Z. Wahrscheinlichkeitstheorie verw. Geb. 17, 168–180 (1971)

Multiple Channel Queues in Heavy Traffic

IV. Law of the Iterated Logarithm

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1. Introduction and Summary

The previous papers ([3, 4, 7]) in this series have dealt with functional central limit theorems for the stochastic processes characterizing multiple channel queues in heavy traffic. In this paper we develop functional laws of the iterated logarithm for these processes. We shall restrict our study to the simplest multiple channel queueing systems (see [3, 4, 7], and [8] for background and a complete description of these and more complicated systems).

The queueing systems considered consist of r independent arrival channels and s independent service channels, where as usual the arrival and service channels are independent. Arriving customers from a single queue and are served in the order of their arrival without defections. We shall treat two distinct modes of operation for the service channels. In the standard system a waiting customer is assigned to the first available service channel and the servers (servers ≡ service channels) are shut off when they are idle. Thus the classical $GI/G/s$ system is a special case of our standard system. In the modified system a waiting customer is assigned to the service channel that can complete his service first and the servers are not shut off when they are idle. Let λ_i denote the arrival rate (reciprocal of the mean interarrival time) in the i^{th} arrival channel and μ_j the service rate (reciprocal of the mean service time) in the j^{th} service channel. Then $\lambda = \sum_{i=1}^{r} \lambda_i$ is the total arrival rate to the system and $\mu = \sum_{j=1}^{s} \mu_i$ is the maximum service rate of the system. As a measure of congestion we define the traffic intensity $\rho = \lambda/\mu$. We shall restrict our attention to systems in which $\rho \geqq 1$, which we shall refer to as heavy traffic.

The principal tools in our analysis are the functional law of the iterated logarithm (f.l.i.l.) for Brownian motion obtained by Strassen (1964) and the well-known representation of Skorohod (1965). We summarize these results in Section 2 and also develop a f.l.i.l. for random partial sums and renewal processes. In Section 3 a f.l.i.l. is obtained for the process $Q'(t)$, the number of customers in the modified system at time t. If we let $\varphi(t) = (2t \log\log t)^{\frac{1}{2}}$, then as an easy consequence of the f.l.i.l. we have for $\rho = 1$ $(\lambda = \mu)$

$$\overline{\lim_{t\to\infty}}\, Q'(t)/\varphi(t) = \gamma \qquad \text{a.s.,}$$

and

$$\underline{\lim_{t\to\infty}}\, Q'(t)/\varphi(t) = 0 \qquad \text{a.s.,}$$

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* This work was supported by Office of Naval Research Contract N00014-67-A-0112-0031 and National Science Foundation Grant GP-8790.

where γ is a positive constant to be specified later. On the other hand, for $\rho>1$ $(\lambda>\mu)$

$$\overline{\lim_{t\to\infty}}\,[Q'(t)-(\lambda-\mu)\,t]/\varphi(t)=\gamma \quad \text{a.s.}$$

and

$$\underline{\lim_{t\to\infty}}\,[Q'(t)-(\lambda-\mu)\,t]/\varphi(t)=-\gamma \quad \text{a.s.}$$

Similar results are obtained in Section 3 for the standard system when $\rho\geqq 1$. Section 4 is devoted to the f.l.i.l. for the departure processes of the standard and modified systems for all values of ρ. Section 5 deals with the f.l.i.l. for the load process, virtual waiting time, and waiting time of the n^{th} customer. Finally, in Section 6, following Strassen, we study the process $v(t)=t^{-1}m\{\tau\colon e\leqq\tau\leqq t,\, Q'(\tau)\geqq c\,\varphi(\tau)\}$ for $\rho=1$, where $m\{\cdot\}$ is Lebesgue measure and $0\leqq c\leqq\gamma$. In particular, we show that

$$\overline{\lim_{t\to\infty}}\, v(t)=1-\exp\{-4((\gamma^2/c^2)-1)\} \quad \text{a.s.}$$

