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On Non-Singular Transformations that Arise from Kolmogoroff Systems*

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We begin by restating the definitions. A non-singular transformation – an automorphism – of a Lebesgue measure space $(X, \mathscr{B}, \mu)$ is a bi-measurable 1-1 mapping of X onto X that carries null sets onto null sets. We set for an automorphism T and a σ-finite measure $\nu \sim \mu$

$$T\nu(A) = \nu(T^{-1}A), \qquad A \in \mathscr{B}.$$

The full group $[\mathscr{G}]$ of a countable group $\mathscr{G}$ of automorphisms is defined as the group of all automorphisms S of $(X, \mathscr{B}, \mu)$ such that

$$Sx \in \{Ux\colon U \in \mathscr{G}\}, \quad \text{f.a.a. } x \in X.$$

Two groups $\mathscr{G}$ and $\mathscr{G}'$ are weakly equivalent if $[\mathscr{G}]$ and $[\mathscr{G}']$ are conjugate subgroups of the group of automorphisms of $(X, \mathscr{B}, \mu)$. A group is called hyperfinite if it is weakly equivalent to a cyclic group. We recall the notion of the ratio set ([7], comp. also [1]). The ratio set $r(\mathscr{G})$ of a group $\mathscr{G}$ of automorphisms is defined as the set of all $\alpha \geqq 0$ with the following property: For all $A \in \mathscr{B}$, $\mu(A) > 0$, and all $\varepsilon > 0$, there exists a set $B \subset A$, $\mu(B) > 0$, and an $S \in [\mathscr{G}]$ such that $SB \subset A$ and

$$\left| \frac{dS^{-1}\mu}{d\mu}(x) - \alpha \right| < \varepsilon, \quad \text{f.a.a. } x \in B. \tag{1}$$

In (1) every $\nu \sim \mu$ can be used instead of the μ (comp. [6, Lemma 2.1]). $r(\mathscr{G})$ is a closed subset of $[0, \infty)$, $r(\mathscr{G}) \cap (0, \infty)$ a group. For all $\alpha > 1$, all ergodic automorphisms whose ratio set is equal to $\{0\} \cup \{\alpha^i\colon i \in \mathbb{Z}\}$ are weakly equivalent [6, Theorem 2.4]. All ergodic automorphisms whose ratio set is equal to $\{1\}$ admit an invariant measure. The ergodic automorphisms that admit a finite invariant measure form a weak equivalence class and so do the ergodic automorphisms that admit an infinite σ-finite invariant measure. Also all ergodic automorphisms whose ratio set is equal to $[0, \infty)$ form a weak equivalence class (see [4, § 4] and [6, Theorem 2.8]).

In this paper we study a quasi-local measure theoretic structure that is invariant under the action of a group Γ. We formulate our results for the case of a countably infinite Γ. We shall be given a standard Borel structure $(\Omega, \mathscr{A})$ as a state space. We set

$$(X, \mathscr{B}) = \prod_{g \in \Gamma} (\Omega, \mathscr{A}),$$

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and for $x=(x_f)_{f\in\Gamma}\in X$ we set

$$\tau_g x=(x_{gf})_{f\in\Gamma}, \quad g\in\Gamma.$$

Let e denote the unit element of Γ, let

$$\mathscr{B}_{\{e\}}=\{\{x\in X: x_e\in B\}: B\in\mathscr{A}\},$$

and let for a set $\Lambda\subset\Gamma$, $\mathscr{B}_\Lambda$ be the σ-algebra that is generated by $\bigcup_{g\in\Lambda}\tau_g\mathscr{B}_{\{e\}}$.

