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Age-Dependent Birth and Death Processes

R. A. Doney

1. Introduction

The oldest mathematical model for the stochastic evolution of a population is the Galton-Watson process, whose study was first motivated by Galton's famous problem on the extinction of family names. This process is only concerned with the sizes of successive generations of the population, so it essentially treats a situation where objects live for a fixed time and are replaced on their death by a random number of children. In the age-dependent branching process the objects live for a random time, but again children are only born when an existing object dies. In the age-dependent birth and death process, which apparently was introduced by Kendall [5], each object is replaced by a random number of children born at various random times throughout its life. The process, which we denote by $Z(t)$, is specified by two quantities, the birth rate $\lambda(x)$ and the death rate $\mu(x)$; an object of age x at time t has probability $\lambda(x)\,dt$ of giving birth to a child in the time interval $[t, t+dt]$, and probability $\mu(x)\,dt$ of dying in this interval. If λ and μ are constants, $Z(t)$ is Markovian, and is in fact the Yule-Furry process. Other wise $Z(t)$ is non-Markovian, and there seems to be little known about it. It shares with age-dependent branching processes the property that the sizes of successive generations form a Galton-Watson process, and one might expect it to have similar properties. In this paper we establish many such similarities (e.g. extinction probability, asymptotic behaviour of first moment, convergence in mean square), but we also establish some interesting differences. In particular (Theorem 7.6) we show that in the supercritical case there is a necessary and sufficient condition for the convergence in distribution of $Z(t)/E(Z(t))$ to a non-degenerate distribution which is weaker than the corresponding condition for the associated Galton-Watson process. The key to all these results is the integral equation satisfied by $F(s,t)(=E\{s^{Z(t)}\})$, which is found in Theorem 3.3. This is rather more complicated than the corresponding equation for branching processes, but we show that it can be exploited in much the same way.

2. The Basic Set-Up and the Generation Sizes

Let J be the collection of all elements $\mathbf{i}$, where each $\mathbf{i}$ is either zero or a finite sequence $\langle i_1, i_2, \ldots, i_k\rangle$ of positive integers. Then by interpreting $\langle 0\rangle$ as the ancestor and $\langle i_1, \ldots, i_k\rangle$ as the i_k-th child of the i_{k-1}-th child of ... of the i_1-th child of the ancestor, we can use J to label all possible descendants of the ancestor. If we prescribe the time of birth and the length of life of each such descendant, then we can trace out the history of the family of the ancestor. This suggests

Definition 2.1. Let $\mathbf{i}_0(=\langle 0\rangle), \mathbf{i}_1, \mathbf{i}_2, \ldots$ be some fixed enumeration of J; then by a *family history* ω we mean a sequence $\{l, b_{\mathbf{i}_1}, l_{\mathbf{i}_1}, b_{\mathbf{i}_2}, l_{\mathbf{i}_2}, \ldots\}$ where each term of the sequence is a non-negative real number such that if $\mathbf{i}=\langle i_1, \ldots, i_k\rangle$ with $i_k>1$ then $b_{\mathbf{i}} \geqq b_{\mathbf{i}'}$, $\mathbf{i}'=\langle i_1, i_2, \ldots, i_{k-1}, i_k-1\rangle$.

If we interpret $l_{\mathbf{i}}$ as the length of $\mathbf{i}$'s life and $b_{\mathbf{i}}$ as the age of the parent of $\mathbf{i}$ when $\mathbf{i}$ is born, then each ω defines a *family* in the following way; for $\mathbf{i}=\langle i_1, \ldots, i_k\rangle$ set

$$v_{\mathbf{i}}(\omega)=\max\{j \mid b_{\mathbf{j}} \leqq l_{\mathbf{i}}, \text{ where } \mathbf{j}=\langle i_1, \ldots, i_k, j\rangle\}. \tag{2.1}$$

Then we can define inductively the *generations*, $I_0(\omega), I_1(\omega)$, by setting $I_0(\omega)=\{\langle 0\rangle\}$, $I_{k+1}(\omega)=\{\mathbf{j}=\langle i_1, \ldots, i_k, j\rangle \mid \mathbf{i}\in I_k(\omega) \text{ and } j \leqq v_{\mathbf{i}}\}$ for $k=0, 1, \ldots$. The *family* $I(\omega)=\bigcup_{k=0}^{\infty} I_k(\omega)$, consists of all descendants of the ancestor which are actually born (note that ω contains information about objects which are never born). To define the population size $Z(t, \omega)$ at time $t>0$, assuming that $Z(0, \omega)=1$, set $Z_{\mathbf{i}}(t, \omega)=1$ if $\mathbf{i}=\langle i_1, \ldots, i_k\rangle \in I_k(\omega)$ and

$$\begin{aligned} &\text{(a)} \quad b_{\langle i_1\rangle}+b_{\langle i_1, i_2\rangle}+\cdots+b_{\mathbf{i}} \leqq t, \\ &\text{(b)} \quad b_{\langle i_1\rangle}+b_{\langle i_1, i_2\rangle}+\cdots+b_{\mathbf{i}}+l_{\mathbf{i}}>t, \end{aligned} \tag{2.2}$$

$Z_{\mathbf{i}}(t, \omega)=0$ otherwise, $Z_k(t, \omega)=\sum_{\mathbf{i}\in I_k(\omega)} Z_{\mathbf{i}}(t, \omega)$ for $k \geqq 1$, $Z_0(t, \omega)=1$ if $l>t$, $=0$ if $l \leqq t$. Then $Z_k(t, \omega)$ is the number of members of the k-th generation which are alive at time t, and so $Z(t, \omega)=\sum_{k=0}^{\infty} Z_k(t, \omega)$ is the population size.

It is useful to note that we can write $Z(t, \omega)=B(t, \omega)-D(t, \omega)$, where $B(t, \omega)$ is the number of objects which are born in $[0, t]$ (including the ancestor) and $D(t, \omega)$ is the number which die in $[0, t]$, so that for each fixed ω $B(t, \omega)$ and $D(t, \omega)$ are *monotone increasing* functions of t. To define these functions rigorously, for $\mathbf{i}\in I_k(\omega)$ put $B_{\mathbf{i}}(t, \omega)=1$ or 0 according as (2.2a) holds or not, $D_{\mathbf{i}}(t, \omega)=1$ if (2.2a) holds and (2.2b) fails, $=0$ otherwise, $B_k(t, \omega)=\sum_{\mathbf{i}\in I_k(\omega)} B_{\mathbf{i}}(t, \omega)$, $D_k(t, \omega)=\sum_{\mathbf{i}\in I_k(\omega)} D_{\mathbf{i}}(t, \omega)$ for $k \geqq 1$, $B_0(t, \omega)=1$, $D_0(t, \omega)=1$ or 0 according as $l \leqq t$ or $l>t$, and

$$B(t, \omega)=\sum_{k=0}^{\infty} B_k(t, \omega), \qquad D(t, \omega)=\sum_{k=0}^{\infty} D_k(t, \omega).$$

We now introduce a probability measure P on the space Ω of all family histories ω which expresses our underlying assumptions. These are that the ancestor is of age zero at time zero; all objects evolve independently of each other; each object is certain to die, and at age x has probability $\mu(x)\,dt$ of dying in $[t, t+dt]$ and probability $\lambda(x)\,dt$ of giving birth to a child in $[t, t+dt]$, these events also being independent. Thus each $l_{\mathbf{i}}$ is the waiting time till the first event in a Poisson process of parameter $D(x)=\int_0^x \mu(y)\,dy$, each $b_{\mathbf{i}}$ is the waiting time till the i_k-th event in a Poisson process of parameter $L(x)=\int_0^x \lambda(y)\,dy$, and we are lead to

Definition 2.2. The propability measure P on Ω is defined by the two assumptions

A. The random variables $l, l_{\mathbf{i}_1}, l_{\mathbf{i}_2}, \ldots$ are independent, identically distributed and $P\{l \leqq x\} = 1 - e^{-\int_0^x \mu(y)\,dy}$, where μ is continuous, $\geqq 0$, and $\int_0^\infty \mu(y)\,dy = +\infty$.

B. The random variables $b_{\mathbf{i}}$ are independent of the $l_{\mathbf{i}}$ and the sequences $\{b_{\langle \mathbf{i},1\rangle}, b_{\langle \mathbf{i},2\rangle}, \ldots\}$ are independent and have the same distribution as $\{b_1, b_2, \ldots\}$, where

$$P\{b_1 \leqq x_1, \ldots, b_n \leqq x_n\} = \int_0^{x_1} \int_{y_1}^{x_2} \cdots \int_{y_{n-1}}^{x_n} \lambda(y_1) \ldots \lambda(y_n)\, e^{-L(y_n)}\, dy_1 \ldots dy_n \qquad (2.3)$$

for $x_n \geqq x_{n-1} \geqq \cdots \geqq x_1 \geqq 0$, where $L(x) = \int_0^x \lambda(y)\,dy$, $\lambda(x)$ is continuous and $\geqq 0$.

Note. A and B together specify a consistent set of finite-dimensional distributions, so Kolmogorov's theorem implies that there is a uniquely defined probability measure P on $(\Omega, \overset{\alpha}{F})$ where $\overset{\alpha}{F}$ is the Borel extension of the cylinder sets in Ω. From the definition of $Z(t, \omega)$ it can be seen that $Z(t, \omega)$ is measurable as a function of (t, ω).

Now let $\xi_k(\omega)$ be the number of objects in the k-th generation, $I_k(\omega)$. We have, writing $G(t) = 1 - e^{-\int_0^t \mu(x)\,dx}$,

Theorem 2.3. *$\{\xi_n(\omega), n \geqq 0\}$ is a Galton-Watson process with generating function* $h(s) = \int_0^\infty e^{(s-1)L(t)}\, dG(t)$.

Proof. Clearly the random variables $v_{\mathbf{i}}$ defined in (2.1) are independent, identically distributed and

$$P\{v = k\} = \int_0^\infty P\{k \text{ births occur in } [0, t]\}\, dG(t).$$

Now $\sum_{k=0}^\infty s^k P\{k \text{ births occur in } [0, t]\} = e^{(s-1)L(t)}$, so it follows that each $v_{\mathbf{i}}$ has generating function $h(s)$, and since $\xi_1(\omega) = v(\omega)$, we have $E(s^{\xi_1(\omega)}) = h(s)$. Also

$$E\{s^{\xi_2(\omega)}\} = \sum_{k=0}^\infty E\{s^{\xi_k(\omega)} \mid \xi_1 = k\}\, P\{\xi_1 = k\} = \sum_{k=0}^\infty E\{s^{v\langle 1\rangle + v\langle 2\rangle + \cdots + v\langle k\rangle}\}\, P\{\xi_1 = k\} = h(h(s)).$$

In the same way one can check that $E(s^{\xi_k(\omega)}) = h(E(s^{\xi_{k-1}(\omega)}))$, and this property is characteristic of the Galton-Watson process.