2. Functional Laws of the Iterated Logarithm

Let C be the metric space of all real-valued continuous functions on $[0,1]$ with the uniform metric $\rho(x,y)=\sup_{0\leqq t\leqq 1}|x(t)-y(t)|$ for $x,y\in C$. Denote by $\mathscr{C}$ the Borel sets of C. Let C_k be the product space of k $(\geqq 2)$ copies of C with the product topology and let $|\cdot|$ be the Euclidean norm in R^k. Denote by $\mathscr{C}_k$ the Borel sets of C_k. Following Strassen (1964) we let $K_k(\delta)$, $\delta>0$, be the set of absolutely continuous functions $\mathbf{x}\in C_k$ such that

$$\mathbf{x}(0)=0$$

and

$$\int_0^1 [\dot{\mathbf{x}}(t)]^2\,dt\leqq\delta^2,$$

where $\dot{\mathbf{x}}$ denotes the derivative of $\mathbf{x}$ which exists almost everywhere with respect to Lebesgue measure and the square denotes inner product. Strassen has shown that $K_k(\delta)$ is compact in C_k and that for $\mathbf{x}\in K_k(\delta)$ and $0\leqq a\leqq b\leqq 1$

$$|\mathbf{x}(b)-\mathbf{x}(a)|\leqq\delta(b-a)^{\frac{1}{2}}. \tag{2.1}$$

While Strassen worked entirely in C and C_k it is more natural for our queueing processes to work in the space of functions with jump discontinuities. Let D be the space of all real-valued right-continuous functions on $[0,1]$ having left limits and endowed with the Skorohod topology induced by the metric d (under which D is complete and separable); see Billingsley (1968, Chapter 3) for a complete discussion of D. We let D_k be the product of k copies of D with the product topology and Borel sets $\mathscr{D}_k$.

The important fact about $(D,\mathscr{D})$ for us is that the Skorohod topology on D relativized to C coincides with the uniform topology there. Let $A\subset C$ be relatively compact in $(C,\mathscr{C})$ and suppose its set of limit points in $(C,\mathscr{C})$ is the compact set K. Then A is relatively compact in $(D,\mathscr{D})$; its set of limit points in $(D,\mathscr{D})$ is K and K is compact in $(D,\mathscr{D})$ since $\mathscr{D}$ relativized to C is $\mathscr{C}$. Similar remarks hold for $(C_k,\mathscr{C}_k)$ and $(D_k,\mathscr{D}_k)$. These facts allow us to state Strassen's f.l.i.l. for Brownian motion in terms of $(D_k,\mathscr{D}_k)$ rather than the original $(C_k,\mathscr{C}_k)$.

Now let ξ be k-dimensional standard Brownian motion and define the random functions ξ_n of D_k by

$$\xi_n(t) = \xi(nt)/\varphi(n)$$

for $t \in [0, 1]$ and $n \geqq 3$; for the obvious extension of Wiener measure to $(D_k, \mathscr{D}_k)$ see Billingsley (1968, p. 137). Then Strassen's f.l.i.l. for $\{\xi_n: n \geqq 3\}$ in $(D_k, \mathscr{D}_k)$ is

Theorem 2.1 (Strassen). *With probability one the sequence $\{\xi_n: n \geqq 3\}$ is relatively compact in $(D_k, \mathscr{D}_k)$ and the set of its limit points coincides with $K_k(1)$.*

Having this result Strassen then uses the Skorohod (1965) representation to obtain a similar result for the natural sequence of functions generated by partial sums of independent, identically distributed (i.i.d.) random variables (r.v.'s). We state Skorohod's result next.

Theorem 2.2 (Skorohod). *Let $\{x_n: n \geqq 1\}$ be a sequence of i.i.d. r.v.'s with $E\{x_1\} = 0$ and $E\{x_1^2\} = \sigma^2 < \infty$ and $\{\xi(t): t \geqq 0\}$ an independent standard Brownian motion both defined on a common complete probability space. Then there exists a sequence $\{\tau_n: n \geqq 1\}$ of non-negative i.i.d. r.v.'s defined on the same space such that $E\{\tau_1\} = \sigma^2$ and the processes $\{x_n: n \geqq 1\}$ and $\left\{\xi\left(\sum_{i=1}^{n} \tau_i\right) - \xi\left(\sum_{i=1}^{n-1} \tau_i\right): n \geqq 1\right\}$ have the same joint distributions.*

We shall use Theorems 2.1 and 2.2 and Strassen's method to derive a f.l.i.l. for processes generated by a renewal process. Let $\{u_n: n \geqq 1\}$ be a sequence of non-negative i.i.d. r.v.'s with $E\{u_1\} = \lambda^{-1} > 0$ and $\sigma^2\{u_1\} = v^2 < \infty$. Let the r.v.'s $x_n \equiv u_n - \lambda^{-1}$ play the role of the $\{x_n\}$ sequence in Theorem 2.2. Then if we let $\tilde{u}_n = \lambda^{-1} + \xi\left(\sum_{i=1}^{n} \tau_i\right) - \xi\left(\sum_{i=1}^{n-1} \tau_i\right)$, the processes $\{u_n: n \geqq 1\}$ and $\{\tilde{u}_n: n \geqq 1\}$ have the same joint distributions. Form the partial sums $S_0 = 0$, $S_n = u_1 + \cdots + u_n$, $n \geqq 1$, and introduce the renewal process

$$N(t) = \begin{cases} \max\{k: S_k \leqq t\}, & u_1 \leqq t \\ 0, & u_1 > t. \end{cases}$$

