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Limit Distribution of the Last Exit Time for Stationary Random Sequences

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Summary. For a strictly stationary random sequence $(X_i)_{i\geqq 0}$ we find sufficient conditions such that the distribution of the last exit time $t_\beta = \max\{i\colon X_i > \beta i\}$ $(\beta > 0)$ tends weakly to a nondegenerate limit distribution as $\beta \downarrow 0$.

1. Introduction and Result

Let $(X_i)_{i\geqq 0}$ be a sequence of identically distributed random variables. Define the *last exit time* by $t_\beta = t_\beta(X) = \max\{i \geqq 0\colon X_i > \beta i\}$, if such an i exists, and $t_\beta = 0$ else, $\beta > 0$. In [2] we proved that its limit distribution for $\beta \downarrow 0$ is highly connected with the limit distribution of the maximum $Z_n = \max_{i\leqq n} X_i$. In this paper we derive limit distributions of t_β for strictly stationary sequences of not necessarily independent random variables. We give sufficient conditions in order that the limit of $P\{\beta t_\beta(X) \leqq a_\beta x + b_\beta\}$ is the same as the limit of $P\{\beta t_\beta(X^*) \leqq a_\beta x + b_\beta\}$, where X^* is a sequence of independent, identically distributed random variables $(X_i^*)_{i\geqq 0}$ with the same marginal distribution as X_i:

$$P\{X_i \leqq x\} = P\{X_i^* \leqq x\} = F(x) \quad \text{for all } i \geqq 0,$$

and a_β and b_β are norming values, depending on F and β (see [2]). According to the mentioned connection with extremal theory, we use some ideas from Leadbetter [4].

We assume that the sequence X is strictly stationary, which means that the finite-dimensional distributions

$$F_{i_1,\ldots,i_n}(x_1,\ldots,x_n) = P\{X_{i_1} \leqq x_1, \ldots, X_{i_n} \leqq x_n\}$$

fulfill the condition

$$F_{i_1+l,\ldots,i_n+l}(x_1,\ldots,x_n) = F_{i_1,\ldots,i_n}(x_1,\ldots,x_n)$$

for all $l \geqq 0$, $i_j \geqq 0$, $n \geqq 1$ and $x_j \in \mathbb{R}$. Therefore we have $F_i(x) = F(x)$ for all $i \geqq 0$.

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We also assume $EX_i^+ < \infty (\forall\, i \geqq 0)$, which implies that t_β a.s. Thus t_β is well-defined.

In the following we set $\bar{F}(x) = 1 - F(x)$, $x_0 = \sup\{x\colon F(x) < 1\} \leqq \infty$ and $H(x) = \int_x^{x_0} \bar{F}(y)\,dy$. The integral exists since $EX_i^+ < \infty$.

We prove the theorem under the conditions (A) and (B). Condition (A) restricts the dependence of events which are widely separated in time; it is some kind of a mixing condition. Condition (B) considers the dependence of events which are close in time. Condition (A) and (B) are fulfilled by a stationary Gaussian sequence with correlation function $r(n)$ where $r(n)\log n \to 0$ as $n \to \infty$. This verification is given in [3]. The result in this case which was proved in [1], is in a way the best possible, since if $r(n)\log n \to \gamma > 0$ as $n \to \infty$, the limit distribution of t_β is no more equal to $\exp(-e^{-x})$, which is the limit distribution of t_β for independent normal variables.

We formulate Condition (A) and (B) for any sequence u_N which will be later of the form $a_{1/N}x + b_{1/N}$ $(N \to \infty)$.

Condition (A). *Let for $n, K, m \in \mathbb{N}$ with $m < K$ and $N = nK$:*

$$A_{l,n} = \left\{X_{lK+i} \leqq u_N + \frac{lK+i}{N},\ i = 1, \ldots, K-m\right\}, \qquad l \geqq 0,$$

$$\left|P\left(\bigcap_{k=0}^{l} A_{k,n}\right) - P\left(\bigcap_{k=0}^{l-1} A_{k,n}\right) P(A_{l,n})\right| \leqq \alpha_{n,m,K,l}.$$

Then we assume

$$\sum_{l=1}^{\infty} \alpha_{n,m,K,l} < \infty \quad \text{and} \quad \lim_{\substack{K\to\infty \\ m_K \to \infty}} \lim_{n\to\infty} \sum_{l=1}^{\infty} \alpha_{n,m_K,K,l} = 0,$$

where $m_K/K \to 0$ as $K \to \infty$.