Remarks. (1) It is well known that $P\{\lim_{n\to\infty} \xi_n(\omega)\} = q$, where q is the smallest root in $[0, 1]$ of $h(s) = s$. Clearly $\lim_{n\to\infty} \xi_n(\omega) = 0 \Rightarrow \lim_{t\to\infty} Z(t, \omega) = 0$; we show later in fact $P\{\lim_{t\to\infty} Z(t, \omega) = 0\} = q$.

(2) Note that $P\{\mathbf{i} \text{ is alive at time } t \mid \mathbf{i} \in I_k(\omega)\}$ is not the same for all members of $I_k(\omega)$; this means that, in contrast to the situation for the age-dependent branching process, we cannot construct the process $\{Z(t, \omega), t \geqq 0\}$ from $\{\xi_n(\omega), n \geqq 0\}$. This suggests that properties of $\xi_n(\omega)$ might not be too reliable as a guide to properties of $Z(t, \omega)$.

If we now set $m = E\{\xi_1(\omega)\} = h'(1) \leqq \infty$, then it is clear that $E\{\xi_n(\omega)\} = m^n < \infty$ for all n, provided $m < \infty$. However, even in the case $m = \infty$, we still have $E\{Z(t, \omega)\} < \infty$ for all $t \geqq 0$, , as we can see from

Lemma 2.4. *Under the basic assumptions,*

$$m(t) = E\{Z(t, \omega)\} = e^{-D(t)} + \sum_{k=1}^{\infty} \iint\cdots\int_{\{0 \leqq x_1 + \cdots + x_k \leqq t\}} e^{-D(t - x_1 - x_2 - \cdots - x_k)} \cdot \prod_{j=1}^{k} \lambda(x_j)\, e^{-D(x_j)}\, dx_1 \ldots dx_k.$$

Proof. We use the decomposition $Z(t, \omega) = B(t, \omega) - D(t, \omega)$, and note that

$$E\big(B(t, \omega)\big) = 1 + \sum_1^{\infty} E\big(B_k(t, \omega)\big) = 1 + \sum_1^{\infty} \beta_k,$$

say. Now, by assumption (A)

$$\begin{aligned}
\beta_k &= \sum_{\mathbf{i} \in J} P\{\mathbf{i} \in I_k(\omega) \text{ and } \mathbf{i} \text{ is born in } [0, t]\} \\
&= \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} P\{b_{\langle i_1 \rangle} \leqq l;\, b_{\langle i_1, i_2 \rangle} \leqq l_{i_1};\, \ldots;\, b_{\langle i_1, \ldots, i_k \rangle} \leqq l_{\langle i_1, \ldots, i_{k-1} \rangle}; \\
&\qquad\qquad b_{\langle i_1 \rangle} + \cdots + b_{\langle i_1, \ldots, i_k \rangle} \leqq t\} \qquad (2.4) \\
&= \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \int_{y_1=0}^{\infty} \cdots \int_{y_k=0}^{\infty} P\{b_{\langle i_1 \rangle} \leqq y_1;\, \ldots;\, b_{\langle i_1, \ldots, i_k \rangle} \leqq y_k;\, b_{\langle i_1 \rangle} + \cdots + b_{\langle i_1, \ldots, i_k \rangle} \leqq t\} \\
&\quad \cdot \prod_1^k \mu(y_j)\, e^{-D(y_j)}\, dy_1 \ldots dy_k.
\end{aligned}$$

From assumption (B) it follows that $b_{\langle i_1, \ldots, i_j \rangle}$ has probability density function $\lambda(x)\, e^{-L(x)} \{L(x)\}^{i_j - 1}/(i_j - 1)!$ and since the b's are independent the probability on the R.H.S. of (2.4) is equal to

$$\int\cdots\int_{\left\{\substack{0 \leqq x_1 \leqq y_1, \ldots, 0 \leqq x_k \leqq y_k \\ x_1 + \cdots + x_k \leqq t}\right\}} \prod_{j=1}^{k} \{\lambda(x_j)\, e^{-L(x_j)} \{L(x_j)\}^{i_j - 1}/(i_j - 1)!\}\, dx_1 \ldots dx_k.$$

Putting this into (2.4) we see that we can interchange the order of integration and perform the y_j integrations to get

$$\begin{aligned}
\beta_k &= \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \int\cdots\int_{\{0 \leqq x_1 + \cdots + x_k \leqq t\}} \prod_{j=1}^{k} \{e^{-D(x_j)} \lambda(x_j)\, e^{-L(x_j)} \{L(x_j)\}^{i_j - 1}/(i_j - 1)!\}\, dx_1 \ldots dx_k \\
&= \int\cdots\int_{\{0 \leqq x_1 + \cdots + x_k \leqq t\}} \prod_{j=1}^{k} e^{-D(x_j)} \lambda(x_j)\, dx_1 \ldots dx_k.
\end{aligned} \qquad (2.5)$$

If we argue in the same way for $D(t, \omega)$, we find that

$$E\big(D(t, \omega)\big) = 1 - e^{-D(t)} + \sum_{k=1}^{\infty} \int\cdots\int_{\{0 \leqq x_1 + \cdots + x_k \leqq t\}} (1 - e^{-D(t - x_1 - \cdots - x_k)}) \cdot \prod_{j=1}^{k} \lambda(x_j)\, e^{-D(x_j)}\, dx_1 \ldots dx_k,$$

which, together with (2.5), proves the lemma.

From this follow several useful facts, which we collect together in

Lemma 2.5. (i) $P\{Z(t,\omega)<\infty$ *for all* $t\geqq 0\}=1$;

(ii) $m(t)$ *is continuous on* $(0,\infty)$;

(iii) $F(s,t)=E(s^{Z(t,\omega)})$ *is continuous on* $[0,\infty)$ *for fixed* s, $|s|\leqq 1$;

(iv) *For each* $n\geqq 0$ $P\{Z(t)=n\}$ *is continuous on* $[0,\infty)$.

Proof. (i) This will follow from $P\{B(t,\omega)<\infty$ for all $t\geqq 0\}=1$, and since $B(t,\omega)$ is monotone in t for fixed ω this will follow if we can show that $E(B(t,\omega))<\infty$ for all $t\geqq 0$. Now $\lambda(x)$ is continuous, so $C_t=\sup_{x\in[0,t]}\lambda(x)<\infty$, and from (2.5) we see that

$$E(B(t,\omega))\leqq 1+\sum_{k=1} C_t^k \int_{\{0\leqq x_1+\cdots+x_k\leqq t\}}\cdots\int dx_1\ldots dx_k=\exp(t\,C_t)<\infty.$$

(ii) As above, we see that

$$\lim_{h\downarrow 0}|m(t)-m(t-h)|\leqq \sum_{k=1}^{\infty} C_t^k \lim_{h\downarrow 0}\int_{\{t-h<x_1+\cdots+x_k\leqq t\}}\cdots\int dx_1\ldots dx_k=0.$$

Similarly $\lim_{h\downarrow 0}|m(t+h)-m(t)|=0$.

(iii) The elementary inequality $|s^a-s^b|\leqq|a-b|$ for any $|s|\leqq 1$, real a and b, gives $|F(s,t_1)-F(s,t_2)|\leqq E\{|Z(t_1)-Z(t_2)|\}$.

Suppose $t_1>t_2$; then since $B(t_1,\omega)\geqq B(t_2,\omega)$, $D(t_1,\omega)\geqq D(t_2,\omega)$ we have

$$E\{|Z(t_1,\omega)-Z(t_2,\omega)|\}\leqq E\{B(t_1,\omega)-B(t_2,\omega)\}+E\{D(t_1,\omega)-D(t_2,\omega)\}\to 0$$

as $t_1\downarrow t_2$ by the argument in (ii).

(iv) By (iii) we know that $Z(t)\xrightarrow{D} Z(t_0)$ as $t\to t_0$ ($\xrightarrow{D}$ means convergence in distribution) and by (i) we know that the distribution of $Z(t)$ is concentrated on the non-negative integers, so this is immediate. □

3. The Equation for $F(s,t)$

We start by formulating mathematically the idea that, for $t>0$, $Z(t,\omega)$ can be considered as the sum of all the living descendants of the children of the ancestor. If $\omega=(l,b_{i_1},l_{i_1},\ldots)$ let $\omega_k=(l_{\langle k\rangle};\,b_{\langle k,i_1\rangle};\,l_{\langle k,i_1\rangle};\ldots)$ denote, for $k\geqq 1$, the family of the k-th child of the ancestor, and let $\omega_0=(l,b_{\langle 1\rangle},b_{\langle 2\rangle},\ldots)$. Then each ω is equivalent to a sequence $(\omega_0,\omega_1,\omega_2,\ldots)$, although if $k>v(\omega)$ none of the family ω_k ever exist. Write $b_{\langle k\rangle}=b_k$; then we have

Theorem 3.1. *For* $t>0$ *set* $v(t,\omega)=\max\{k\colon b_k\leqq t,\,k\leqq v(\omega)\}$, $A_t=\{\omega\colon l(\omega)\geqq t\}$ *and let* $I_{A_t}(\omega)$ *be the indicator function of* A_t. *Then*

$$Z(t,\omega)=\sum_{k=1}^{v(t,\omega)} Z(t-b_k,\omega_k)+I_{A_t}(\omega), \tag{3.1}$$

$$B(t,\omega)=\sum_{k=1}^{v(t,\omega)} B(t-b_k,\omega_k)+1, \tag{3.2}$$

$$D(t,\omega)=\sum_{k=1}^{v(t,\omega)} D(t-b_k,\omega_k)+1-I_{A_t}(\omega). \tag{3.3}$$

Proof. It is easy to check that when $k \leqq v(t,\omega)$, $B_{\langle k,\mathbf{i}\rangle}(t,\omega)=B_{\mathbf{i}}(t-b_k,\omega_k)$; summing over all $\mathbf{i}\in J$ and $k\leqq v(t,\omega)$ yields (3.2), if we remember that $B_{\mathbf{0}}(t,\omega)=1$. A similar argument yields (3.3), and subtracting these two results gives (3.1). $\square$

Now writing $\omega=(\omega_0,\omega_1,\omega_2,\dots)$, $\Omega_j=\{\text{all }\omega_j\}$ we have $\Omega=\prod_0^\infty \Omega_j$, and in the same way we can write $(\Omega,\overset{\alpha}{F},P)=\prod_0^\infty(\Omega_i,\overset{\alpha}{F_i},P_i)$. Also, for $i\geqq i$ the spaces $(\Omega_i,\overset{\alpha}{F_i},P_i)$ will be replicas of the basic space $(\Omega,\overset{\alpha}{F},P)$. Since the random variables $v(t,\omega)$, $b_k(\omega)$ and $I_{A_t}(\omega)$ are $\overset{\alpha}{F_0}$ measurable for fixed t, it is clear that for $|s|\leqq 1$,

$$E\left\{s^{\sum_1^{v(t,\omega)} Z(t-b_k(\omega),\omega_k)+I_{A_t}(\omega)}\,\Big|\,\overset{\alpha}{F_0}\right\}=\left\{\prod_1^{v(t,\omega)} F(s,t-b_k(\omega))\right\}\{s^{I_{A_t}(\omega)}\}.$$

It follows therefore, from (3.1), that $|s|\leqq 1$,

$$F(s,t)=E\left\{\prod_{k=1}^{v(t,\omega)} F(s,t-b_k(\omega))\cdot[s I_{A_t}(\omega)+I_{\bar{A}_t}(\omega)]\right\}, \tag{3.4}$$

where $\bar{A}_t=\Omega-A_t=\{l\leqq t\}$.