We are concerned with a probability measure μ on $(X,\mathscr{B})$ that is invariant under the action of Γ:

$$\mu(A)=\mu(\tau_g A), \quad g\in\Gamma,\ A\in\mathscr{B}. \tag{2}$$

For the role that such μ play as states in statistical mechanics see [7]. We will also be given an automorphism T of $(\Omega,\mathscr{A},\nu)$, where

$$\nu(A)=\mu(\{x\in X: x_e\in A\}), \quad A\in\mathscr{A}.$$

We set

$$(T_e x)_g=\begin{cases} Tx_e, & \text{if } g=e,\\ x_g, & \text{if } g\neq e,\end{cases}$$

$$T_g x=\tau_g^{-1} T_e \tau_g x, \quad \text{f.a.a. } x\in X.$$

The group $\mathscr{G}_{\mu,T}$ that is generated by $\{T_g: g\in\Gamma\}$ is a hyperfinite group. (See [4, Section 4].) We make the hypothesis that we have a Kolmogoroff system at hand, that is we assume that a 0-1 law holds: $\mathscr{G}_{\mu,T}$ is assumed to be an ergodic group. T itself is then ergodic. Our aim is to identify the weak equivalence type of $\mathscr{G}_{\mu,T}$.

Denote for a finite set $\Phi\subset\Gamma$

$$\mathscr{G}_\Phi=\{\prod_{g\in\Phi} T_g^{a_g}: a=(a_g)_{g\in\Phi}\in\mathbb{Z}^\Phi\}.$$

We need a familiar lemma.

Lemma. *Let $\Phi(i)$, $i\in\mathbb{N}$, be an increasing sequence of finite subsets of Γ,*

$$\Gamma=\bigcup_{i=1}^{\infty}\Phi(i).$$

If the 0-1 law holds then for all $B\in\mathscr{B}$

$$\lim_{i\to\infty}\sup_{A\in\mathscr{B}_{\Gamma-\Phi(i)}}|\mu(A\cap B)-\mu(A)\,\mu(B)|=0.$$

Proof. If T is ergodic then the group $\mathscr{G}_{\Phi(i)}$ is ergodic on $\mathscr{B}_{\Phi(i)}$ for all $i\in\mathbb{N}$. We conclude that a set $C\in B$ that is invariant under the group $\mathscr{G}_{\Phi(i)}$ is contained in $\mathscr{B}_{\Gamma-\Phi(i)}$, $i\in\mathbb{N}$. This means that a set $C\in B$ is $\mathscr{G}_{\mu,T}$-invariant if and only if

$$C\in\bigcap_{i=1}^{\infty}\mathscr{B}_{\Gamma-\Phi(i)}.$$

Hence the ergodicity of $\mathscr{G}_{\mu,T}$ implies that the σ-field $\bigcap_{i=1}^{\infty}\mathscr{B}_{\Gamma-\Phi(i)}$ is trivial, and the lemma follows from [2, Section 5.5]. Q.e.d.

Theorem. *Let $\Phi(i)$, $i \in \mathbb{N}$, be an increasing sequence of finite subsets of Γ,*

$$\Gamma = \bigcup_{i=1}^{\infty} \Phi(i),$$

and let $\mathcal{G}_{\mu,T}$ be ergodic. Then $r(\mathcal{G}_{\mu,T}) \cap (0, \infty)$ is equal to

$$\Xi = \bigcap_{\varepsilon>0} \bigcup_{i=1}^{\infty} \bigcup_{U \in \mathcal{G}_{\Phi(i)}} \left\{\rho > 0 \colon \mu\left\{x \in X \colon e^{-\varepsilon}\rho < \frac{dU^{-1}\mu}{d\mu}(x) < e^{\varepsilon}\rho\right\} > 0\right\}.$$

Proof. Let $\rho \in \Xi$, and let $\varepsilon > 0$. Our task is to show that for all $A \in \mathcal{B}$, $\mu(A) > 0$, there is a $B \subset A$, $\mu(B) > 0$, and a $W \in [\mathcal{G}_{\mu,T}]$ such that

$$WB \subset A, \qquad e^{-\varepsilon}\rho < \frac{dW^{-1}\mu}{d\mu}(x) < e^{\varepsilon}\rho, \quad \text{f.a.a. } x \in B.$$