Condition (B). *Let n, K, m, N be as in Condition (A) and*

$$\sum\sum_{1 \leqq i < j \leqq K-m} P\left\{X_i > u_N + \frac{lK+i}{N},\ X_j > u_N + \frac{lK+j}{N}\right\} = \alpha^*_{n,m,K,l}.$$

Then we assume $\lim_{K\to\infty} \lim_{n\to\infty} \sum_{l=0}^{\infty} \alpha^*_{n,m,K,l} = 0$ *for all sufficiently large m.*

Instead of Condition (B) we could assume Condition (B′)

$NH(u_N) = O(1)$ and

$$\lim_{K\to\infty} \lim_{n\to\infty} \sum_{l=0}^{\infty} \sum\sum_{1 \leqq i < j \leqq K-m} \left|P\left\{X_i > u_N + \frac{lK+i}{N},\ X_j > u_N + \frac{lK+j}{N}\right\} - \bar{F}\left(u_N + \frac{lK+i}{N}\right) \bar{F}\left(u_N + \frac{lK+j}{N}\right)\right| = 0.$$

For

$$\sum_{l=0}^{\infty} \sum_{1 \leqq i < j \leqq K-m} \bar{F}\left(u_N + \frac{lK+i}{N}\right) \bar{F}\left(u_N + \frac{lK+j}{N}\right)$$
$$\leqq \sum_{l=0}^{\infty} \left(\sum_{i=1}^{K} \bar{F}\left(u_N + \frac{lK+i}{N}\right)\right)^2$$
$$\leqq \sum_{l=0}^{\infty} N^2 \left\{H\left(u_N + \frac{lK}{N}\right) - H\left(u_N + \frac{lK+K}{N}\right)\right\}^2.$$

Because of the monotony of the integrand

$$H\left(u_N + \frac{lK}{N}\right) - H\left(u_N + \frac{lK+K}{N}\right) \leqq \int_{u_N}^{u_N + K/N} \bar{F}(y)\, dy \leqq \frac{1}{n} \bar{F}(u_N),$$

the above sum is smaller than

$$N^2 \frac{1}{n} \bar{F}(u_N) \sum_{l=0}^{\infty} \left\{H\left(u_N + \frac{l}{n}\right) - H\left(u_N + \frac{l+1}{n}\right)\right\} \leqq K \bar{F}(u_N)\, N H(u_N) \to 0$$

as $n \to \infty$ and $K \to \infty$, for any m.

To simplify our formulation of the theorem we define

Condition (C). *We assume that $F(x)$ is such that as $\beta \downarrow 0$ $P\{\beta t_\beta(X^*) \leqq a_\beta x + b_\beta\}$ tends weakly to a nondegenerate limit distribution $\phi(x)$, where X^* is a sequence of independent, identically distributed random variables with distribution $F(x)$ and where $a_\beta > 0$ and b_β are norming values.*

For the discussion of these limit distributions see [2].

Theorem. *Let $(X_i)_{i \geqq 0}$ be a stationary random sequence with $EX_i^+ < \infty$ and the marginal distribution $F(x)$, such that the conditions (A), (B) and (C) are fulfilled with $u_n = a_{1/n} x + b_{1/n}$, where $a_{1/n}$ and $b_{1/n}$ are the values of Condition (C), $\forall x \in \mathbb{R}$. Then*

$$\lim_{\beta \downarrow 0} P\{\beta t_\beta \leqq u_{[1/\beta]}\} = \phi(x)$$

for all points of continuity x of ϕ.

The proof of the theorem is given in the next section by the following steps:

1) we may restrict our calculation to sequences $\beta_n = n^{-1}$,

2) the probability of the event $\{t_{1/n} \leqq n u_n\}$ is asymptotically equal to $P\left(\bigcap_{l=0}^{\infty} A_{l,n}\right)$, $A_{l,n}$ as in Condition (A),

3) $P\left(\bigcap_{l=0}^{\infty} A_{l,n}\right)$ is asymptotically equal to the product $\prod_{l=0}^{\infty} P(A_{l,n})$,

4) the last product tends to $\phi(x)$.

2. Proof of the Theorem

As mentioned we give the proof by showing the four steps.