Similarly, define the partial sums $\tilde{S}_0 = 0$, $\tilde{S}_n = \tilde{u}_1 + \cdots + \tilde{u}_n$, $n \geqq 1$, and let $\tilde{N}(t)$ be the associated renewal process. Now define the random function of D, N_n, for $n \geqq 3$ by

$$N_n(t) = [N(nt) - \lambda nt]/\varphi(n), \qquad t \in [0, 1].$$

The random function $\tilde{N}_n$ is defined similarly in terms of $\tilde{N}(t)$. The random functions N_n and $\tilde{N}_n$ will have the same distribution by our construction. Let $\xi_n(t) = \xi(n\,t)/\varphi(n)$. The next result is comparable to Theorem 2.1 but for the sequence $\{N_n: n \geqq 3\}$.

Theorem 2.3. *With probability one the sequence $\{N_n: n \geqq 3\}$ is relatively compact in $(D, \mathscr{D})$ and the set of its limit points coincides with $K_1(v\lambda^{\frac{3}{2}})$.*

Proof. We shall show that $d(\tilde{N}_n, v\lambda^{\frac{3}{2}}\xi_n) \to 0$ a.s. and the result will follow from Theorem 2.1. Since $d(x, y) \leqq \rho(x, y)$, it will suffice to show that $\rho(\tilde{N}_n, v\lambda^{\frac{3}{2}}\xi_n) \to 0$ a.s. By a well-known set of inequalities we have

$$\lambda \tilde{S}_{\tilde{N}(t)} - \tilde{N}(t) \leqq \lambda t - \tilde{N}(t) \leqq \lambda \tilde{S}_{\tilde{N}(t)} - \tilde{N}(t) + \lambda \tilde{u}_{\tilde{N}(t)+1},$$

or

$$\lambda \xi \left(\sum_{i=1}^{\tilde{N}(t)} \tau_i\right) \leqq \lambda t - \tilde{N}(t) \leqq \lambda \xi \left(\sum_{i=1}^{\tilde{N}(t)} \tau_i\right) + \lambda \tilde{u}_{\tilde{N}(t)+1}.$$

Since the sequence $\{\tilde{u}_n: n \geqq 1\}$ is i.i.d. with finite second moment it is well known that $\tilde{u}_{\tilde{N}(t)+1}/\varphi(t) \to 0$ a.s. By the strong law for partial sums and renewal processes we have $\sum_{i=1}^{\tilde{N}(t)} \tau_i/t \to v^2 \lambda$ a.s. From this point we can follow Strassen (1964, p. 217) to obtain

$$Pr\{\lim_{t\to\infty} |[\tilde{N}(t) - \lambda t] - v \lambda^{\frac{3}{2}} \xi(t)|/\varphi(t) = 0\} = 1.$$

This implies that $\rho(\tilde{N}_n, v \lambda^{\frac{3}{2}} \xi_n) \to 0$ a.s. and completes the proof.

To prepare for our study of queueing processes we next extend Theorem 2.4 to cover the linear combination of a finite number of independent renewal processes. Assume now that we are given $r+s$ independent sequence of non-negative, i.i.d. r.v.'s with finite variance: $\{u_n^i: n \geqq 1\}$ $(i=1, \ldots, r)$ and $\{v_n^j: n \geqq 1\}$ $(j=1, 2, \ldots, s)$ all defined on a common complete probability space $(\Omega, \mathscr{F}, P)$. Also assume that $r+s+1$ independent standard Brownian motions $\{\xi^j(t): t \geqq 0\}$ $(j=0, 1, 2, \ldots, r+s)$ which are independent of the sequence $\{u_n^i\}$ and $\{v_n^j\}$ are defined on $(\Omega, \mathscr{F}, P)$; this construction, of course, can always be carried out. Let $\{A^i(t): t \geqq 0\}$ $(i=1, 2, \ldots, r)$ be the renewal processes associated with the sequences $\{u_n^i: n \geqq 1\}$ $(i=1, \ldots, r)$ and $\{S^j(t): t \geqq 0\}$ be those associated with the $\{v_n^j: n \geqq 1\}$ $(j=1, 2, \ldots, s)$. We shall let

