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Contours of Brownian Processes with Several-dimensional Times

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Summary. Two generalisations of Brownian motion to several-dimensional time are considered and the topology of their level sets is analysed. It is shown that for these maps non-trivial contours are quite rare - their union has Lebesgue measure zero. The boundedness of all contours is established for the generalisation due to Lévy. For the other, the Brownian sheet, a partial result concerning the behaviour of the zero contour near the boundary is established.

Introduction

Contours are frequently used in everyday life to indicate the shape and nature of a surface or a function. This paper is devoted to investigating the nature of the contours of two random maps, the Brownian sheet W and Lévy's multitime generalisation X of Brownian motion. The work was begun to answer a question asked by Pyke in [11] about the topological or dimensional properties of the level sets of W. Adler, in [1], has investigated the Hausdorff dimension of the level sets of W. However although the answers obtained there make it clear that the level sets are most irregular they throw little light on the topology of the level sets. In this paper the connected components of the level sets - the true contours - are considered both for W and for X.

A level set of one of these processes is a closed subset of $\mathbb{R}^N$, being the set of points on which the process attains a certain value. Given a particular point we consider the connected component of the level set passing through this point. The connected component containing the point in question we call the contour of the process at this point. A contour which is a one-point set is called *trivial*.

The questions to be considered are

how likely is a given contour to be trivial?

are there any unbounded contours, and, in the case of W,

what is the behaviour of the contours at the boundary?

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The rather curious answer to the first question is that in both cases the contour at a specified point is almost surely trivial. Nevertheless by a topological argument it can be shown that non-trivial contours must exist. (This argument is sketched out after the proof of Corollary 1.3 in the paper.)

Although the second question is an easy corollary of the first in the case of X, in the case of W the lack of symmetry makes it very hard. Only a partial answer has been obtained for W, and that only in dimension 2. Nevertheless the answer (given in the theorem at (1.4)) provides some insight into the nature of the trajectory of a 2-time Brownian sheet.

Section 1 deals with the Brownian sheet and Sect. 2 with Lévy's generalisation. The results of Sect. 2 can be obtained as corollaries to the results of Pitt and Tran in [10] concerning rather more general processes. However the results are derived in Sect. 2 by means of simple geometrical arguments exploiting symmetry and a white noise representation due to Čentsov [3].

1. The Brownian Sheet

Let $\mathbb{T}=\{\mathbf{t}=(t_1,\ldots,t_N)\in\mathbb{R}^N\colon\ t_1,\ldots,t_N\geqq 0\}$ and let $\partial\mathbb{T}=\{\mathbf{t}\in\mathbb{T}\colon$ at least one of the t_i is zero$\}$. Then the N-parameter Brownian sheet (or Wiener sheet) W is a Gaussian process $\{W(\mathbf{t})\colon\ \mathbf{t}\in\mathbb{T}\}$ of continuous trajectories and of covariance structure given by

$$EW(\mathbf{t})=0 \quad\text{and}\quad EW(\mathbf{t})\cdot W(\mathbf{s})=\prod_{m=1}^{N}(s_m\wedge t_m)$$

$$\text{for } \mathbf{s}=(s_1,\ldots,s_N) \quad\text{and}\quad \mathbf{t}=(t_1,\ldots,t_N) \text{ in } \mathbb{T}.$$

In [2] Čentsov remarked that W could be represented by a white noise η, a representation that aids the exposition of the following work considerably. In fact we can write

$$W(\mathbf{t})=\eta\{\mathbf{s}\text{ in }\mathbb{T}\colon s_m\leqq t_m\text{ for all }m\}$$

where η is a Gaussian random measure on $\mathbb{T}$ with

$$E\eta(G)=0 \quad\text{and}\quad E\eta(G)\cdot\eta(F)=m(G\cap F)$$

for any Borel sets F, $G\subset\mathbb{T}$ and for m Lebesgue measure on $\mathbb{T}$.