Lemma 1. *If* $u_n \to x_0$ and $\lim_{n\to\infty} P\{t_{1/n} \leqq n u_n\} = \theta > 0$, *then with* $u^*(\beta) = u_n$ *for* $n \leqq 1/\beta < n+1$

$$\lim_{\beta\downarrow 0} P\{\beta t_\beta \leqq u^*(\beta)\} = \theta.$$

Proof. Since $P\{\beta t_\beta \leqq u^*(\beta)\} = P\{X_i \leqq \beta i,\ i > \beta^{-1} u^*(\beta)\}$ we have

$$\begin{aligned} P\{\beta t_\beta \leqq u^*(\beta)\} &\leqq P\{X_i \leqq u^*(\beta) + \beta i, i \geqq 1\} \\ &\geqq P\{X_i \leqq u^*(\beta) + \beta(i-1), i \geqq 1\} \end{aligned}$$

by using the stationarity of (X_i) and the monotony of the boundary. The difference between the two approximations tends to zero, since with the Bonferroni inequality

$$\begin{aligned} 0 &\leqq P\{X_i \leqq u^*(\beta) + \beta i, i \geqq 1\} - P\{X_i \leqq u^*(\beta) + \beta(i-1), i \geqq 1\} \\ &\leqq \sum_{i=1}^{\infty} P\{u^*(\beta) + \beta(i-1) < X_i \leqq u^*(\beta) + \beta i\} \\ &= \sum_{i=1}^{\infty} \{\bar F(u^*(\beta) + \beta(i-1)) - \bar F(u^*(\beta) + \beta i)\} = \bar F(u^*(\beta)) \to 0 \quad \text{as} \quad \beta \downarrow 0. \end{aligned}$$

Let n be such that $n \leqq 1/\beta < n+1$. With the same argument we estimate

$$P\{X_i \leqq u^*(\beta) + \beta i, i \geqq 1\} \leqq P\left\{X_i \leqq u_n + \frac{i}{n}, i \geqq 1\right\} = P\{t_{1/n} \leqq n u_n\} + o(1)$$

and conversely

$$\begin{aligned} P\{X_i \leqq u^*(\beta) + \beta i, i \geqq 1\} &\geqq P\left\{X_i \leqq u_n + \frac{i}{n+1}, i \geqq 1\right\} \\ &\geqq P\left\{X_i \leqq u_n + \frac{i}{n}, i \geqq 1\right\} - \sum_{i=1}^{\infty} P\left\{u_n + \frac{i}{n+1} < X_i \leqq u_n + \frac{i}{n}\right\} \\ &= P\{t_{1/n} \leqq n u_n\} + o(1) \end{aligned}$$

since the sum is equal to

$$\begin{aligned} &\sum_{i=1}^{\infty} \left\{\bar F\left(u_n + \frac{i}{n+1}\right) - \bar F\left(u_n + \frac{i}{n}\right)\right\} \\ &\leqq (n+1) \int_{u_n}^{x_0} \bar F(y)\,dy - n \int_{u_n}^{x_0} \bar F(y)\,dy + \bar F(u_n) \\ &= H(u_n) + \bar F(u_n) = o(1) \quad \text{as} \quad n \to \infty. \end{aligned}$$

Remark. For $u^*(\beta)$ we may obviously use any measurable function which is bounded by u_{n+1} and u_n on $[(n+1)^{-1}, n^{-1}]$.

Lemma 2. *Let $NH(u_N)\to\theta>0$ as $N\to\infty$, $A_{l,n}$ and n, m, K, N be as in Condition (A). Then*

$$P\left\{X_i\leqq u_N+\frac{i}{N}, i\geqq 1\right\}-P\left(\bigcap_{l=0}^{\infty}A_{l,n}\right)=o(1) \quad as \quad n\to\infty,\ K\to\infty.$$

Proof. Define $A^*_{l,n}=\left\{X_{lK+i}\leqq u_N+\frac{lK+i}{N}, i=K-m+1,\dots,K\right\}$, $l\geqq 0$. Thus

$$P\left\{X_i\leqq u_N+\frac{i}{N}, i\geqq 1\right\}=P\left(\bigcap_{l=0}^{\infty}(A_{l,n}\cap A^*_{l,n})\right).$$