$$\lambda_i = 1/E\{u_1^i\}, \quad \mu_j = 1/E\{v_1^j\}, \quad \lambda = \sum_{i=1}^r \lambda_i, \quad \rho = \lambda/\mu, \quad \alpha_i^2 = \lambda_i^3 \sigma^2\{u_1^i\},$$

and $\sigma_j^2 = \mu_j^3 \sigma^2 \{v_1^j\}$, and

$$\gamma^2 = \sum_{i=1}^r \alpha_i^2 + \sum_{j=1}^s \sigma_j^2.$$

Next define the random functions $A_n^i \equiv [A^i(nt) - \lambda^i nt]/\varphi(n)$ $(i=1, \ldots, r)$ and $S_n^j \equiv [S^j(nt) - \mu^j nt]/\varphi(n)$ $(j=1, \ldots, s)$. Let the random function $X_n \equiv A_n - S_n$, where $A_n = \sum_{i=1}^r A_n^i$ and $S_n = \sum_{j=1}^s S_n^j$. Then we easily obtain the following corollary.

Corollary 2.1. *With probability one the sequence $\{X_n: n \geqq 3\}$ is relatively compact in $(D, \mathscr{D})$ and the set of its limit points coincides with $K_1(\gamma)$.*

Proof. Form the random functions $\tilde{A}_n^i$, $\tilde{S}_n^j$, and $\tilde{X}_n$ as in Theorem 2.3. From the triangle inequality and the proof of Theorem 2.3 we have

$$\rho\left(\tilde{X}_n, \sum_{i=1}^r \alpha_i \xi_n^i - \sum_{j=1}^s \sigma_j \xi_n^{r+j}\right) \leqq \sum_{i=1}^r \rho(\tilde{A}_n^i, \alpha_i \xi_n^i) + \sum_{j=1}^s \rho(\tilde{S}_n^j, \sigma_j \xi_n^{r+j}) \to 0 \quad \text{a.s.}$$

But $\sum_{i=1}^r \alpha_i \xi_n^i - \sum_{j=1}^s \sigma_j \xi_n^j$ has the same distribution as $\gamma \xi_n^0$. Using Theorem 2.1 the result follows for $\{\tilde{X}_n: n \geqq e\}$ and therefore also for $\{X_n: n \geqq e\}$.

The next corollary is only needed to study the departure process of our queueing system when $\rho = 1$.

Corollary 2.2. *With probability one the sequence* $\{(A_n, S_n): n \geqq 3\}$ *is relatively compact in* $(D_2, \mathscr{D}_2)$ *and the set of its limit points coincides with* $K_2(\gamma)$.

Proof. From Theorem 2.3 we have $\rho(\tilde{A}_n, \alpha\,\xi_n^1) \to 0$ a.s. and $\rho(\tilde{S}_n, \sigma\,\xi_n^2) \to 0$ a.s., where ξ_n^1 and ξ_n^2 are based on independent Brownian motions, $\alpha^2 = (\alpha_1^2 + \cdots + \alpha_r)$, and $\sigma^2 = (\sigma_1^2 + \cdots + \sigma_s^2)$. Hence $\rho_2\{(\tilde{A}_n, \tilde{S}_n), (\alpha\,\xi_n^1, \sigma\,\xi_n^2)\} \to 0$ a.s. The result follows then from Theorem 2.1.

We close this section by stating a useful result indicated by Strassen.

Lemma 2.1 (Strassen). *Let* $\{x_n: n \geqq 1\}$ *be a relatively compact sequence in* $(D, \mathscr{D})$ *with the compact set* K *as its set of limit points. If* h *is a continuous mapping from* $(D, \mathscr{D})$ *into some metric space* S *with Borel sets* $\mathscr{S}$, *then the sequence* $\{h(x_n): n \geqq 1\}$ *is relatively compact in* $(S, \mathscr{S})$ *and the set of its limit points coincides with* $h(K)$, *a compact set.*

3. The Queue Length Process for Modified and Standard Systems

This section shall be devoted to obtaining the f.l.i.l. for the processes representing the total number of customers in the system under both the modified and standard service disciplines. The modified system is introduced basically as a tool to get a handle on the standard system. For a complete discussion of this device see [3, Section 2].