If $L(\mathbf{t})=\{\mathbf{s}\in\mathbb{T}\colon\ W(\mathbf{s})=W(\mathbf{t})\}$ and if $C(\mathbf{t})$ is the connected component of $L(\mathbf{t})$ containing $\mathbf{t}$ then we consider the questions

for a fixed $\mathbf{t}$ in $(0,\infty)^N$ is $C(\mathbf{t})$ trivial?

what can be said about $C(0)$? (Apart from the obvious fact that it contains $\partial\mathbb{T}$)

The first question can be answered completely. By scaling symmetries it suffices to take $\mathbf{t}=\mathbf{1}=(1,\ldots,1)$ and consider the contour $C(\mathbf{1})$. The following $0-1$ law will be useful in the discussion of $C(\mathbf{1})$;

if $\mathscr{H}_n=\sigma\{W(\mathbf{s})-W(\mathbf{1})$ for $\mathbf{s}$ in $\mathbb{T}$ such

that $\max\{|s_m-1|\}\leqq 1/n\}$

then the germ σ-field $\bigcap_{n=1} \mathscr{H}_n$ satisfies a $0-1$ law. This is a straightforward consequence of the $0-1$ law of Orey and Pruitt [8 p 140].

The following rather surprising result answers the first question.

(1.1) **Theorem.**

$$P\{C(\mathbf{1})=\{\mathbf{1}\}\}=1$$

so that the contour at $\mathbf{1}$ *is almost surely trivial.*

Proof. The event $\{C(\mathbf{1})=\{\mathbf{1}\}\}$ contains the event

$$\lim_{H\to 0} \bigcup_{h<H} \{W(\mathbf{1}+h\mathbf{s})<W(\mathbf{1}) \text{ for all } \mathbf{s} \text{ with } \max\{|s_m|\}=1\}$$

because if the second event holds then $\mathbf{1}$ is surrounded by an infinite number of arbitrarily small boxes on the surface of which W is smaller than $W(\mathbf{1})$. Since the second event is in the germ σ-field mentioned above an application of Fatou's lemma will complete the proof if it can be shown that the event

$$\{W(\mathbf{1}+h\mathbf{s})<W(\mathbf{1}) \text{ for all } \mathbf{s} \text{ with } \max\{|s_m|\}=1\}$$

has positive probability bounded away from zero for all suitably small h.

To prove this we exploit the white noise representation of W. We can decompose the Brownian sheet by

$$\begin{aligned} W(\mathbf{u})= & W((1-h)\mathbf{1})+\sum_{k=1}^{N} B_k(u_k-1+h) \\ & +\sum_{m=2}^{N} \sum_{j_1,\dots,j_m} W_{j_1,\dots,j_m}(u_{j_1}-1+h,\dots,u_{j_m}-1+h) \end{aligned}$$

where the third sum is taken over the distinct m-tuples $j_1,\dots,j_m$ with $1\leqq j_1<j_2<\dots<j_m\leqq N$ and the N processes B_k are Brownian motions while the $\binom{N}{m}$ processes $W_{j_1,\dots,j_m}$ are m-parameter Brownian sheets. The processes are defined in terms of η by

$$B_k(r)=\eta\{\mathbf{v}\in\mathbb{T}\colon v_k\leqq r+1-h,\ v_k>1-h \text{ and } v_q\leqq 1-h \text{ for all } q\neq k\}$$

and

$$\begin{aligned} & W_{j_1,\dots,j_m}(r_1,\dots,\dots r_m) \\ & \quad =\eta\{\mathbf{v}\in\mathbb{T}\colon v_{j_k}\leqq r_k+1-h,\ v_{j_k}>1-h \text{ for all } k=1,\dots,m \\ & \qquad \text{and } v_q\leqq 1-h \text{ for all } q\neq j_1,\dots,j_m\}. \end{aligned}$$

Because the sets used to define different processes are always disjoint, the processes in the decomposition are all independent.

When h is small the contributions of the Brownian sheet terms may be ignored, at least with a positive probability. For by scaling properties and continuity there is an $M>0$ and a $p'>0$ such that for any $j_1,\dots,j_m$ and any $h>0$

$$P\{|W_{j_1,\ldots,j_m}(r_1,\ldots,r_m)| \leqq M\cdot h^{m/2}$$
$$\text{when } r_k\in[0,2h] \text{ for all } k\}\geqq p'.$$

Consequently by the independence of the sheets the probability of

$$\left\{\left|W(\mathbf{u})-W((1-h)\mathbf{1})-\sum_{k=1}^{N} B_k(u_k-1+h)\right| \leqq \sum_{m=2}^{N}\binom{N}{m} h^{m/2}\cdot M \right.$$
$$\left.\text{for all } \mathbf{u} \text{ in } \mathbb{T} \text{ with } u_1,\ldots,u_N \text{ in } [0,2h]\right\}$$

is greater than $(p')^{2^N-N-1}$. Moreover this event is independent of the Brownian motions $B_1,\ldots,B_N$.