Let $N(u,\omega)=\sup\{n\colon b_n(\omega)\leqq u\}$; then if $l(\omega)=l$ and $\tau=\min\{l,t\}$ we see that, for $u\in[0,\tau]$, $N(u,\omega)$ is a step function which increases by one at the points $b_1,\dots,b_{v(t,\omega)}$. Now $F(s,t)$ is a continuous function of t (Lemma 2.5) so it follows that

$$E\left\{\prod_{k=1}^{v(t,\omega)} F(s,t-b_k(\omega))\,|\,l(\omega)=l\right\}=E\left\{\exp\int_0^\tau \log F(s,t-u)\,dN(u,\omega)\right\}. \tag{3.5}$$

We calculate the R.H.S. of (3.5) in

Lemma 3.2. *If $\phi(u)$ is any piecewise continuous, complex-valued function and $\psi(u)=e^{\phi(u)}$ we have*

$$E\left\{\exp\int_0^t \phi(u)\,dN(u,\omega)\right\}=\exp\int_0^t \{\psi(u)-1\}\,\lambda(u)\,du. \tag{3.6}$$

Proof. If we set $N_n^r(\omega)=N((r+1)\delta,\omega)-N(r\delta,\omega)$ for $r=0,1,\dots,n-1$, where $\delta=t/n$, then outside of a set of probability zero (on which $\exists$ a common point of discontinuity of $\phi(u)$ and $N(u,\omega)$) we have

$$\int_0^t \phi(u)\,dN(u,\omega)=\lim_{n\to\infty}\sum_{r=0}^{n-1}\phi(r\delta)\,N_n^r(\omega).$$

Also

$$\{N(t,\omega)=k\}=\bigcup_{0\leqq r_1<r_2<\cdots<r_k<n} A_n(r_1,r_2,\dots,r_k)+E_n^k,$$

where

$$A_n(r_1,r_2,\dots,r_k)=\{N_n^r(\omega)=1 \text{ for } r=r_j, j=1,2,\dots,k;\ N_n^r(\omega)=0 \text{ for } r\neq r_j\},$$

and $\bigcup_{k=0}^{\infty} E_n^k \subset E_n = \{\text{at least one of } N_n^r(\omega) > 1\}$. Now $N(u, \omega)$ is a Poisson Process, so $\lim_{n\to\infty} P\{E_n\} = 0$, and since $\sum_{r=1}^{n-1} \phi(r\delta) N_n^r(\omega) = \sum_{j=1}^{k} \phi(r_j\delta)$ when $\omega \in A_n(r_1, \ldots, r_k)$ we have

$$E\left\{\exp\int_0^t \phi(u)\, dN(u,\omega)\right\} = P\{N(t,\omega)=0\} + \lim_{n\to\infty} \sum_{k=1}^{\infty} \sum_{0\leq r_1<\cdots<r_k<n} \prod_{j=1}^{k} \psi(r_j\delta)\, P\{A_n(r_1,\ldots,r_k)\}. \tag{3.7}$$

Clearly $P\{N(t,\omega)=0\} = P\{b_1 > t\} = e^{-L(t)}$ and

$$P\{A_n(r_1,\ldots,r_k)\} = P\{b_j \in (r_j\delta, (r_j+1)\delta), j=1,2,\ldots,k, b_{k+1} > t\} = e^{-L(t)} \prod_{j=1}^{k} \delta\lambda(r_j\delta) + o(1),$$

where $o(1)$ denotes a term which tends to 0 uniformly in k as $n\to\infty$. Putting this into (3.7), we see that

$$E\left\{\exp\int_0^t \phi(u)\, dN(u,\omega)\right\} = e^{-L(t)} \lim_{n\to\infty}\left\{1 + \sum_{k=1}^{\infty} I_n^k\right\}, \tag{3.8}$$

where $I_n^k = \sum_{0\leq r_1<\cdots<r_k\leq n} \prod_{j=1}^{k} \delta\psi(r_j\delta)\lambda(r_j\delta)$.

Now I_n^k is a Riemann sum approximating

$$I_k = \int\cdots\int_{0\leq u_1\leq\cdots\leq u_k\leq t} \prod_{j=1}^{k} \psi(u_j)\lambda(u_j)\, du_j = \left\{\int_0^t \psi(u)\lambda(u)\, du\right\}^k \Big/ k!,$$

so (3.6) will follow if we can justify letting $n\to\infty$ in (3.8). However, for some $0<M<\infty$ we have $|\psi(u)\lambda(u)| \leq M$ for $u\in[0,t]$, so that $|I_n^k| \leq (Mt)^k/k!$, and the result follows by dominated convergence. □

This leads us immediately to the basic

Theorem 3.3. *If* $|s|\leq 1$, *then for* $t\geq 0$ $F(s,t)$ *satisfies*

$$F(s,t) = \int_0^t \exp\int_0^u (F(s,t-v)-1)\lambda(v)\, dv\, dG(u) + s(1-G(t)) \cdot \exp\left\{\int_0^t (F(s,t-v)-1)\lambda(v)\, dv\right\}. \tag{3.9}$$

Moreover, if $0<s\leq 1$, *there is only one solution of* (3.9) *with* $0\leq F(s,t)\leq 1$ *for* $t\geq 0$.

Proof. From (3.4) and (3.5) it follows that

$$F(s,t) = \int_0^t E\left\{\exp\int_0^u \log F(s,t-u)\, dN(u,\omega)\right\} dG(u) + E\left\{\exp\int_0^t \log F(s,t-u)\, dN(u,\omega)\right\}\int_t^{\infty} dG(u). \tag{3.10}$$

Taking s and t fixed and writing $\psi(u) = F(s,t-u)$, we see that (3.9) follows from (3.10), using Lemma 3.2.

To establish the uniqueness result,we define a sequence of functions $F_k(s,t)$ as follows. Let $T\{F(s,t)\}$ denote the R.H.S. of (3.9); put $F_0(s,t)\equiv 0$, $F_{k+1}(s,t)=T\{F_k(s,t)\}$ for $k=0,1,\ldots$. Then we can see by induction that the $F_k(s,t)$ are power series in s with non-negative coefficients, and that $0\leqq F_1(s,t)\leqq F_2(s,t)\leqq\cdots\leqq F_k(s,t)\leqq 1$ for $0\leqq s\leqq 1$ for every k. It follows that $F_k(s,t)\uparrow F^*(s,t)$, where $F^*(s,t)$ is a solution of (3.9) with $0\leqq F^*(s,t)\leqq 1$ for $0\leqq s\leqq 1$. For any fixed $s\in(0,1]$ let $F(s,t)$ be a solution of (3.9) with $0\leqq F(s,t)\leqq 1$. Then it is easy to see that $F(s,t)\geqq F_k(s,t)$ for each k and hence $f(t)=F(s,t)-F^*(s,t)\geqq 0$. If we use the elementary inequality $|e^{-x_1}-e^{-x_2}|\leqq|x_1-x_2|$ for $x_1\geqq 0$, $x_2\geqq 0$, we see from (3.9) that

$$f(t)\leqq\int_0^t\int_0^u f(t-v)\lambda(u)\,dv\,e^{-D(u)}\mu(u)\,du+s\,e^{-D(t)}\int_0^t f(t-u)\lambda(u)\,du.$$

Integrating the first term on the R.H.S. by parts we have

$$\begin{aligned} f(t)&\leqq\int_0^t f(t-u)\lambda(u)e^{-D(u)}\,du+(s-1)e^{-D(t)}\int_0^t f(t-u)\lambda(u)\,du\\ &\leqq\int_0^t f(t-u)\lambda(u)e^{-D(u)}\,du.\end{aligned}\tag{3.11}$$

Clearly $\exists\, 0<K<\infty$ with $\int_0^K\lambda(u)e^{-D(u)}<1$; it follows from (3.11) that if $0<m_1=\sup_{t\in[0,K]}f(t)$ then $m_1<m_1$; thus $m_1=0$, i.e. $f(t)=0$ for $t\in[0,K]$. Similarly, if $0<m_2=\sup_{t\in[K,2K]}f(t)$, (3.11) yields $m_2\leqq\int_0^K m_2\lambda(u)e^{-D(u)}\,du<m_2$, so $m_2=0$. Proceeding in this way we see that $f(t)\equiv 0$, i.e. $F(s,t)=F^*(s,t)$. □

Clearly we can use Theorem 3.1 and Lemma 3.2 to find equations which are satisfied by $F_B(s,t)=E(s^{B(t,\omega)})$ and $F_D(s,t)=E(s^{D(t,\omega)})$; the results are given in

Theorem 3.4. *For* $t\geqq 0$, $|s|\leqq 1$, $F_B(s,t)$ *and* $F_D(s,t)$ *satisfy* (3.12) *and* (3.13) *respectively.*

$$\begin{aligned} F_B(s,t)=&\int_0^t\exp\int_0^u[\{F_B(s,t-v)-1\}\lambda(v)-\mu(v)]\,dv\,\mu(u)\,du\\ &+\exp\int_0^t[\{F_B(s,t-u)-1\}\lambda(u)-\mu(u)]\,du,\end{aligned}\tag{3.12}$$

$$\begin{aligned} F_D(s,t)=s&\int_0^t\exp\int_0^u[\{F_D(s,t-v)-1\}\lambda(v)-\mu(v)]\,dv\,\mu(u)\,du\\ &+\exp\int_0^t[\{F_D(s,t-u)-1\}\lambda(u)-\mu(u)]\,du.\end{aligned}\tag{3.13}$$

Another use of Lemma 3.1 is in investigating the joint distribution of $Z(t_1,\omega)$, $Z(t_2,\omega),\ldots,Z(t_k,\omega)$. The most important case is $k=2$, when we have

Theorem 3.5. $F_2(s_1,s_2,t_1,t_2)=E\{s_1^{Z(t_1,\omega)}s_2^{Z(t_1+t_2,\omega)}\}$ *satisfies, for* $t_1\geqq 0$, $t_2\geqq 0$, $|s_1|\leqq 1$, $|s_2|\leqq 1$,

$$\begin{aligned} F_2(s_1,s_2,t_1,t_2)=&\int_0^{t_1}G(s_1,s_2,t_1,t_2,u)\mu(u)\,du+s_1G(s_1,s_2,t_1,t_2,t_1)\\ &\cdot\int_{t_1}^{t_1+t_2}H(s_2,t_1,t_2,u)\mu(u)\,du+s_1s_2G(s_1,s_2,t_1,t_2,t_1)H(s_2,t_1,t_2,t_1+t_2),\end{aligned}\tag{3.14}$$

where

$$G(s_1, s_2, t_1, t_2, u) = \exp \int_0^u [\{F_2(s_1, s_2, t_1 - v, t_2) - 1\} \lambda(v) - \mu(v)] \, dv$$

for $u \in [0, t_1]$,

$$H(s, t_1, t_2, u) = \exp \int_{t_1}^u [\{F(s, t_1 + t_2 - u) - 1\} \lambda(v) - \mu(v)] \, dv$$

for $u \in [t_1, t_1 + t_2]$.