From $\rho \in \Xi$ we have that for some $k \in \mathbb{N}$ there is a $U \in \mathcal{G}_{\Phi(i)}$ such that the set

$$E = \left\{x \in X \colon e^{-\varepsilon}\rho < \frac{dU^{-1}\mu}{d\mu}(x) < e^{-\varepsilon}\rho\right\}$$

has positive μ-measure. Also for some $l \geqq i$ there is a $D \in \mathcal{B}_{\Phi(l)}$ such that

$$\frac{\mu(D \cap A)}{\mu(D)} > 2\left(2 + \rho\, e^{-\varepsilon} \mu(E)^2\right)^{-1}. \tag{3}$$

We apply the lemma and find that for some $h \in \Gamma$

$$h\Phi_l \cap \Phi_l = \emptyset \tag{4}$$

and

$$2\mu(A \cap D \cap \tau_h E) > \mu(A \cap D)\, \mu(E). \tag{5}$$

From (4) we have that all sets in $\mathcal{B}_{\Phi(l)}$ are invariant under $\tau_h \mathcal{G}_{\Phi(l)} \tau_h^{-1}$, hence

$$F = \tau_h U \tau_h^{-1}(A \cap D \cap \tau_h E) \subset D. \tag{6}$$

We have from (2) that

$$\mu(F) > \rho\, e^{-\varepsilon} \mu(A \cap D \cap \tau_h E). \tag{7}$$

Hence from (5)

$$2\mu(F) > \rho\, e^{-\varepsilon} \mu(A \cap D)\, \mu(E).$$

And from (3), (6) and (7)

$$\mu(A \cap D \cap F) > \mu(A \cap D) - \left(\mu(D) - \mu(F)\right) > 0.$$

There is a $W \in [\mathcal{G}_{\mu,T}]$ such that for

$$B = \tau_h U^{-1} \tau_h^{-1}(A \cap D \cap F) \subset A,$$

$$Wx = \tau_h U \tau_h^{-1} x, \qquad \text{f.a.a. } x \in B.$$

We have $WB \subset A$ and

$$e^{-\varepsilon}\rho < \frac{dW^{-1}\mu}{d\mu}(x) < e^{\varepsilon}\rho, \qquad \text{f.a.a. } x \in B. \quad \text{Q.e.d.}$$

We see from this theorem that for an ergodic $\mathscr{G}_{\mu,T}$ always $r(\mathscr{G}_{\mu,T}) \neq \{0,1\}$. We have that $r(\mathscr{G}_{\mu,T}) = [0,\infty)$ if the distribution of the random variable $\frac{dT\nu}{d\nu}$ is not discrete. If $\frac{dT\nu}{d\nu}$ has discrete distribution then $\mathscr{G}_{\mu,T}$ contains μ, that is the group

$$\{S \in [\mathscr{G}_{\mu,T}] : S\mu = \mu\}$$

is ergodic (comp. [3] and [4, § 1]).

For an example where $r(\mathscr{G}_{\mu,T}) = \{0\} \cup \{\alpha^i : i \in \mathbb{Z}\}$, for some $\alpha > 1$, and where μ is not a product measure, let $\Omega = \{0,1\}$, $\Gamma = \mathbb{Z}$, $T0 = 1$, $T1 = 0$, and let μ be the Markoff measure with initial distribution

$$p(0) = (1+\alpha)^{-1}, \qquad p(1) = \alpha(1+\alpha)^{-1}$$

and transition matrix

$$\begin{pmatrix} s & 1-s \\ \alpha^{-1}(1-s) & 1-\alpha^{-1}s \end{pmatrix}, \qquad 0 < s < 1.$$

The theorem allows us to identify for the ergodic $\mathscr{G}_{\mu,T}$ the isomorphy type of the factor that they produce (comp. [6, § 2]). These factors will be equal to one of the $\mathscr{A}_\alpha$, $\alpha \geqq 1$ or to $\mathscr{A}_\infty$. For a description of these factors see [8, 1, 5].

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