By the "Forgery Theorem" of Lévy (proved in [4, p 30, 1.3 Thm. 38]) and the scaling property of Brownian motion there is a $p''>0$ such that, for any $h>0$,

$$P\{B_k(r)\leqq h^{\frac{1}{2}} \text{ for } r \text{ in } [0,2h]$$
$$B_k(0) \text{ and } B_k(2h)\leqq h^{\frac{1}{2}}/4$$
$$B_k(h)>(1-1/(4N))h^{\frac{1}{2}}\}$$

equals p'' for $k=1,\ldots,N$. By the independence of the B_k this means

$$\left\{\sum_{k=1}^{N} B_k(r_k)\leqq(N-3/4)h^{\frac{1}{2}} \text{ if all of } r_k \text{ are in } [0,2h]\right.$$
$$\left.\text{and precisely one of } |r_k| \text{ equals } h;\ \sum_{k=1}^{N} B_k(h)>(N-1/4)h^{\frac{1}{2}}\right\}$$

has probability greater than $(p'')^N$.

Combining these two independent events we see that with positive probability bounded away from zero the absolute error in the approximation

$$W(\mathbf{u})-W((1-h)\mathbf{1})\sim\sum_{k=1}^{N} B_k(u_k-1+h)$$

is less than $M[(1+h^{\frac{1}{2}})^N-1-N\cdot h^{\frac{1}{2}}]=o(h^{\frac{1}{2}})$ while

$$\mathbf{u}\mapsto\sum_{k=1}^{N} B_k(u_k-1+h)$$

is less than $(N-3/4)h^{\frac{1}{2}}$ on the box of side $2h$ and centre $\mathbf{1}$ but also greater than $(N-1/4)h^{\frac{1}{2}}$ when $\mathbf{u}=\mathbf{1}$. Consequently there is a positive probability that

$$W(\mathbf{1}+h\mathbf{s})<W(\mathbf{1}) \qquad \text{for all } \mathbf{s} \quad \text{with} \quad \max\{|s_m|\}=1$$

for sufficiently small h. Moreover, for h smaller than a certain amount this probability will be bounded away from zero. □

(1.2) **Corollary.** *The union of the non-trivial contours of W is a Lebesgue null-set of $\mathbb{T}$.*

Proof. The function $\mathbf{t} \mapsto P\{C(\mathbf{t}) \neq \{\mathbf{t}\}\}$ is zero everywhere in $(0, \infty)^N$. Therefore

$$E \int_{\mathbb{T}} 1_{[C(\mathbf{t}) \neq \{\mathbf{t}\}]} dm(\mathbf{t}) = 0. \quad \square$$

(1.3) **Corollary.** *With probability* 1 *the set of local maxima of the N-parameter Brownian sheet is dense.*

Proof. Clearly if $W(\mathbf{t}+h\mathbf{s}) < W(\mathbf{t})$ for all $\mathbf{s}$ with $\max\{|s_m|\}=1$ then there is a local maximum of W within $\sqrt{N} \cdot h$ of $\mathbf{t}$. The proof of the result above shows that this occurs for arbitrarily small h with probability 1. Hence with probability 1 every $\mathbf{t}$ with rational coordinates is an accumulation point for the set of local maxima of W. The proof is easily completed. $\square$

This last result was proved for the case $N=2$ by Tran in [12] in the course of an analysis of the local maxima of the 2-time Brownian sheet.

The first corollary raises the question of whether there are any non-trivial contours apart from $C(\mathbf{0})$. If there were none then a level set such as $L(\mathbf{1})$ would be totally disconnected and hence of dimension 0. Such a set is incapable of disconnecting $\mathbb{T}$ when $N \geqq 2$ (see [5] Theorem IV 4 p48). But

$$\mathbb{T} \setminus L(\mathbf{1}) = \{\mathbf{t}: W(\mathbf{t}) < W(\mathbf{1})\} \cup \{\mathbf{t}: W(\mathbf{t}) > W(\mathbf{1})\}$$

is disconnected with probability 1. Therefore non-trivial contours must exist.