Proof. If we write

$$Y = Y(s_1, s_2, t_1, t_2, \omega) = \prod_{k=1}^{\nu(t_1, \omega)} F_2(s_1, s_2, t_1 - b_k(\omega), t_2) \prod_{k=\nu(t,\omega)+1}^{\nu(t_1+t_2, \omega)} F(s_2, t_1 + t_2 - b_k(\omega))$$

then, from Theorem 3.1 we can deduce that

$$F_2(s_1, s_2, t_1, t_2) = E_{A_0}(Y) + s_1 E_{A_1}(Y) + s_1 s_2 E_{A_2}(Y), \tag{3.15}$$

where $A_0 = \{l < t_1\}$, $A_1 = \{t_1 \leqq l < t_1 + t_2\}$, and $A_2 = \{l \geqq t_1 + t_2\}$. If we write $\psi(u) = F_2(s_1, s_2, t_1 - u, t_2)$ for $u \in [0, t_1]$, $\psi(u) = F(s_2, t_1 + t_2 - u)$ for $u \in (t_1, t_1 + t_2]$, then $\psi(u)$ is piecewise continuous and

$$E\{Y \mid l(\omega) = v\} = E\left\{\exp \int_0^{\min(v, t_1 + t_2)} \log \psi(u) \, dN(u, \omega)\right\},$$

so an application of Lemma 3.2 to (3.15) yields (3.14). □

4. The Extinction Probability

As our first application of Theorem 3.3, we show that the extinction probability for $Z(t, \omega)$ is the same as for $\xi_n(\omega)$.

Theorem 4.1. *Let* $q(t) = P\{Z(t, \omega) = 0\}$ *and* $q = P\{\lim_{n \to \infty} \xi_n(\omega) = 0\}$, (*so that* q *is the smallest root in* $[0, 1]$ *of* $h(s) = s$). *Then if* $q = 0$, $q(t) \equiv 0$; *if* $q > 0$ *then* $q(t) \uparrow q$ *as* $t \to \infty$.

Proof. Since $q(t) = F(0, t)$ it follows from Theorem 3.3 that

$$q(t) = \int_0^t \exp \int_0^u [\{q(t - v) - 1\} \lambda(v) - \mu(v)] \, dv \, \mu(u) \, du. \tag{4.1}$$

Since $Z(t, \omega) = 0 \Rightarrow Z(t', \omega) = 0$ for $t' \geqq t$, $q(t) \uparrow q_0 \leqq 1$ as $t \to \infty$. By the monotone convergence theorem, we can let $t \to \infty$ in (4.1) to get

$$q_0 = \int_0^\infty \exp(q_0 - 1) \, L(u) \, dG(u) = h(q_0).$$

Thus $q_0 \geqq q$; to show that $q_0 \leqq q$, let $t_0 = \sup\{t : q(t) < q\}$ and assume that t_0 is finite. Now by Lemma 2.5, $q(t)$ is continuous, so $q(t_0) = q$. However, from (4.1) we have

$$q(t_0) \leqq \int_0^{t_0} \exp(q - 1) \, L(u) \, dG(u),$$

and we know that

$$q=h(q)=\int_0^\infty \exp(q-1)\,L(u)\,dG(u),$$

so it follows that $G(t_0)=1$. This contradicts our basic assumption that $\mu(u)$ is continuous for $u\in[0,\infty)$, so we must have $t_0=+\infty$. Thus $q_0=q$ as asserted. □

Corollary 1. *If* $A=\{\xi_n(\omega)>0$ *for all* $n\}$, $B=\{Z(t,\omega)>0$ *all* $t\geqq 0\}$ *and* $P(A)>0$ *then* $P\{B|A\}=1$.

Corollary 2. *Let* $m=E\{\xi_1(\omega)\}=h'(1)\leqq\infty$. *Then if* $m\leqq 1$, $P\{Z(t,\omega)=0$ *for large enough* $t\}=1$; *if* $m>1$ $P\{Z(t,\omega)>0$ *all* $t\}=1-q>0$.

5. The Moments of $Z(t,\omega)$

We know from Lemma 2.4 that $m(t)=E(Z(t,\omega))$ is bounded on each finite interval, so we can differentiate (3.9) w.r.t. s and let $s\uparrow 1$ to get

$$m(t)=\int_0^t\int_0^u m(t-v)\,\lambda(v)\,dv\,e^{-D(u)}\,\mu(u)\,du+e^{-D(t)}\left\{1+\int_0^t m(t-u)\,\lambda(u)\,du\right\}. \quad (5.1)$$

If we integrate the first integral by parts we find that

$$m(t)=\int_0^t m(t-u)\,\lambda(u)\,e^{-D(u)}\,du+e^{-D(t)}. \quad (5.2)$$

Suppose now $\exists\,\alpha$ s.t. $e^{-\alpha u}\lambda(u)\,e^{-D(u)}=v(u)$, where $\int_0^\infty v(u)\,du=1$, then it is easy to see that $m^*(t)=e^{-\alpha t}\,m(t)$ satisfies

$$m^*(t)=\int_0^t m^*(t-u)\,v(u)\,du+e^{-\alpha t}\,e^{-D(t)}, \quad (5.3)$$

which is the integral equation of renewal theory. Now if

$$1<m=\int_0^\infty \lambda(u)\,e^{-D(u)}\,du<\infty,$$

then clearly α exists and is positive; also if $m<1$, α may exist and if it does it is negative, finally if $m=+\infty$ then α may or may not exist; if it does it is positive. We can therefore employ standard methods to prove

Theorem 5.1. (i) *If* $1<m<\infty$ *or* $m=\infty$ *and* α *exists then* $m(t)\searrow a\,e^{\alpha t}$ *as* $t\to\infty$, *where* $a=\int_0^\infty e^{-\alpha t}\,e^{-D(t)}\,dt\Big/\int_0^\infty t\,v(t)\,dt$ *is finite.*

(ii) *If* $0<m<1$ *then* $m(t)\to 0$ *as* $t\to\infty$; *if* α *exists then* $m(t)\searrow a\,e^{\alpha t}$ *as* $t\to\infty$ *provided that* $e^{-\alpha t}\cdot e^{-D(t)}$ *is Directly Riemann integrable over* $[0,\infty]$

$$\left\{a=0 \quad \text{if} \quad \int_0^\infty t\,v(t)\,dt=+\infty\right\}.$$

(iii) *If* $m=1$ *then* $\alpha=0$, *and if* $\int_0^\infty e^{-D(t)}\,dt<\infty$, $\lim_{t\to\infty} m(t)=a$, *where again* $a=0$ *if* $\int_0^\infty t\,v(t)\,dt=+\infty$. *If* $\int_0^\infty e^{-D(t)}\,dt=+\infty$ *and* $\int_0^\infty t\,v(t)\,dt<\infty$, *then* $m(t)\to+\infty$ *as* $t\to\infty$.

Proof. All of these statements follow from Theorem 2, p. 349 of [2], except for the last part of (iii). To see this, note that if $m_T(t)$ is the solution of

$$m_T(t)=e^{-D(t)}+\int_0^t m_T(t-u)\,v(u)\,du,\qquad t\leqq T,$$

$$m_T(t)=\int_0^t m_T(t-u)\,v(u)\,du,\qquad t>T,$$

then $m(t)\geqq m_T(t)$. But $m_T(t)\to\int_0^T e^{-D(t)}\,dt\Big/\int_0^\infty t\,v(t)\,dt$, so $m(t)\to+\infty$. □

It we differentiate Eq. (3.14) w.r.t. s_1 and s_2, then let $s_1\uparrow 1$ and $s_2\uparrow 1$ we find that $M(t,\tau)=E(Z(t,\omega)\,Z(t+\tau,\omega))$ satisfies

$$M(t,\tau)=\int_0^t M(t-u,\tau)\,\lambda(u)\,e^{-D(u)}\,du+f(t,\tau),\qquad \text{for } t\geqq 0,\ \tau\geqq 0 \tag{5.4}$$

where

$$\begin{aligned}f(t,\tau)=\;&e^{-D(t+\tau)}+e^{-D(t)}\int_0^t m(t+\tau-u)\,\lambda(u)\,du+e^{-D(t+\tau)}\int_0^t m(t-u)\,\lambda(u)\,du\\&+\int_0^t e^{-D(u)}\lambda(u)\left\{m(t+\tau-u)\int_0^u m(t-v)\,\lambda(v)\,dv+m(t-u)\int_0^u m(t+\tau-v)\,\lambda(v)\,dv\right\}du\\&+\left(1+\int_0^t m(t-u)\,\lambda(u)\,du\right)\int_t^{t+\tau} e^{-D(u)}\,\lambda(u)\,m(t+\tau-u)\,du.\end{aligned}$$

6. The Supercritical Case; Mean Square Convergence

In this section we assume $1<m<\infty$, and investigate the mean square convergence of $W(t)=Z(t)/a\,e^{\alpha t}$. To do this, we need to know the asymptotic behaviour of $M(t,\tau)$; if we write $\overline{M}(t,\tau)=e^{-2\alpha t-\alpha\tau}M(t,\tau)$, $\overline{m}=\int_0^\infty e^{-2\alpha u}\,\lambda(u)\,e^{-D(u)}\,du$, $\overline{v}(u)=e^{-2\alpha u}\,\lambda(u)\,e^{-D(u)}/\overline{m}$, $\overline{m}(t)=e^{-\alpha t}m(t)$ and $\bar{f}(t,\tau)=e^{-2\alpha t-\alpha\tau}f(t,\tau)$, then (5.4) may be written as

$$\overline{M}(t,\tau)=\bar{f}(t,\tau)+\overline{m}\int_0^t \overline{M}(t-u,\tau)\,\overline{v}(u)\,du. \tag{6.1}$$

For fixed τ, this is a renewal equation; to deduce the asymptotic behaviour of $\overline{M}(t,\tau)$ we need the preliminary

Lemma 6.1. *If* $1<m<\infty$ *and* $E(\xi_1^2(\omega))<\infty$ *then* $\lim_{t\to\infty}\bar{f}(t,\tau)=a^2K$ *uniformly in* $\tau\geqq 0$, *where* $K=2\int_0^\infty e^{-\alpha u-D(u)}\,\lambda(u)\int_0^u e^{-\alpha v}\,\lambda(v)\,dv\,du$.