We assume now that we are given as the basic data for our problem the $r+s$ independent sequences of non-negative, i.i.d. r.v.'s with finite variance all defined on a common complete probability space $(\Omega, \mathscr{F}, P)$ as described in Section 2. The variable u_n^i represents the interarrival time between the $(n-1)^{\text{st}}$ and n^{th} customers in the i^{th} arrival channel and the variable v_n^j represents the n^{th} potential service time of the j^{th} server. As in [3] the v_n^j's are associated with the j^{th} server rather than with the n^{th} customer which is usually the case in queueing theory. With this interpretation for the u_n^i's and v_n^j's the renewal process $A^i(t)$ represents the total number of arrivals in the i^{th} arrival channel in the interval $(0, t]$ and $S^j(t)$ represents the total number of potential service times in the j^{th} service channel in $(0, t]$. Because of the service discipline in the modified system it is particularly easy to express the queue length process, $Q'(t)$, in terms of these basic renewal processes. We assume that the system is initially empty, although as in [3] our limit theorems do not depend on this condition. For each $\omega \in \Omega$ and $t \geqq 0$, we have

$$Q'(t) = X(t) - \inf\{X(s): 0 \leqq s \leqq t\}, \tag{3.1}$$

where

$$A(t) = A^1(t) + \cdots + A^r(t),$$

$$S(t) = S^1(t) + \cdots + S^s(t),$$

and

$$X(t) = A(t) - S(t).$$

Let A_n^i $(i = 1, \ldots, r)$, S_n^j $(j = 1, \ldots, s)$, and X_n be as defined in Section 2. Let Q_n' be the random function in D defined by

$$Q_n' \equiv [Q'(nt) - [\lambda - \mu]^+ \, nt]/\varphi(n),$$

where $t \in [0, 1]$.

To obtain the f.l.i.l. for Q'_n it is convenient to introduce the continuous mapping $f\colon D \to D$ which corresponds to an impenetrable barrier at the origin. For $x \in D$, f is defined by $f(x)(t) = x(t) - \inf_{0 \leqq s \leqq t} \{x(s)\}$, $t \in [0, 1]$. The f.l.i.l. for Q'_n when $\rho = 1$ is contained in

Theorem 3.1. *If $\rho = 1$, then with probability one the sequence $\{Q'_n\colon n \geqq 3\}$ is relatively compact in $(D, \mathscr{D})$ and the set of its limit points coincides with $f[K_1(\gamma)]$.*

Proof. Since $\rho = 1$, the translation terms in both Q'_n and X_n are zero. Hence from (3.1) we have $Q'_n = f(X_n)$. Since f is a continuous mapping the result follows immediately from Corollary 2.1 and Lemma 2.1.

The next two corollaries follow easily from Theorem 3.1.

Corollary 3.1. *If $\rho = 1$, then the $\overline{\lim}_{t \to \infty} Q'(t)/\varphi(t) = \gamma$ a.s.*

Proof. First we observe that $\sup_{x \in f[K(\gamma)]} x(1) = \gamma$ from (2.1), where the supremum is actually attained for the function $x(t) = \gamma t$. From Theorem 3.1, we have $\overline{\lim}_{n \to \infty} Q'(n)/\varphi(n) = \gamma$ a.s. However, as remarked by Strassen (1964, p. 215), we could just as well have considered the net $\{Q'_u\colon u \geqq e\}$ and obtained a result comparable to Theorem 3.1. This would establish our result.

Corollary 3.2. *If $\rho = 1$, then $\underline{\lim}_{t \to \infty} Q'(t)/\varphi(t) = 0$ a.s.*

Proof. Use the method of Corollary 3.1 and observe that the $\inf_{x \in f[K(\gamma)]} x(1) = 0$ which is attained for the function $x(t) \equiv 0$.

Actually a good deal more is true for the limit inferior. From Corollary 2.1 we know that the $\overline{\lim}_{t \to \infty} X(t) = +\infty$ a.s. and the $\underline{\lim}_{t \to \infty} X(t) = -\infty$ a.s. Hence since $Q'(t) = f(X)(t)$, $Q'(t) = 0$ for arbitrarily large values of t a.s.

Next we turn to the corresponding results for $\rho > 1$.