An extension of this argument shows that the Lebesgue null-set of non-trivial contours is nonetheless dense in $\mathbb{T}$.

We turn to the second question about contours, concerning the nature of $C(\mathbf{0})$; namely if

$$\partial \mathbb{T} = \{\mathbf{t} \in \mathbb{T}: \text{at least one of the coordinates is zero}\}$$

then does $C(\mathbf{0})$ equal $\partial \mathbb{T}$? We shall only be able to establish a partial answer to this question, and that only for the case $N=2$. The argument works in dimension 2 because of the topology of the plane.

Two basic properties of Brownian sheets will be invoked repeatedly. One is the inversion symmetry; that the process

$$\mathbf{t} = (t_1, \ldots, t_N) \mapsto \begin{cases} 0 & \text{if } t_1 = 0, \\ t_1 \cdot W(1/t_1, t_2, \ldots, t_N) & \text{otherwise} \end{cases}$$

is itself a Brownian sheet. This and further symmetries obtained by inverting other coordinates enable the behaviour of the zero-contours at infinity to be deduced from their behaviour at the origin.

The other property is the Cameron-Martin formula for the Brownian sheet. This has been obtained by a number of workers. For our purposes we refer to [9] and state an easy extension; if $g \in L^2(\mathbb{T})$ then the law of the process

$$\mathbf{t} = (t_1, \ldots, t_N) \mapsto W(\mathbf{t}) + \int_0^{t_1} \ldots \int_0^{t_N} g(\mathbf{x})\, dx_N \ldots dx_1$$

is mutually absolutely continuous to the law of W with a density with respect to the latter process of

$$\exp(\int g\,dW - \tfrac{1}{2}\int g^2).$$

(1.4) **Theorem.** *For a fixed $T<\infty$ let $C^*(\mathbf{0})$ be the connected component containing $\mathbf{0}$ of $C(\mathbf{0})\cap[0,T]^2$. Then*

$$C^*(\mathbf{0}) = \partial\mathbb{T}\cap[0,T]^2.$$

Proof. By scaling symmetries we may take $T=1$.

The first stage in the proof is to note that the topological nature of $\mathbb{T} = [0,\infty)^2$ restricts the possible behaviour of connected regions on which W is of a constant sign. We state an interior lemma;

(1.5) **Lemma.** *Almost surely there is no continuous*

$$f\colon \mathbb{R}\to((0,S]\times(0,S^{-1}])\cup([S,\infty)\times[S^{-1},\infty))$$

satisfying the three conditions following;

(i) *$W(f(u))$ is positive for all u;*

(ii) $\liminf_{u\to\infty}\max\{f_1(u), f_2(u)\}=0$;

(iii) $\limsup_{u\to\infty}\min\{f_1(-u), f_2(-u)\}=\infty$.

Thus no connected region of positivity of W can adhere to both $(0,0)$ and (∞,∞).

Proof of Lemma. Suppose that there is a positive probability of existence for such f. We derive a contradiction.

By the $0-1$ law of Orey and Pruitt referred to above there is a $0-1$ law for $\bigcap_{S>0}\sigma\{W(s,t)\colon s\leqq S \text{ or } t\geqq S^{-1}\}$. Using the inversion symmetry, this $0-1$ law, and the fact that $-W$ has the same law as W we may infer that with probability 1 for some $S_1>0$ there will exist a continuous

$$f'\colon \mathbb{R}\to([S_1^{-1},\infty)\times(0,S_1^{-1}])\cup((0,S_1^{-1}]\times[S_1^{-1},\infty))$$

with $W(f'(u))$ negative for every u and with

$$\liminf_{u\to\infty}\max\{1/f_1'(u), f_2'(u)\}=0,$$

$$\limsup_{u\to\infty}\min\{1/f_1'(-u), f_2'(-u)\}=\infty.$$

The image of f will separate points sufficiently close to $(0,\infty)$ from points sufficiently close to $(\infty,0)$, while the curve f' will connect some points from both of these regions. Consequently it is impossible for such an f and such an f' to coexist. This gives the desired contradiction. □

We return to the proof of the theorem. Once more we use the method of contradiction. Suppose $C^*(\mathbf{0})\neq\partial\mathbb{T}\cap[0,1]^2$. Then in the set $[0,1]^2$ there is a connected region of positivity for the translated process $(s,t)\mapsto W(s,t)+st$ with closure intersecting the boundary $\partial\mathbb{T}\cap[0,1]^2$. By the Cameron-Martin formula this last event will have probability zero if almost surely there is no continuous f: $[0,\infty)\to(0,1)^2$ with $W(f(u))$ positive for all u and $\liminf_{u\to\infty}\min\{f_1(u),f_2(u)\}=0$.