Proof. Since $\bar{m}(t)$ is continuous and converges to a as $t\to\infty$, $\exists A<\infty$ s.t. $\bar{m}(t)\leqq A$ for all $t\geqq 0$. Using (5.2), we see that

$$e^{-D(t)}\int_0^t m(t+\tau-u)\,\lambda(u)\,du \leqq \int_0^{t+\tau} m(t+\tau-u)\,\lambda(u)\,e^{-D(u)}\,du \leqq m(t+\tau)\leqq A\,e^{\alpha(t+\tau)},$$

$$e^{-D(t+\tau)}\int_0^t m(t-u)\,\lambda(u)\,du \leqq \int_0^t m(t-u)\,e^{-D(u)}\,\lambda(u)\,du \leqq A\,e^{\alpha t},$$

and

$$\int_t^{t+\tau} e^{-D(u)}\,\lambda(u)\,m(t+\tau-u)\leqq m(t+\tau)\leqq A\,e^{\alpha(t+\tau)}.$$

Also

$$\int_0^\infty e^{-D(u)}\,\lambda(u)\,L(u)\,du = E\big(\xi_1(\omega)(\xi_1(\omega)-1)\big)<\infty,$$

so that

$$\begin{aligned} &e^{-\alpha(2t+\tau)}\int_0^t m(t-u)\,\lambda(u)\,du \int_t^{t+\tau} m(t+\tau-u)\,e^{-D(u)}\,\lambda(u)\,du\\ &\leqq A^2\int_0^t \lambda(u)\,du \int_t^{t+\tau} e^{-D(u)}\,\lambda(u)\,du\\ &\leqq A^2 \int_t^{t+\tau} e^{-D(u)}\,\lambda(u)\,L(u)\,du\\ &\leqq A^2 \int_t^\infty e^{-D(u)}\,\lambda(u)\,L(u)\,du \to 0 \quad \text{as} \quad t\to\infty. \end{aligned}$$

Thus

$$\begin{aligned} \bar{f}(t,\tau) = \int_0^t e^{-D(u)-\alpha u}\Big\{&\bar{m}(t+\tau-u)\int_0^u \bar{m}(t-v)\,e^{-\alpha v}\,\lambda(v)\,dv\\ &+\bar{m}(t-u)\int_0^u \bar{m}(t+\tau-v)\,e^{-\alpha v}\,\lambda(v)\,dv\Big\}\,\lambda(u)\,du + o(1), \end{aligned}$$

where $o(1)$ denotes a term which tends to 0 uniformly for $\tau\geqq 0$ as $t\to\infty$. Also for fixed u the bracketed term in the integral tends to $2a^2\int_0^u e^{-\alpha v}\,\lambda(v)\,dv$ uniformly in τ, and is dominated by $2A^2\,L(u)$. Thus, uniformly in $\tau\geqq 0$, $\lim_{t\to\infty}\bar{f}(t,\tau)=a^2 K<\infty$. □

Applying a standard renewal theory result (see [3], p. 163), we have the immediate

Corollary. $M(t,\tau)\sim\dfrac{a^2 K\,e^{\alpha(2t+\tau)}}{1-\bar{m}}$ *as* $t\to\infty$, *uniformly in* $\tau\geqq 0$. (Note that necessarily $0<\bar{m}<1$.)

This leads us to

Theorem 6.2. *If* $1<m<\infty$ *and* $E(\xi_1^2(\omega))<\infty$, *then* $W(t)$ *converges in mean square (and hence in distribution) as* $t\to\infty$ *to some random variable* W, *where* $E(W)=1$, $\operatorname{Var}(W)=\{K/(1-\bar{m})\}-1>0$, *and* $\phi(p)=E(e^{-pW})$ *satisfies, for* $p\geqq 0$

$$\phi(p)=\int_0^\infty \exp\int_0^u \{(\phi(p\,e^{-\alpha v})-1)\,\lambda(v)-\mu(v)\}\,dv\,\mu(u)\,du. \tag{6.2}$$

Proof. It follows from the above corollary that $E\{W_{t+\tau}-W_t\}^2 \to 0$ uniformly in τ as $t\to\infty$, so $\exists\, W$ such that $E(W_t-W)^2\to 0$; clearly $E(W)=1$ and $E(W^2)=K/(1-\bar{m})$. To see that $\operatorname{Var}(W)>0$ we must check that $K\geqq 1$. Writing

$$\Lambda(u)=\int_0^u e^{-\alpha v}\,\lambda(v)\,du,$$

we have

$$\begin{aligned}K&=2\int_0^\infty e^{-D(u)}\,\Lambda(u)\,\Lambda'(u)\,du=2\int_0^\infty \Lambda(u)\,\Lambda'(u)\int_{v=u}^\infty e^{-D(v)}\,\mu(v)\,dv\\&=\int_0^\infty e^{-D(v)}\,\mu(v)\{\Lambda(v)\}^2\,dv\geqq\left\{\int_0^\infty \Lambda(v)\,dG(v)\right\}^2\end{aligned}$$

by Schwartz's inequality. But a simple calculation gives

$$\int_0^\infty \Lambda(v)\,dG(v)=\int_0^\infty e^{-\alpha v}\,\lambda(v)\,e^{-D(v)}\,dv=1.$$

To establish (6.2) we merely feed into (3.9) the fact that $E(e^{-pW_t})=F(e^{-p/ae^{\alpha t}},t)\to\phi(p)$ as $t\to\infty$. □

Just as in the case of the age-dependent branching process, it is possible to show that under the assumptions of Theorem 6.2 $W(t)$ converges a.s. to W. The proof consists of three steps; firstly, recalling the decomposition $Z(t)=B(t)-D(t)$ it is necessary to prove theorems analogous to Theorem 6.2 for the variables $W^+(t)=B(t)/a\,e^{\alpha t}$, $W^-(t)=D(t)/a\,e^{\alpha t}$; in view of (3.12), (3.13) and analogoes to (3.14), this is straight forward. Secondly, the method used by Harris {[3], p. 147} to deduce the a.s. convergence of his W_t to his W from the corresponding mean square result under the assumption that $\int_0^\infty E(W_t-W)^2\,dt<\infty$, must be applied to our case. Finally we must show that in fact $\int_0^\infty E(W_t-W)^2\,dt<\infty$, this also can be done in the same way that Jaegers [4] did it for the branching process. Thus,

Theorem 6.3. *Under the assumptions of Theorem* 6.2, $W_t \xrightarrow{\text{a.s.}} W$ *as* $t\to\infty$.

7. Convergence in Distribution of W_t for the Supercritical Case

Throughout this section we assume that either $1<m<\infty$, in which case $\exists\,\alpha>0$ such that $v(u)=e^{-\alpha u}\,\lambda(u)\,e^{-D(u)}$ is a probability density function, or $m=\infty$ and $\exists\,\alpha>0$ such that $v(u)$ is a probability density function. In either case we know from Theorem 5.1 that $E(W_t)=m(t)/a\,e^{\alpha t}\to 1$ as $t\to\infty$, and from Theorem 3.3 we see that if $W_t \xrightarrow{D} W$ then $E(e^{-pW})=\phi(p)$ satisfies

$$\phi(p)=\int_0^\infty \exp\int_0^u\{\phi(p\,e^{-\alpha v})-1\}\,\lambda(v)\,e^{-D(u)}\,\mu(u)\,du,\qquad p\geqq 0. \tag{7.1}$$

Our first step is to find when (7.1) has a non-trivial solution (clearly $\phi(p)=1$ is always a solution) and in this the following is useful.

Lemma 7.1. *Let f be any real-valued, measurable function and Y a random variable with $P\{Y \leqq t\} = 1 - e^{-D(t)}$. Then if Z denotes a random variable with $P\{Z \leqq t\} = \int_0^t v(u)\,du$ we have*

$$E\left\{\int_0^Y f(x)\, e^{-\alpha x}\, \lambda(x)\, dx\right\} = E\{f(Z)\}.$$

Proof.

$$\begin{aligned} E\left\{\int_0^Y f(x)\, e^{-\alpha x}\, \lambda(x)\, dx\right\} &= \int_0^\infty \int_0^y f(x)\, e^{-\alpha x}\, \lambda(x)\, dx\, e^{-D(y)}\, \mu(y) \\ &= \int_0^\infty \int_{y=x}^\infty e^{-D(y)}\, \mu(y)\, f(x)\, e^{-\alpha x}\, \lambda(x)\, dx \\ &= \int_0^\infty f(x)\, v(x)\, dx \\ &= E\{f(Z)\}. \quad \square \end{aligned}$$

We use this to show that (7.1) cannot have 2 solutions corresponding to different non-degenerate distributions with mean one.

Theorem 7.2. *Suppose that ϕ_1, ϕ_2 are continuous solutions of* (7.1) *which satisfy*

(a) $0 < \phi(p) \leqq 1,\ p \geqq 0,$

(b) $\phi(0) = 1,$ (7.2)

(c) $\lim\limits_{p \downarrow 0} \dfrac{1 - \phi(p)}{p} = 1.$

Then $\phi_1 \equiv \phi_2$.