Theorem 3.2. *If $\rho > 1$, then with probability one the sequence $\{Q'_n\colon n \geqq 3\}$ is relatively compact in $(D, \mathscr{D})$ and the set of its limit points coincides with $K_1(\gamma)$.*

Proof. We shall show that $d(X_n, Q'_n) \to 0$ a.s. and the result will follow from Corollary 2.1. Recall that $d(X_n, Q'_n) \leqq \rho(X_n, Q'_n) = \sup_{0 \leqq s \leqq n} \{-X(s)\}/\varphi(n) \geqq 0$. By the strong law for renewal processes the $\lim_{t \to \infty} [-X(t)/t] = (\mu - \lambda) < 0$ a.s. Thus, for any $\delta > 0$ and $\varepsilon > 0$ there exists a s_0 such that

$$P\left\{\sup_{s_0 \leqq t} \left| \frac{-X(t)}{t} - (\mu - \lambda) \right| \leqq \varepsilon \right\} \geqq 1 - \delta/2.$$

Hence, the

$$P\{\sup_{s_0 \leqq s \leqq n} \{-X(s)\}/\varphi(n) \leqq \varepsilon \text{ for all } n \geqq [s_0] + 1\} \geqq 1 - \delta/2.$$

On the other hand since $X(t)$ is a finite r.v. there exists a n_0 such that the

$$P\{\sup_{0 \leqq s \leqq s_0} \{-X(s)\}/\varphi(n) \leqq \varepsilon \text{ for } n \geqq n_0\} \geqq 1 - \delta/2.$$

Thus if we take $n_1 = n_0 \vee ([s_0]+1)$ we have shown that the

$$P\{\sup_{0\leqq s\leqq n} \{-X(s)\}/\varphi(n) \leqq \varepsilon \text{ for all } n\geqq n_1\} \geqq 1-\delta$$

and the theorem is proved.

The same technique used for Corollaries 3.1 and 3.2 yields

Corollary 3.3. *If* $\rho>1$, *then*

(i) $\overline{\lim}_{t\to\infty} [Q'(t)-(\lambda-\mu)\,t]/\varphi(t) = +\gamma$ *a.s.*

and

(ii) $\underline{\lim}_{t\to\infty} [Q'(t)-(\lambda-\mu)\,t]/\varphi(t) = -\gamma$ *a.s.*

We turn now to the standard system and seek identical results for the queue-length process $Q(t)$ again for $\rho\geqq 1$. The central idea, due to Borovkov (1965, Section 5), is to define the standard system in terms of the same basic sequences of random variables already used for the modified system. This device was used in [3] and yields the following inequalities: for all $\omega\in\Omega$ and $t\geqq 0$ there exists a $t_0\leqq t$ such that

$$Q(t)\leqq Q'(t)+\sum_{j=1}^{s}[S^j(t_0+\theta_j(t_0))-S^j(t_0)] \tag{3.2}$$

and

$$Q'(t)\leqq Q(t)+\sum_{j=1}^{s}[S^j(t+\theta_j(t))-S^j(t)], \tag{3.3}$$

where $\theta_j(t)$ denotes the shift in channel j at time t and is related to the excess random variable of a renewal process based on a subsequence $\{v^j_{n_k}\}$; $\theta_j(t)$ is explained in detail in [3, Section 3]. The method of proof will be to show that $d(Q_n, Q'_n)\to 0$ a.s., where $Q_n \equiv [Q(nt)-[\lambda-\mu]^+\, nt]/\varphi(n)$ for $t\in[0,1]$. Our proof of this fact is valid for all values of ρ.

Theorem 3.3. *If* $\rho\geqq 1$, *then with probability one the sequence* $\{Q_n\colon n\geqq 3\}$ *is relatively compact in* $(D,\mathscr{D})$ *and the set of its limit points coincides with* $f[K_1(\gamma)]$ *when* $\rho=1$ *and with* $K_1(\gamma)$ *when* $\rho>1$.

Proof. From (3.2) and (3.3) we have the estimate

$$d(Q_n, Q'_n)\leqq \rho(Q_n, Q'_n)\leqq \sup_{0\leqq t\leqq 1}\frac{s+\sum_{j=1}^{s}[S^j(nt+\theta_j(nt))-S^j(nt)]}{\varphi(n)}.$$

Hence to show $d(Q_n, Q'_n)\to 0$ a.s. it will suffice to show that for all j, $1\leqq j\leqq s$, that

$$\sup_{0\leqq t\leqq 1}\frac{S^j(nt+\theta_j(nt))-S^j(nt)}{\varphi(n)}\to 0 \quad \text{a.s.} \tag{3.4}$$