Therefore we have to bound the difference

$$P\left(\bigcap_{l=0}^{\infty}A_{l,n}\right)-P\left(\bigcap_{l=0}^{\infty}(A_{l,n}\cap A^*_{l,n})\right)\leqq\sum_{l=0}^{\infty}P(A_{l,n}\cap A^{*c}_{l,n})$$

$$\leqq\sum_{l=0}^{\infty}P(A^{*c}_{l,n})\leqq\sum_{l=0}^{\infty}\sum_{i=K-m+1}^{K}\bar F\left(u_N+\frac{lK+i}{N}\right)\leqq\sum_{l=0}^{\infty}m\bar F\left(u_N+\frac{l}{n}\right)$$

by using the monotony of $\bar F$

$$\leqq m\bar F(u_N)+mn\int_{u_N}^{x_0}\bar F(y)\,dy=o(1)+mnH(u_N)$$

$$=o(1)+m\frac{\theta+o(1)}{K}=o(1) \quad \text{as} \quad n\to\infty,\ K\to\infty.$$

Lemma 3. *Assume Condition (A), then as $n\to\infty$, $K\to\infty$, $m_K\to\infty$ with $m_K/K\to 0$*

$$P\left(\bigcap_{l=0}^{\infty}A_{l,n}\right)-\prod_{l=0}^{\infty}P(A_{l,n})=o(1).$$

Proof. Let L be a positive integer. Then by adding and subtracting we derive

$$\left|P\left(\bigcap_{l=0}^{L}A_{l,n}\right)-\prod_{l=0}^{L}P(A_{l,n})\right|=\left|P\left(\bigcap_{l=0}^{L}A_{l,n}\right)-P\left(\bigcap_{l=0}^{L-1}A_{l,n}\right)P(A_{L,n})\right.$$

$$+P\left(\bigcap_{l=0}^{L-1}A_{l,n}\right)P(A_{L,n})-P\left(\bigcap_{l=0}^{L-2}A_{l,n}\right)P(A_{L-1,n})P(A_{L,n})+-\dots$$

$$\left.+P\left(\bigcap_{l=0}^{1}A_{l,n}\right)P(A_{2,n})P(A_{3,n})\dots P(A_{L,n})-\prod_{l=0}^{L}P(A_{l,n})\right|$$

$$\leqq\sum_{l=1}^{L}\left|P\left(\bigcap_{l'=0}^{l}A_{l',n}\right)-P\left(\bigcap_{l'=0}^{l-1}A_{l',n}\right)P(A_{l,n})\right|$$

$$\leqq\sum_{l=1}^{L}\alpha_{n,m,K,l}\leqq\sum_{l=1}^{\infty}\alpha_{n,m,K,l}=o(1)$$

by assumption for any L.

Lemma 4. *Assume Condition (B) and $NH(u_N)\to\theta>0$. Then*

$$\lim_{K\to\infty}\lim_{n\to\infty}\prod_{l=0}^{\infty}P(A_{l,n})=e^{-\theta}.$$

Proof. First we approximate $\sum_{l=0}^{\infty}(1-P(A_{l,n}))$.

$$\begin{aligned}1-P(A_{l,n})&=P\left\{\exists i: X_i>u_N+\frac{lK+i}{N}\text{ with }1\leqq i\leqq K-m\right\}\\&\leqq\sum_{i=1}^{K-m}\bar F\left(u_N+\frac{lK+i}{N}\right)\leqq N\int_0^{1/n}\bar F\left(u_N+\frac{l}{n}+y\right)dy\\&=N\left\{H\left(u_N+\frac{l}{n}\right)-H\left(u_N+\frac{l+1}{n}\right)\right\}\end{aligned}$$

by using again the monotony of $\bar F$. Therefore

$$\begin{aligned}&\sum_{l=0}^{\infty}(1-P(A_{l,n}))\\&\quad\leqq N\sum_{l=0}^{\infty}\left\{H\left(u_N+\frac{l}{n}\right)-H\left(u_N+\frac{l+1}{n}\right)\right\}=NH(u_N)\to\theta\quad\text{as}\quad n\to\infty.\end{aligned}$$

With the inequality of Bonferroni we approximate conversely

$$\begin{aligned}&1-P(A_{l,n})\\&\geqq\sum_{i=1}^{K-m}\bar F\left(u_N+\frac{lK+i}{N}\right)-\sum\sum_{1\leqq i<j\leqq K-m}P\left\{X_i>u_N+\frac{lK+i}{N},X_j>u_N+\frac{lK+j}{N}\right\}\\&\geqq N\left\{H\left(u_N+\frac{l}{n}\right)-H\left(u_N+\frac{l+1}{n}\right)\right\}-(m+1)\bar F\left(u_N+\frac{l}{n}\right)-\alpha^*_{n,m,K,l}.\end{aligned}$$

