$$\sum_{n=0}^{\infty} [\phi_{\xi,T}(1/2^{n+2})]^{-1} E\Big[\sup_{\Delta_n(\xi,T)} |\langle X_{t_j^n}, \xi\rangle - \langle X_{t_{j-1}^n}, \xi\rangle|^{\alpha_{\xi,T}}\Big]^{\lambda_{\xi,T}}$$
$$\leqq \sum_{n=0}^{\infty} [\phi_{\xi,T}(1/2^{n+2})]^{-1} E\Big[\sum_{\Delta_n(\xi,T)} |\langle X_{t_j^n}, \xi\rangle - \langle X_{t_{j-1}^n}, \xi\rangle|^{\alpha_{\xi,T}}\Big]^{\lambda_{\xi,T}}$$
$$\leqq \sum_{n=0}^{\infty} [\phi_{\xi,T}(1/2^{n+2})]^{-1} \Big(\sum_{\Delta_n(\xi,T)} \Psi_{\xi,T}(V_{t_j^n}(\xi) - V_{t_{j-1}^n}(\xi))\Big)^{\lambda_{\xi,T}} < \infty.$$

By Theorem 2, for each $T > 0$ there exists a natural number p_T such that $\{X_t; t \in [0, T]\}$ has a $\|\cdot\|_{-p_T}$-continuous version, which completes the proof.

Proof of Theorem 3. Since $\Psi_{\xi,T}$ is non-decreasing in a neighborhood of 0, it follows from the integral condition (1.6) that

$$\sum_{n=0}^{\infty} (2^{n+1} \Psi_{\xi,T}(1/2^n))^{\lambda_{\xi,T}} < \infty. \tag{2.18}$$

Set up $U_t(\xi) = V_t(\xi) + t$. Then $U_t(\xi)$ is a strictly monotone increasing function of t, so that if we define $\Delta_n(\xi, T) = \{t_j^n; t_j^n$ is the unique solution of the equation $U_t(\xi) = U_0(\xi) + j(U_T(\xi) - U_0(\xi))/2^n\}$, (2.15) and (2.16) are satisfied. Since $\Psi_{\xi,T}$ is non-decreasing in a neighborhood of 0, by (2.18) and a way similar to the proof of Theorem 2.3 of [3] there exists a non-negative, non-decreasing function $\phi_{\xi,T}$ on $[0,1]$ such that $\phi_{\xi,T}(h)\to 0$ as $h\to 0$ and

$$\sum_{n=0}^{\infty}[\phi_{\xi,T}(1/2^{n+2})]^{-1}\Big(\sum_{\Delta_n(\xi,T)}\Psi_{\xi,T}(V_{t_j^n}(\xi)-V_{t_{j-1}^n}(\xi))\Big)^{\lambda_{\xi,T}}<\infty .$$

Thus Theorem 3 is a corollary of Proposition 6.

3. Examples

For each $T>0$ let α_T be positive number and $\Psi_T(t,s)$ a non-negative function of $t,s\in[0,\infty)$ which are independent of ξ in E. For each $T>0$ and each ξ in E let $\gamma_{\xi,T}$ be a positive number.

Proposition 7. *Let $\mathbf{X}=\{X_t; t\in[0,\infty)\}$ be an E'-valued stochastic process.*

(I) *For each $T>0$ if*

$$E[|\langle X_t,\xi\rangle-\langle X_s,\xi\rangle|^{\alpha_T}]\leqq\gamma_{\xi,T}\Psi_T(t,s),\quad 0\leqq s\leqq t\leqq T \ \textit{for any}\ \xi \tag{3.1}$$

in E, then for each $T>0$ there exist a natural number p_T and a positive number A_T such that

$$E[\|X_t-X_s\|_{-p_T}^{\alpha_T}]\leqq A_T\Psi_T(t,s),\qquad 0\leqq s\leqq t\leqq T. \tag{3.2}$$

(II) *In addition to the assumption of* (I) *if there exists a positive locally bounded function $f(t)$ on $[0,\infty)$ such that*

$$\sup_{0\leqq T<\infty}\gamma_{\xi,T}/f(T)<\infty\qquad \textit{for any}\ \xi\ \textit{in}\ E, \tag{3.3}$$

then for each $T>0$ there exist a natural number p which is independent of T and a positive number B_T such that

$$E[\|X_t-X_s\|_{-p}^{\alpha_T}]\leqq B_T\Psi_T(t,s),\qquad 0\leqq s\leqq t\leqq T. \tag{3.4}$$

In the above cases the same type of criteria on α_T and Ψ_T with those of real cases are applicable for the norm ($\|\cdot\|_{-p_T}$ or $\|\cdot\|_{-p}$) continuity of $\mathbf{X}$. For example we have

Example 8. Let $\mathbf{X}=\{X_t; t\in[0,\infty)\}$ be an E'-valued Gaussian process with mean 0. If for each $T>0$ there exists a positive number β_T such that the condition (3.1) with $\alpha_T=2$ and $\Psi_T(t,s)=(t-s)^{\beta_T}$ is satisfied, then by (3.2) and Kolmogorov's criterion for continuity of real Gaussian processes (Theorem 21.3 of [10]), $\{X_t; t\in[0,T]\}$ has a $\|\cdot\|_{-p_T}$-continuous version for each $T>0$. Moreover if the condition (3.3) is satisfied, by making use of (3.4), $\mathbf{X}$ has a $\|\cdot\|_{-p}$-continuous version, where p is a natural number which is independent of T and ξ. Theorem 7 of [7] is the case where $\beta_T=1$.

Before proving Proposition 7 we give a lemma whose proof is similar to that of Lemma 2.1.

Lemma 3.1. (i) *Under the condition* (3.1) *for each* $T>0$ *there exist a natural number* q_T *and a positive number* M_T *which are independent of* ξ *such that*

$$E[|\langle X_t, \xi\rangle - \langle X_s, \xi\rangle|^{\alpha_T}] \leqq M_T \|\xi\|_{q_T}^{\alpha_T} \Psi_T(t,s), \qquad 0 \leqq s \leqq t \leqq T.$$

(ii) *Under the conditions* (3.1) *and* (3.3) *for each* $T>0$ *there exist a natural number* q *which is independent of* T *and* ξ *and a positive number* N_T *which is independent of* ξ *such that*

$$E[|\langle X_t, \xi\rangle - \langle X_s, \xi\rangle|^{\alpha_T}] \leqq N_T f(T) \|\xi\|_q^{\alpha_T} \Psi_T(t,s), \qquad 0 \leqq s \leqq t \leqq T.$$

Proof of Proposition 7. By (i) of Lemma 3.1, (3.2) follows from the estimation similar with (2.7) if $\alpha_T \leqq 2$. If $\alpha_T > 2$ we choose an integer $p_T > q_T$ such that $\sum_{j=1}^{\infty} \|\xi_j\|_{q_T} < \infty$ for a C.O.N.S. $\{\xi_j\}$ of E_{p_T}. Let m be the first natural number such that $\alpha_T/2^{m+1} \leqq 1$. Put $A_T = \left(\sum_{j=1}^{\infty} (M_T \|\xi_j\|_{q_T}^{\alpha_T})^{1/2^m}\right)^{2^m}$, then (3.2) follows by a simple estimation. By (ii) of Lemma 3.1, (II) is proved similarly.

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