If there is a positive probability of such an f then the probability is still positive under the further restriction that $f(0)=\mathbf{1}$. (For some $\mathbf{t}$ in $(0,1]^2$ and some $M>0$ we may take $f(0)=\mathbf{t}$ and $W>-M$ over the segment from $\mathbf{t}$ to $\mathbf{1}$. The Cameron-Martin formula with $g=1_{[0,t_1]\times[0,t_2]}\cdot M/(t_1 t_2)$ shows we may take $\mathbf{t}=\mathbf{1}$.)

Given ε, with probability 1 there is a sequence $x_n\to 0$ such that $W(s,x_n)>0$ for all s in $[\varepsilon,1]$ for all n. This follows from a $0-1$ law for $\bigcap_{h>0}\sigma\{W(s,t);\ s\leqq 1, t\leqq h\}$ obtained by applying the Orey-Pruitt $0-1$ law. Consequently we may further restrict f by requiring $\liminf_{u\to\infty}\sup\{f_1(u),f_2(u)\}=0$.

Finally the independence of $W|_{[1,\infty)^2}$ and $W|_{[0,1]^2}$ given $W(\mathbf{1})$ and the inversion symmetry of W may be applied to show the positive probability of an f existing as developed above together with a corresponding curve in $[1,\infty)^2$. But two such curves put together form a curve as described in the lemma at (1.5) and this contradicts the positive probability of their existence. So the initial premise must be wrong and consequently

$$P\{C^*(\mathbf{0})=\partial\mathbb{T}\cap[0,1]^2\}=1. \quad \square$$

The proof of the interior lemma at (1.5) depends on the topology of the plane – in particular on the fact that curves can disconnect the plane. This hinders the generalisation of the theorem to higher dimensions.

2. Lévy's Brownian Process with Multidimensional Time

The N-parameter generalisation of the Brownian process due to Lévy is a Gaussian process $\{X(\mathbf{u})\colon \mathbf{u}\in\mathbb{R}^N\}$ of continuous trajectories, satisfying

$$X(\mathbf{0})=0, \qquad EX(\mathbf{u})=0,$$

and

$$E(X(\mathbf{u})-X(\mathbf{v}))^2=\|\mathbf{u}-\mathbf{v}\| \qquad \text{for all } \mathbf{u} \text{ and } \mathbf{v} \text{ in } \mathbb{R}^N.$$

As before, a white noise generalisation is useful. The white noise ξ is defined on the space M of hyperplanes equipped with an invariant measure μ inherited from the action on M of the group of rigid motions. So μ is unique up to a constant multiple. The process ξ is Gaussian with

$$E\xi(G)=0 \quad \text{and} \quad E\xi(G)\cdot\xi(F)=\mu(G\cap F)$$

for F and G Borel sets in $\mathbb{R}^N$.

If the right constant multiple is chosen for μ then the process $X(\mathbf{u})=\xi\{h\in M\colon h \text{ separates } \mathbf{u} \text{ from } \mathbf{0}\}$ has the finite-dimensional distributions of the Lévy Brownian process. So it can be modified to have continuous trajectories and so to be a Lévy Brownian process itself.

This representation is due to Čentsov [3].

As before the level set of X at $\mathbf{u}$ is defined to be

$$L(\mathbf{u}) = \{\mathbf{v} \in \mathbb{R}^N: X(\mathbf{v}) = X(\mathbf{u})\}$$

and the contour of X at $\mathbf{u}$ is the connected component $C(\mathbf{u})$ of $L(\mathbf{u})$ containing $\mathbf{u}$. As with the Brownian sheet we direct our attention to the question of the triviality of $C(\mathbf{u})$.