As observed in [3, proof of Theorem 3.1], $\sup_{0\leqq t\leqq 1}\theta_j(nt)\leqq \max_{1\leqq k\leqq \tilde{S}^j(n)+1} v^j_{n_k}$, where $\tilde{S}^j(t)$ is the renewal process generated by the subsequence, $\{v^j_{n_k}\}$, of unused potential service times in the j^{th} service channel. Since the subsequence $\{v_{n_k}^j\}$ is i.i.d., we have $\max_{1 \leqq k \leqq S^j(n)+1} \{v_{n_k}^j/n^{\frac{1}{2}}\} \to 0$ a.s. as we have remarked previously. Hence

$$\sup_{0 \leqq t \leqq 1} \theta_j(nt)/n^{\frac{1}{2}} \to 0 \qquad \text{a.s.} \tag{3.5}$$

From the proofs of Theorems 2.3 and 2.4 we know that $\rho(S_n^j, \sigma_j \xi_n) \to 0$ a.s. Strassen (1964, p. 214) has shown that for every $\varepsilon > 0$, $\rho(\sigma_j \xi_n, K(\sigma_j)) < \varepsilon$ eventually with probability one. Combining these two facts we see that for $\varepsilon > 0$ there exists n_0 such that with probability close to one $\rho(S_n^j, K(\sigma_j)) < \varepsilon$ for $n \geqq n_0$. Let $w_x(\delta)$ denote the modulus of continuity for C with the uniform metric; namely

$$w_x(\delta) = \sup_{\substack{0 \leqq s, t \leqq 1 \\ |s-t| \leqq \delta}} |x(t) - x(s)| \qquad \text{for } x \in C.$$

Then using (2.1) we have for ε, $\eta > 0$ an n_1 with the property that

$$P\{w_{S_n^j}(\delta) \leqq 2\varepsilon + \sigma_j \delta^{\frac{1}{2}}, \text{ for all } n \geqq n_1\}\ 1 - \eta. \tag{3.6}$$

Hence $\lim_{\delta \to 0} \overline{\lim}_{n \to \infty} w_{S_n^j}(\delta) = 0$ a.s. Putting together (3.5) and (3.6) yields (3.4) and completes the proof.

Results comparable to Corollaries 3.1, 3.2, and 3.3 but for the standard system follow in the same way from Theorem 3.3.

4. The Departure Processes

Let the departure processes for the standard and modified systems be denoted by $\{D(t): t \geqq 0\}$ and $\{D'(t): t \geqq 0\}$ respectively, where each process measures the number of departures for the corresponding system in the interval $(0, t]$. Define the random function D_n by

$$D_n \equiv [D(nt) - (\lambda \wedge \mu) nt] / \varphi(n), \qquad t \in [0, 1]$$

and D'_n in a similar manner. In this section our goal is to develop a f.l.i.l. for both $\{D_n: n \geqq 3\}$ and $\{D'_n: n \geqq 3\}$ when $\rho > 0$.

From the definition of the departure processes, $D(t) = A(t) - Q(t)$ and $D'(t) = A(t) - Q'(t)$. Using the definition of $Q'(t)$ we obtain

$$D'(t) = S(t) + \inf_{0 \leqq s \leqq t} [A(s) - S(s)].$$

Next we introduce the continuous mapping $g: D \times D \to D$ defined by

$$g(x, y)(t) = y(t) + \inf_{0 \leqq s \leqq t} [x(s) - y(s)], \qquad t \in [0, 1].$$

Recall that in the proof of Theorem 3.3 we showed that $d(Q_n, Q'_n) \to 0$ a.s. for all values of ρ. This in turn implies that $d(D_n, D'_n) \to 0$ a.s. So any result established for $\{D'_n: n \geqq 3\}$ will hold for $\{D_n: n \geqq 3\}$. Hence we shall work with the more tractable modified system. We look first at the case $\rho < 1$.

Theorem 4.1. *If $\rho<1$, then with probability one the sequences $\{D_n'\colon n\geqq 3\}$ and $\{D_n\colon n\geqq 3\}$ are relatively compact in $(D,\mathscr{D})$ and the set of their limit points coincides with $K_1(\alpha)$.*

Proof. We shall show that $\rho(Q_n',0)\to 0$ a.s. which implies that $d(D_n',A_n)\to 0$ a.s. This in turn implies the result using a slight variant of Corollary 2.1. We proceed now to show $\rho(Q_n',0)\to 0$ a.s. From the definition of Q_n' we have

$$\rho(Q_n',0)=\sup_{0\leqq t\leqq 1}\big[X(nt)-\inf_{0\leqq s\leqq t}\{X(ns)\}\big]/\varphi(n).$$