Proof. Let $\phi^*(p) = |\phi_1(p) - \phi_2(p)|/p$ for $p > 0$; then using the inequality $|e^{-x_1} - e^{-x_2}| \leqq |x_1 - x_2|$ (which holds for $x_1 \geqq 0,\ x_2 \geqq 0$) we see from (7.1) that for $p > 0$

$$\begin{aligned} \phi^*(p) &\leqq \int_0^\infty \int_0^u \phi^*(p\, e^{-\alpha v})\, e^{-\alpha v}\, \lambda(v)\, dv\, e^{-D(u)}\, \mu(u), \\ &= E\left\{\int_0^Y \phi^*(p\, e^{-\alpha v})\, e^{-\alpha v}\, \lambda(v)\, dv\right\}, \\ &= E\{\phi^*(p\, e^{-\alpha Z})\}, \end{aligned}$$

by Lemma 7.1. Iterating this we see that $\phi^*(p) \leqq \lim\limits_{n \to \infty} E\{\phi^*(p\, e^{-\alpha S_n})\}$, where S_n is the sum of n independent copies of Z. Now clearly $E(Z) > 0$, so that, by the Strong Law, $e^{-\alpha S_n} \xrightarrow{\text{a.s.}} 0$, and we get $\phi^*(p) \leqq \phi^*(0+)$ by the dominated convergence theorem. However it follows from (7.2c) that $\phi^*(0+) = 0$, so $\phi_1(p) = \phi_2(p)$ for $p > 0$. $\square$

The key to our neccessary and sufficient condition for the existence of a solution to (7.1) lies in the following simple lemma, which is due to Athreya [1].

Lemma 7.3. *Let X be any non-negative random variable with $E(X) = 1$. Then for all $\delta > 0$ $\int_0^\delta \{E(e^{-uX}) - e^{-u}\}/u^2\, du < \infty$ iff $E(X \log |X|) < \infty$.*

Corollary. *Let Y be as in Lemma 7.1, $\Lambda(x)=\int_0^x e^{-\alpha y}\lambda(y)\,dy$ (so that $E(\Lambda(Y))=1$ by Lemma 7.1) and $\psi(p)=\frac{1}{p}E\{e^{-p\Lambda(Y)}-(1-p)\}$ for $p>0$. Then the following are equivalent;*

$$\int_0^\delta \psi(p)/p\,dp<\infty \qquad \text{for all } \delta>0, \tag{7.3}$$

$$\sum_0^\infty \psi(\delta r^n)<\infty \qquad \text{for all } \delta>0,\ 0<r<1, \tag{7.4}$$

$$E(\Lambda(Y)\log|\Lambda(Y)|)<\infty. \tag{7.5}$$

Proof. Since $\int_0^\delta \{e^{-p}-(1-p)\}/p^2\,dp<\infty$ for all $\delta>0$, the equivalence of (7.3) and (7.5) follows from Lemma 7.3. To see that (7.3) and (7.4) are equivalent, note that $\psi(p)\geqq 0$ and $\psi(p)$ increases as p increases, so that

$$\begin{aligned}\log\frac{1}{r}\sum_{n=1}^\infty \psi(\delta r^{n-1}) &= \sum_{n=1}^\infty \psi(\delta r^{n-1})\int_{\delta r^n}^{\delta r^{n-1}}\frac{dp}{p}\\ &\geqq \int_0^\delta \frac{\psi(p)}{p}\,dp\\ &\geqq \sum_{n=1}^\infty \psi(\delta r^n)\int_{\delta r^n}^{\delta r^{n-1}}\frac{dp}{p}\\ &=\log\frac{1}{r}\sum_{n=1}^\infty \psi(\delta r^n).\end{aligned}$$

Theorem 7.4. *There exists a continuous solution of (7.1) satisfying (7.2) iff*

$$E\{\Lambda(Y)\log|\Lambda(Y)|\}<\infty. \tag{7.5}$$

Proof. Suppose (7.5) holds and set $\phi_0(p)=1-p$ for $p\in[0,\frac{1}{2}]$, $\phi_0(p)=\frac{1}{2}$ for $p>\frac{1}{2}$, and $\phi_n(p)=T\{\phi_{n-1}(p)\}$ for $n=1,2,\ldots$, where $T\{\phi(p)\}$ denotes R.H.S. of (7.1). Clearly $0\leqq\phi_n(p)\leqq 1$ for each n. For $p>0$ set $\psi_n(p)=|\phi_n(p)-\phi_{n-1}(p)|/p$ for $n=1, 2,\ldots$. Then we see from (7.1) that for $p>0$

$$\begin{aligned}0\leqq\psi_{n+1}(p)&\leqq E\left\{\int_0^Y \psi_n(p\,e^{-\alpha v})\,e^{-\alpha v}\lambda(v)\,dv\right\}\\ &=E\{\psi_n(p\,e^{-\alpha Z})\}\leqq E\{\psi_1(p\,e^{-\alpha S_n})\}.\end{aligned} \tag{7.6}$$

Now for $p\in(0,\frac{1}{2}]$

$$T\{\phi_0(p)\}=\int_0^\infty \exp\left\{-p\int_0^u e^{-\alpha v}\lambda(v)\,dv\right\}e^{-D(u)}\mu(u)\,du=E\{e^{-p\Lambda(Y)}\},$$

and hence $\psi_1(p)=|E(e^{-p\Lambda(Y)})-(1-p)|/p=\{E(e^{-p\Lambda(Y)})-(1-p)\}/p=\psi(p)$; since $\psi(p)$ is an increasing function it also follows that if $\psi^*(p)=\sup_{0<u\leqq p}\psi_1(u)$ then $\psi^*(p)=\psi(p)$ when $p\in(0,\frac{1}{2}]$.

Let $\Psi(p)=\sum_{n=1}^{\infty}\psi_n(p)$; then we show that $\Psi(p)<\infty$ for all $p>0$. Clearly $0<E(e^{-Z})<1$, so we can choose $\gamma>0$ with $e^{\gamma}E\{e^{-Z}\}<1$; then

$$\begin{aligned}\sum_{n=1}^{\infty}P\{S\leqq n\gamma\}&=\sum_{n=1}^{\infty}P\{e^{-S_n}\geqq e^{-n\gamma}\}\\&\leqq\sum_{n=1}^{\infty}e^{n\gamma}E\{e^{-S_n}\}=\sum_{n=1}^{\infty}\{e^{\gamma}E(e^{-Z})\}^n<\infty.\end{aligned}\tag{7.7}$$

Let $N(p)$ denote the smallest integer n with $p\,e^{-n\alpha\gamma}\leqq\frac{1}{2}$; then writing $S_0\equiv 0$ it follows from (7.6), (7.7) and (7.4) that

$$\begin{aligned}\Psi(p)&\leqq\sum_{n=0}^{\infty}E\{\psi_1(p\,e^{-\alpha S_n})\}\\&\leqq\psi^*(p)\sum_{n=0}^{\infty}P\{S_n\leqq n\gamma\}+\sum_{n=0}^{\infty}\psi^*(p\,e^{-n\alpha\gamma})\,P\{S_n>n\gamma\}\\&\leqq\psi^*(p)\sum_{n=0}^{\infty}P\{S_n\leqq n\gamma\}+N(p)\,\psi^*(p)+\sum_{n=N(p)}^{\infty}\psi(p\,e^{-n\alpha\gamma})<\infty.\end{aligned}\tag{7.8}$$

This means that $\lim_{n\to\infty}\phi_n(p)=\phi(p)$ exists and satisfies $\phi=T\phi$, i.e. (7.1). Clearly ϕ satisfies (7.2a) and (7.2b); to check (7.2c) note that, since $\psi(p)\downarrow 0$ and $N(p)\downarrow 0$ as $p\downarrow 0$, it follows from (7.8) that $\Psi(0+)=0$. Since $|\phi(p)-\phi_0(p)|/p=\lim_{n\to\infty}\psi_n(p)\leqq\Psi(p)$ for $p>0$, and $\phi_0(p)=1-p$ for $p\in(0,\frac{1}{2}]$, this means that $\lim_{p\downarrow 0}\left\{\dfrac{\phi(p)-(1-p)}{p}\right\}=0$, as required.

To see that (7.5) is a necessary condition, let us assume that $E(\Lambda(Y)\log|\Lambda(Y)|)=+\infty$ and that $\phi(p)$ is a continuous solution of (7.1) which satisfies (7.2). Let $g(p)=\{1-\phi(p)\}/p$ for $p>0$; then $g(p)\geqq 0$ and $g(0+)=0$. Let $g_1(p)=\sup_{[0,p]}g(u)$, $g_2(p)=\inf_{[0,p]}g(u)$; then $\exists\,\delta>0$ such that $\frac{1}{2}\leqq g_2(p)\leqq g_1(p)\leqq\frac{3}{2}$ for $p\in[0,\delta]$. For $u\geqq 0$ write $A(u)=e^{-u}+u-1$; then $A(u)\geqq 0$ and $A(u)$ increases as u increases. Since $\psi(p)=\dfrac{1}{p}E\{A(\Lambda(Y))\}$ we see from (7.1) that for $p\in(0,\delta]$

$$\begin{aligned}g(p)&=\frac{1}{p}E\{1-e^{-p\int_0^Y g(pe^{-\alpha u})e^{-\alpha u}\lambda(u)du}\}\\&=E\left\{\int_0^Y g(p\,e^{-\alpha u})\,e^{-\alpha u}\lambda(u)\,du\right\}-\frac{1}{p}E\left\{A\left\{p\int_0^Y g(p\,e^{-\alpha u})\,e^{-\alpha u}\lambda(u)\,du\right\}\right\}\\&\leqq E\{g(p\,e^{-\alpha Z})\}-\tfrac{1}{2}\psi(\tfrac{1}{2}p).\end{aligned}\tag{7.9}$$

Since $v(u)=e^{-\alpha u}\lambda(u)\,e^{-D(u)}$ is continuous, $\exists\,T_0>0$ with $\int_0^{T_0}v(u)\,du=\frac{1}{2}$; it then follows from (7.9) that for $p\in[0,\delta]$,

$$g_1(p)\leqq\tfrac{1}{2}g_1(p\,e^{-\alpha T_0})+\tfrac{1}{2}g_1(p)-\tfrac{1}{2}\psi(\tfrac{1}{2}p),$$

and hence $g_1(p) \leqq g_1(p\,e^{-\alpha T_0}) - \psi(\frac{1}{2}p)$. Iterating this gives

$$g_1(p\,e^{-n\alpha T_0}) \leqq g_1(p\,e^{-(n+1)\alpha T_0}) - \sum_{r=0}^{n} \psi(\tfrac{1}{2}p\,e^{-r\alpha T_0}).$$

Since $\frac{1}{2} \leqq g_1(p) \leqq \frac{3}{2}$ for $p \in (0, 6]$, this implies that $\sum_{n=0}^{\infty} (\frac{1}{2}p\,e^{-n\alpha T_0}) < \infty$, which contradicts (7.4). □

If (7.5) holds, our next aim is to show that $W_t \xrightarrow{D} W$, where $E(e^{-pW}) = \phi(p)$ is the unique solution of (7.1). We need the preliminary

Lemma 7.5. *If* (7.5) *holds, then* $\lim_{p\downarrow 0} \sup_{t\geqq 0} |H(p,t)| = 0$, *where*

$$H(p,t) = \frac{1}{p} E\{A(p\,W_t)\} = \frac{1}{p} E\{e^{-pW_t} + p\,W_t - 1\}.$$

Proof. Since $\frac{1}{p} A(p) \geqq 0$ and decreases to zero as $p \downarrow 0$, the same is true of $H(p,t)$ for each fixed t. Thus $|H(p,t)| = m(t)/a\,e^{\alpha t} - (1 - G(p,t))/p$, where $G(p,t) = E(e^{-pW_t}) = F(e^{-p/a e^{\alpha t}}, t)$, so that for fixed p, $H(p,t)$ is continuous in t, from Lemma 2.5. This means that if $H_T(p) = \sup_{0\leqq t\leqq T} |H(p,t)|$ then for some $t_0(p) \in [0, T]$ we have $H_T(p) = H(p, t_0(p))$, so $\lim_{p\downarrow 0} H_T(p) = 0$ for each fixed T.