As a preliminary step of some intrinsic interest we establish a $0-1$ law for the germ σ-field

$$\mathscr{G} := \bigcap_{r>0} \sigma\{X(\mathbf{u}): \|\mathbf{u}\| < r\}.$$

Such a $0-1$ law was stated by McKean in [6] and proved by Molčan in [7]. The following proof is geometrical in flavour, and more direct.

(2.1) **Theorem.** *Events in the germ σ-field $\mathscr{G}$ have probability 0 or 1.*

Proof. Let $A_m = \{h \in M$: the perpendicular from $\mathbf{0}$ to h is of length between $1/m$ and $1/(m+1)\}$. Define $\mathscr{G}_m = \sigma\{\xi(G)$: G a Borel subset of $A_m\}$. Clearly $\mathscr{G} \subset \lim_{n\to\infty} \bigvee_{m=n}^{\infty} \mathscr{G}_m$ by the white noise representation. The white noise being independent on disjoint sets, the $\mathscr{G}_m$ are independent of each other. So the result follows from Kolmogoroff's $0-1$ law. □

(2.2) **Theorem.**

$$P\{C(\mathbf{0}) = \{\mathbf{0}\}\} = 1.$$

Proof. By the $0-1$ law at (2.1) it suffices to show

$$P\{C(\mathbf{0}) \neq \mathbf{0}\} < 1.$$

By an argument similar to that of the proof of (1.1), but using spheres rather than boxes and exploiting the scaling symmetry of X, this follows if

$$P\{X \text{ is positive on the sphere of unit radius centred at } \mathbf{0}\}$$

is positive.

Let S be the unit sphere and let m be the unit invariant measure on S. Then symmetry arguments show that Y is independent of the process $X - Y$ on S, if $Y = \int_S X(\mathbf{u})\, dm(\mathbf{u})$. For

$$EY.(X(\mathbf{u}) - Y) = \int_S EY.(X(\mathbf{u}) - Y)\, dm(\mathbf{u}) = EY.(Y - Y) = 0.$$

On the other hand

$$\operatorname{var}(Y) = E(Y^2) = E \int_S \int_S X(\mathbf{u}) \cdot X(\mathbf{v})\, \mathrm{d}m(\mathbf{u})\, dm(\mathbf{v}) > 0$$

since $EX(\mathbf{u}) \cdot X(\mathbf{v}) = \frac{1}{2}[\|\mathbf{u}\| + \|\mathbf{v}\| - \|\mathbf{u} - \mathbf{v}\|] > 0$.

So Y is a nondegenerate Gaussian random variable independent of the continuous process $X-Y$ restricted to S. Therefore

$$P\{\inf_{\|\mathbf{u}\|=1}(X(\mathbf{u})-Y)>-Y\}>0$$

and this completes the proof. $\square$

Because the processes $\mathbf{u}\mapsto X(\mathbf{u}+\mathbf{v})-X(\mathbf{v})$ and

$$\mathbf{u}\mapsto\begin{cases}0 & \text{if } \mathbf{u}=\mathbf{0},\\ \|\mathbf{u}\|\,X(\mathbf{u}/\|\mathbf{u}\|^2) & \text{otherwise}\end{cases}$$

both have the same law as X, it follows from the result at (2.2) that

$$P\{C(\mathbf{u})=\{\mathbf{u}\}\}=1 \qquad \text{for any } \mathbf{u} \text{ in } \mathbb{R}^N$$

and also that with probability 1 all components of $L(\mathbf{u})$ are bounded for all $\mathbf{u}$ in $\mathbb{R}^N$.

As with the Brownian sheet we may conclude that the union of all the non-trivial contours of X is Lebesgue null but still dense in $\mathbb{R}^N$.

Most of the results in this paper were obtained in the course of an S.R.C studentship at the University of Oxford, and appear in the ensuing D. Phil. thesis. I wish to acknowledge the encouragement of my supervisor John Kingman, and the stimulus of correspondence with J.B. Walsh and R. Pyke.

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Note Added in Proof

The author has recently obtained an analogous Baire category result. Given the usual topology for $\mathbb{R}^N \times C(\mathbb{R}^N)$ the set

$$\{(u, X)\text{: the } X\text{-contour at } u \text{ is trivial}\}$$

is a residual set.