Hence by assumption

$$\sum_{l=0}^{\infty}(1-P(A_{l,n}))\geqq NH(u_N)+o(1)\quad\text{as}\quad n\to\infty$$

since

$$\sum_{l=0}^{\infty}(m+1)\bar F\left(u_N+\frac{l}{n}\right)\leqq o(1)+(m+1)nH(u_N)=o(1)\quad\text{as}\quad n\to\infty,\ K\to\infty.$$

Finally we need to know that each term $P(A_{l,n})\to 1$. This is established by using the monotony of the boundary

$$1-P(A_{l,n})\leqq 1-P(A_{0,n})\leqq\sum_{i=1}^{K-m}\bar F\left(u_N+\frac{i}{N}\right)\leqq(K-m)\bar F(u_N)=o(1).$$

This implies now that

$$\prod_{l=0}^{\infty}P(A_{l,n})=\exp\left\{-(1+o(1))\sum_{l=0}^{\infty}(1-P(A_{l,n}))\right\}\to e^{-\theta}$$

as $n\to\infty$, $K\to\infty$.

Proof of the Theorem. Let $\theta = -\log\phi(x) > 0$ for a point of continuity x of ϕ. Condition C implies $nH(u_n) \xrightarrow[n\to\infty]{} \theta$ (see [2]). By Lemma 1 it is sufficient to show that

$$P\{t_{1/n} \leqq n u_n\} \to e^{-\theta} \quad \text{as} \quad n \to \infty.$$

Lemma 2, 3 and 4 imply that $P\left\{X_i \leqq u_N + \frac{i}{N}, i \geqq 1\right\} \to e^{-\theta}$ as $n\to\infty$, $K\to\infty$ and $m_K \to \infty$ with $m_K/K \to 0$, $N = nK$. By the argument of the proof of Lemma 1 it follows that also $P\{t_{1/N} \leqq N u_N\} \to e^{-\theta}$. Therefore the proof is complete if we show that

$$P\left\{X_i \leqq u_n + \frac{i}{n}, i \geqq 1\right\} - P\left\{X_i \leqq u_{rK} + \frac{i}{rK}, i \geqq 1\right\} = o(1) \quad \text{as} \quad n \to \infty,$$

with $rK \leqq n < rK + K$, K any positive integer, $r \to \infty$. We estimate this difference with the argument of the proof of Lemma 1. The difference is bounded by the sum of

$$\left| P\left\{X_i \leqq u_n + \frac{i}{n}, i \geqq 1\right\} - P\left\{X_i \leqq u_n + \frac{i}{rK}, i \geqq 1\right\} \right|$$

and

$$\left| P\left\{X_i \leqq u_n + \frac{i}{rK}, i \geqq 1\right\} - P\left\{X_i \leqq u_{rK} + \frac{i}{rK}, i \geqq 1\right\} \right|.$$

The first term is bounded with the Bonferroni inequality by

$$\begin{aligned}\sum_{i=1}^{\infty} P\left\{u_n + \frac{i}{n} < X_i \leqq u_n + \frac{i}{rK}\right\} &\leqq \sum_{i=1}^{\infty} \left\{\bar{F}\left(u_n + \frac{i}{n}\right) - \bar{F}\left(u_n + \frac{i}{rK}\right)\right\} \\ &\leqq nH(u_n) - rKH(u_n) + \bar{F}(u_n) = o(1),\end{aligned}$$

since $nH(u_n) \to \theta$ and $rK/n \to 1$. For the second term we have to consider the two cases: $u_{rK} \leqq u_n$ and $u_{rK} > u_n$. In the first case the term is bounded again with the same argument by

$$\sum_{i=1}^{\infty} P\left\{u_{rK} + \frac{i}{rK} < X_i \leqq u_n + \frac{i}{rK}\right\} \leqq rKH(u_{rK}) - rKH(u_n) + \bar{F}(u_n) = o(1)$$

Similar in the second case, the term is bounded by

$$rKH(u_n) - rKH(u_{rK}) + \bar{F}(u_{rK}) = o(1).$$

This finishes the proof of the Theorem.

Remark. As mentioned in [2] there is no essential restriction in considering the linear function as boundary instead of more general boundary functions $f(y)$ in the definition of the last exit time.

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