Recalling (3.9) and (5.2) we see that $p > 0$

$$\begin{aligned} H(p,t) = {} & \int_0^t H(p\,e^{-\alpha u}, t-u)\,v(u)\,du + \frac{1}{p}\int_0^t A\{G_t(p,u)\}\,e^{-D(u)}\,\mu(u)\,du \\ & + \frac{1}{p} e^{-D(t)} \{A\{G_t(p,t)\} + A(p/a\,e^{\alpha t})\,e^{-G_t(p,t)} + p/a\,e^{\alpha t}\{1 - e^{-G_t(p,t)}\}\}, \end{aligned} \tag{7.10}$$

where, for $0 \leqq u \leqq t$, $G_t(p,u) = \int_0^u \{1 - G(p\,e^{-\alpha v}, t-v)\}\,\lambda(v)\,dv$.

To estimate the R.H.S. of (7.10) note that from $H(p,t) \geqq 0$ it follows that $1 - G(p,t) \leqq p\,m(t)/a\,e^{\alpha t} \leqq p\,c_1$, for some $0 < c_1 < \infty$, for all p and t, since $m(t)/a\,e^{\alpha t} \to 1$ and is continuous. Thus

$$G_t(p,u) \leqq c_1\,p \int_0^u e^{-\alpha v}\,\lambda(v)\,dv = c_1\,p\,\Lambda(u).$$

Also

$$\frac{1}{p} e^{-D(t)} A(p\,\Lambda(t)) \leqq \frac{1}{p} \int_t^{\infty} A(p\,\Lambda(u))\,e^{-D(u)}\,\mu(u)\,du \leqq E\left\{\frac{1}{p} A(\Lambda(Y))\right\} = \psi(p),$$

so if we use the elementary inequality $A(p) \leqq p^2/2$ and recall that $\Lambda(t)\,e^{-D(t)} \to 0$ as $t \to \infty$ (proof of Theorem 6.2) it follows from (7.10) that for some $0 < c_2 < \infty$,

$$0 \leqq H(p,t) \leqq \int_0^t H(p\,e^{-\alpha u}, t-u)\,v(u)\,du + c_1\,\psi(c_1\,p) + c_2\,p,$$

and hence

$$0 \leqq H_T(p) \leqq \int_0^T H_T(p\, e^{-\alpha u})\, v(u)\, du + c_1 \psi(c_1 p) + c_2 p. \tag{7.11}$$

Now for fixed T, $H_T(p)$ is monotone in p, so

$$\int_0^T H_T(p\, e^{-\alpha u})\, v(u)\, du \leqq \tfrac{1}{2} H_T(p\, e^{-\alpha T_0}) + \tfrac{1}{2} H_T(p)$$

$\left(\text{recall } T_0 > 0 \text{ is such that } \int_0^{T_0} v(u)\, du = \tfrac{1}{2}\right)$ and (7.11) gives

$$H_T(p) \leqq H_T(p\, e^{-\alpha T_0}) + 2c_1 \psi(c_1 p) + 2c_2 p.$$

Iterating this and remembering that $H_T(p) \to 0$ as $p \to 0$ we see that

$$H_T(p) \leqq 2c_1 \sum_{n=0}^{\infty} \psi(c_1 p\, e^{-n\alpha T_0}) + 2c_2 p/(1 - e^{-\alpha T_0}),$$

and as the R.H.S. is independent of T and is finite by Lemma 7.3, we can let $T \to \infty$ and then $p \downarrow 0$ to get

$$\limsup_{p \downarrow 0} \sup_{t \geqq 0} |H(p,t)| \leqq 2c_1 \lim_{p \downarrow 0} \sum_{1}^{\infty} \psi(c_1 p\, e^{-n\alpha T_0}) = 0,$$

since $\psi(p) \downarrow 0$ as $p \downarrow 0$. □

Theorem 7.6. *If* (7.5) *holds then* $\lim_{t \to \infty} G(p,t) = \phi(p)$, *where* $\phi(p)$ *is the unique continuous solution of* (7.1) *satisfying* (7.2).

Proof. For $p > 0$ set $K(p,t) = \{G(p,t) - \phi(p)\}/p$, $K_T(p) = \sup_{t \geqq T} |K(p,t)|$, and $K(p) = \lim_{T \to \infty} K_T(p)$. Then since

$$|K(p,t) - H(p,t)| \leqq |m(t)/a\, e^{\alpha t} - 1| + \left|(1 - \phi(p))/p - 1\right|,$$

it follows from Theorem 5.1, (7.2), and Lemma 7.5 that $K(0+) = 0$. If we write $g(p,u) = \int_0^u \{1 - \phi(p\, e^{-\alpha v})\}\, \lambda(v)\, dv$ and use (3.10) and (7.1) then we can check that $K(p, 2t) = p(I + J)$, where

$$I = \int_0^t \{e^{-G_{2t}(p,u)} - e^{-g(p,u)}\}\, e^{-D(u)}\, \mu(u)\, du,$$

$$J = \int_t^{2t} e^{-G_{2t}(p,u) - D(u)}\, \mu(u)\, du + e^{-p/a e^{\alpha t}}\, e^{-D(2t)}\, e^{-G_{2t}(p,2t)}$$

$$+ \int_t^{\infty} e^{-g(p,u) - D(u)}\, \mu(u)\, du.$$

Clearly $0 \leqq J \leqq \int_t^{2t} e^{-D(u)} \mu(u)\, du + e^{-D(2t)} + e^{-D(t)} \leqq 2e^{-D(t)}$; also

$$\begin{aligned}
|I| &\leqq \int_0^t |G_{2t}(p,u) - g(p,u)|\, e^{-D(u)} \mu(u)\, du \\
&\leqq \int_0^t \int_0^u |G(p\, e^{-\alpha v}, 2t-v) - \phi(p\, e^{-\alpha v})|\, \lambda(v)\, dv\, e^{-D(u)} \mu(u)\, du \\
&= \int_0^t |G(p\, e^{-\alpha u}, 2t-u) - \phi(p\, e^{-\alpha u})|\, \lambda(u)\, e^{-D(u)}\, du \\
&\quad - e^{-D(t)} \int_0^t |G(p\, e^{-\alpha u}, 2t-u) - \phi(p\, e^{-\alpha u})|\, \lambda(u)\, du \\
&\leqq p \int_0^t K(p\, e^{-\alpha u}, 2t-u)\, v(u)\, du.
\end{aligned}$$

It follows, therefore, that

$$K_{2T}(p) \leqq \int_0^\infty K_T(p\, e^{-\alpha u})\, v(u)\, du + 2e^{-D(T)}$$

and, letting $T \to \infty$ this gives $K(p) \leqq E\{K(p\, e^{-\alpha Z})\}$. Since $K(0+) = 0$, it follows by the usual argument that $K(p) \equiv 0$, which proves the theorem. □

If $W_t \xrightarrow{D} W$, and W is non-degenerate, then clearly $\phi(p) = E(e^{-pW})$ will be a continuous solution of (7.1) satisfying (7.2), so (7.5) is a N.A.S.C. for this to happen. Now it is known that a N.A.S.C. for ξ_n/m^n to have a non-degenerate limit distribution (assuming $1 < m < \infty$, of course) is that $E\{\xi_1 \log \xi_1\} < \infty$, and when ξ_1 has the distribution exhibited in Theorem 2.1, it is easy to check that this is equivalent to

$$E\{L(Y) \log |L(Y)|\} < \infty. \tag{7.12}$$

Also, if (7.12) fails (i.e. $E\{\xi_1 \log \xi_1\} = +\infty$) it is known that $\xi_n/m^n \xrightarrow{P} 0$. Now clearly (7.12) implies (7.5), and in the example $\lambda(x) = \lambda > 1$,

$$\mu(x) = c\big(1 + \log(1+x)\big)/(1+x)$$

for suitable c, we have $1 < m < \infty$, (7.5) true and (7.12) false. It is therefore possible for $\xi_n/m^n \xrightarrow{P} 0$ and $W_t \xrightarrow{D} W$ (where W is a non-degenerate random variable) to hold simultaneously; this cannot happen for the age-dependent branching process (see [1]). As yet it is undecided whether or not $W_t \xrightarrow{P} 0$ when (7.5) fails. It is also known that when ξ_n/m^n has a non-degenerate limit distribution, this distribution is absolutely continuous, except for an atom of size q (the extinction probability) at zero. In the remainder of this section we prove an analogous result for the distribution of W;

Theorem 7.7. *If (7.5) holds so that $W_t \xrightarrow{D} W$, then $P\{W=0\} = q$ and $\exists$ a continuous $\omega(x) \geqq 0$ such that*

$$P\{x_1 < W \leqq x_2\} = \int_{x_1}^{x_2} \omega(x)\, dx \qquad \textit{for } 0 < x_1 < x_2 < \infty.$$

Proof. According to Lemma 3 of [1], this result will follow if we can establish that $P\{W=0\}=q$, $\lim_{\theta\to\infty}|f(\theta)-q|=0$, and $\int_0^\infty |f'(\theta)|\,d\theta<\infty$, where $f(\theta)=E(e^{i\theta W})=\phi(-i\theta)$. We do this in a sequence of lemmas.

Lemma 7.8. $P\{W=0\}=q<1$, *and the distribution of* W *is not concentrated at one point.*

Proof. If $q^*=P\{W=0\}$ then $q^*=\lim_{p\to\infty}\phi(p)$, so from (7.1) it follows that $q^*=h(q^*)$. This means that $q^*=q$ or $q^*=1$; but $q^*=1$ contradicts $E(W)=1$, which follows from (7.2c).

If $P\{W=x\}=1$, then again because of $E(W)=1$, we must have $x=1$, so $f(\theta)=e^{i\theta}$. Now since (3.10) holds for $|s|\leqq 1$, and $W_t\to W$, we see that $f(\theta)$ satisfies

$$f(\theta)=E\left\{\exp\int_0^Y (f(\theta e^{-\alpha y})-1)\,\lambda(y)\,dy\right\} \tag{7.13}$$

for all real θ. But if we put $f(\theta)=e^{i\theta}$ in this we see that

$$1=|f(\theta)|\leqq E\left\{\exp\int_0^Y \{\cos(\theta e^{-\alpha y})-1\}\,\lambda(y)\,dy\right\}<1,$$

for $0<\theta<\pi/2$, since Y is not concentrated at zero. □

Lemma 7.9. $\limsup_{\theta\to\infty}|f(\theta)|<1$.

Proof. Since W is not concentrated at one point, $\exists\,\delta>0$ such that $|f(\theta)|<1$ for $0<\theta\leqq\delta$. Now for $\varepsilon>0$ let $A_\varepsilon=\{e^{-\alpha Y}\leqq(1+\varepsilon)^{-1}\}$; then $P(A_\varepsilon)>0$ for ε sufficiently small, and if $0<\theta\leqq(1+\varepsilon)\,\delta$ it follows from (7.13) that

$$|f(\theta)|\leqq E_{A_\varepsilon}\left\{\exp\int_0^Y R(f(\theta e^{-\alpha Y})-1)\,\lambda(y)\,dy\right\}+1-P\{A_\varepsilon\}<1.$$

Repeating this argument we see that $|f(\theta)|<1$ whenever $0<\theta\leqq(1+t)^n\,\delta$ for some n, so that $|f(\theta)|<1$ for all $\theta>0$. Now assume $\limsup_{\theta\to\infty}|f(\theta)|=1$; then since $|f(\theta)|$ is continuous and $|f(\theta)|=1$, for all sufficiently small $\rho>0\ \exists\ 0<\theta_1<\theta_2<\infty$ with $|f(\theta_1)|=|f(\theta_2)|=1-\rho$, and $|f(\theta)|<1-\rho$ for $\theta\in(\theta_1,\theta_2)$. If we put $\varepsilon=\theta_2/\theta_1-1$, then on A_ε we know that $R(f(\theta_2 e^{-\alpha Y}))<1-\rho$, so from (7.13)

$$\begin{aligned}(1-\rho)=|f(\theta_2)|&\leqq E_{A_\varepsilon}\left\{\exp\int_0^Y R\{f(\theta_2 e^{-\alpha y})-1\}\,\lambda(y)\,dy\right\}+1-P(A_\varepsilon)\\&\leqq \int_0^{\frac{1}{\alpha}\log(1+\varepsilon)} e^{-\rho L(y)}\,e^{-D(y)}\,\mu(y)\,dy+\int_{\frac{1}{\alpha}\log(1+\varepsilon)}^{\infty} e^{-D(y)}\,\mu(y)\,dy\\&=1+\int_0^{\frac{1}{\alpha}\log(1+\varepsilon)} (e^{-\rho L(y)}-1)\,dG(y).\end{aligned}$$

Rearranging, this is

$$\int_0^{\frac{1}{\alpha}\log(1+\varepsilon)} \left\{\frac{1-e^{-\rho L(y)}}{\rho}\right\} dG(y) \leqq 1.$$

Now as $\rho\downarrow 0$, $\varepsilon\uparrow\infty$, and $(1-e^{-\rho L(y)})/\rho\uparrow L(y)$; thus by monotone convergence, the L.H.S. tends to $E(L(Y))$ as $\rho\downarrow 0$. However, by interchanging orders of integration we can see immediately that $E\{L(Y)\}=\int_0^\infty \lambda(x)\, e^{-D(y)}\, dx = m>1$; this contradiction means that we must have $\limsup_{\theta\to\infty} |f(\theta)|<1$. □

Lemma 7.10. $\limsup_{\theta\to\infty} |f(\theta)| \leqq q$.

Proof. Set $\sigma_t = \sup_{\theta\geqq t} |f(\theta)|$, $\sigma=\lim_{t\to\infty}\sigma_t$. From (7.13) we have, for $\varepsilon>0$,

$$|f(\theta)| \leqq E\left\{\exp\int_0^Y \{|f(\theta\, e^{-\alpha y})|-1\}\,\lambda(y)\, dy \,|\, e^{-\alpha Y}\geqq\varepsilon\right\} + P\{e^{-\alpha Y}<\varepsilon\}$$

so that

$$\sigma_t \leqq E\{\exp\sigma_{\varepsilon t}-1)\, \Lambda(Y)\} + P\{e^{-\alpha Y}<\varepsilon\} = h(\sigma_{\varepsilon t}) + P\{e^{-\alpha Y}<\varepsilon\}.$$

Now let $t\to\infty$, then $\varepsilon\downarrow 0$, to get $\sigma\leqq h(\sigma)$. Since $\sigma<1$ (Lemma 7.9), it follows that $\sigma\leqq q$. □

Lemma 7.11. $\int_0^\infty |f'(\theta)|\, d\theta<\infty$.

Proof. Since $f(\theta)=E(e^{i\theta W})$ where $E(W)=1$, we know that $f'(\theta)$ exists and is continuous and bounded for all real θ. Since $E\{\Lambda(Y)\}=1$, we may differentiate (7.13) and interchange the order of integration to get

$$\begin{aligned} f'(\theta) &= \int_0^\infty\int_0^y e^{-\alpha x} f'(\theta\, e^{-\alpha x})\,\lambda(x)\, dx \cdot \exp\int_0^y \{f(\theta\, e^{-\lambda z})-1\}\,\lambda(z)\, dz\cdot e^{-D(y)}\,\mu(y)\, dy \\ &= \int_0^\infty e^{-\alpha x} f'(\theta\, e^{-\alpha x})\,\lambda(x)\, F(\theta,x)\, dx, \end{aligned} \tag{7.14}$$

where $F(\theta,x)=\int_x^\infty \exp\int_0^y \{f(\theta\, e^{-\lambda z})-1\}\,\lambda(z)\, dz\cdot e^{-D(y)}\,\mu(y)\, dy$.

Now by Lemma 7.10, given any $\varepsilon>0\ \exists\ \beta$ such that for $\theta\geqq\beta$, $|f(\theta)|\leqq q+\varepsilon$, so that if $y_0(\theta)=\dfrac{1}{\alpha}\log\theta/\beta$, we have

$$\begin{aligned} \left|\exp\int_0^y \{f(\theta\, e^{-\lambda z})-1\}\,\lambda(z)\, dz\right| &\leqq \exp(q+\varepsilon-1)\, L(y) && \text{for } y\leqq y_0(\theta) \\ &\leqq \exp(q+\varepsilon-1)\, L(y_0(\theta)) && \text{for } y>y_0(\theta). \end{aligned}$$

Also, if $q+\varepsilon<1$ then

$$\begin{aligned} &\int_0^\infty \lambda(x)\int_x^\infty \exp\{(q+\varepsilon-1)\, L(y)\}\, e^{-D(y)}\,\mu(y)\, dy\, dx \\ &\qquad = \int_{y=0}^\infty L(y)\exp\{(q+\varepsilon-1)\, L(y)\}\, e^{-D(y)}\,\mu(y)\, dy = h'(q+\varepsilon). \end{aligned}$$

Now $h'(s)$ is continuous, and $h'(q)<1$, so for a suitably chosen ε we have $\lambda(x)\,|F(\theta,x)|\leqq\gamma\,u(x)$, for $\theta\geqq\beta$, where $0<\gamma<1$ and $u(x)\geqq 0$ with $\int_0^\infty u(x)\,dx=1$. If U is a random variable with $P\{U\leqq x\}=\int_0^x u(y)\,dy$, then we see from (7.14) that for $\theta\geqq\beta$

$$|f'(\theta)|\leqq\gamma\,E\{e^{-\alpha U}\,|f'(\theta\,e^{-\alpha U})|\}. \tag{7.15}$$

Now define $M(\theta)=\int_\beta^\theta |f'(x)|\,dx$ for $\theta>\beta$, $M(\theta)=0$ for $\theta\leqq\beta$, and $K=\int_0^\beta |f'(x)|\,dx<\infty$. Then, from (7.15),

$$M(\theta)\leqq\gamma\,E\left\{e^{-\alpha U}\int_\beta^\theta e^{-\alpha U}\,|f'(x\,e^{-\alpha U})|\,dx\right\}$$

$$\leqq\gamma\,E\left\{\int_{\beta e^{-\alpha U}}^{\theta e^{-\alpha U}} |f'(x)|\,dx\right\}\leqq\gamma\{E\{M(\theta\,e^{-\alpha U})\}+K\}.$$

Iterating this and applying the strong law gives $M(\theta)\leqq K(1-\gamma)^{-1}$ for all $\theta>0$, and the result follows. □

Lemma 7.12. $\lim_{\theta\to\infty}|f(\theta)-q|=0$.

Proof. Since $\sup_{\theta_2\geqq\theta_1\geqq\theta}|f(\theta_2)-f(\theta_1)|\leqq\int_\theta^\infty|f'(x)|\,dx\to 0$ as $\theta\to\infty$, it follows that $\lim_{\theta\to\infty} f(\theta)$ exists, $=q^*$ say. Since $0\leqq|f(\theta)|\leqq 1$ we can let $\theta\to\infty$ in (7.13) to find $q^*=h(q^*)$. By Lemma 7.10, we have $|q^*|\leqq q$, so the result will follow if we can show that the equation $z=h(z)$ has only one root in $|z|\leqq q$. Now for $s\in[0,1]$ $h'(s)$ is monotone increasing, so $\exists$ unique $q_0\in[0,1]$ with $h'(q_0)=1$; clearly $q\leqq q_0<1$. Thus if $|z|\leqq q$, $|h'(z)|\leqq h'(|z|)\leqq h'(q)<1$, so the equation $1=h'(z)$ has no roots in $|z|\leqq q$, and the result follows.

Lemmas 7.8, 7.9, 7.10, 7.11 and 7.12 together establish Theorem 7.7.

Added in Proof. It has been brought to my notice that certain of the results in this paper are contained in Crump, K., and Mode, C.J.: A general age-dependent branching process, I and II. J. Math. Anal. Appl. **24**, 494–508 (1968), and **25**, 8–17 (1969).

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Dr. R.A. Doney
Department of Mathematics
University of Manchester
Manchester M13 9PL, Great Britain

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