Now for $\delta\in(0,1)$ we have

$$\begin{aligned}\rho(Q_n',0)\leqq &\sup_{0\leqq t\leqq 1}\big|X(nt)-\inf_{t-\delta\leqq s\leqq t}\{X(ns)\}\big|/\varphi(n)\\ &+\sup_{0\leqq t\leqq 1}\Big|\inf_{0\leqq s\leqq t}\{X(ns)\}-\inf_{t-\delta\leqq s\leqq t}\{X(ns)\}\Big|/\varphi(n).\end{aligned}\tag{4.1}$$

The first term on the right-hand side of (4.1) is less than or equal to $w_{X_n}(\delta)$ and using the same method employed in the proof of Theorem 3.3 $\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\,w_{X_n}(\delta)=0$ a.s. So we can forget about this term and concentrate on the second term. The second term can be written as

$$\sup_{0\leqq t\leqq 1}\big|\min\big[0,\inf_{0\leqq s\leqq t-\delta}\{X(ns)\}-\inf_{t-\delta\leqq s\leqq t}\{X(ns)\}\big]\big|/\varphi(n).$$

Thus our proof will be complete if we can show that for every $\eta>0$ there exists an n_0 such that

$$P\{\sup_{0\leqq t\leqq 1}\big[\inf_{0\leqq s\leqq t-\delta}\{X(ns)\}-X(nt)\big]/\varphi(n)\geqq 0 \text{ for all } n\geqq n_0\}\geqq 1-\eta. \tag{4.2}$$

We have immediately

$$\big[\inf_{0\leqq s\leqq t-\delta}\{X(ns)\}-X(nt)\big]/\varphi(n)\geqq \inf_{0\leqq s\leqq t-\delta}\{X_n(s)\}-X_n(t)-\delta(\lambda-\mu)\,n/\varphi(n) \tag{4.3}$$

just using the definition of X_n. Since for every $\varepsilon>0$ $\rho(X_n,K_1(\gamma))<\varepsilon$ eventually with probability one and $K_1(\gamma)$ is compact, the first two terms on the right-hand side of (4.3) are uniformly bounded in t eventually with probability one. Since $\rho<1$ and $\delta>0$, $-\delta(\lambda-\mu)\,n/\varphi(n)\to+\infty$ as $n\to\infty$. Hence we have shown (4.2) and the proof is completed.

Next we turn to the case $\rho=1$ which requires the use of Corollary 2.2.

Theorem 4.2. *If $\rho=1$, then with probability one the sequences $\{D_n'\colon n\geqq 3\}$ and $\{D_n\colon n\geqq 3\}$ are relatively compact in $(D,\mathscr{D})$ and the set of their limit points coincides with $g[K_2(\gamma)]$.*

Proof. Simply apply Corollaries 2.1 and 2.2.

Theorem 4.3. *If $\rho>1$, then with probability one the sets $\{D_n'\colon n\geqq 3\}$ and $\{D_n\colon n\geqq 3\}$ are relatively compact and the set of their limit points coincides with $K_1(\sigma)$.*

Proof. We have shown in the proof of Theorem 3.2 that $\rho(X_n, Q_n') \to 0$ a.s. But this implies that $\rho(D_n', S_n) \to 0$ a.s. which in turn gives us $d(D_n', S_n) \to 0$ a.s. The desired result follows using the method of Corollary 2.1.

Results comparable to Corollary 3.3 also follow for the departure processes.

5. The Load, Waiting Time, and Queue Length at the i^{th} Service Channel

Let $Q^i(t)$ be the number of customers in the standard system at time t who will be processed through the i^{th} service channel and $L^i(t)$ be the work load (future service time required for all customers in the system) at time t which will be processed through the i^{th} service channel.

Define the corresponding random functions of D as

$$Q_n^i \equiv Q^i(nt)/\varphi(n)$$

and

$$L_n^i \equiv L^i(nt)/\varphi(n)$$

where $t \in [0, 1]$.

The first lemma is an immediate consequence of the queue discipline.

Lemma 5.1. *For all values of* ρ, $\rho(L_n^i, L_n^j) \to 0$ *a.s.*, $i, j = 1, \ldots, s$.

Proof. Since a waiting customer goes to the first available server, $L^i(nt)$ and $L^j(nt)$ can differ at most by a potential service time. Hence

$$\rho(L_n^i, L_n^j) \leqq \Big(\max_{1 \leqq k \leqq S^i(n+\theta_i(n))} \{v_k^i/\varphi(n)\}\Big) \vee \Big(\max_{1 \leqq k \leqq S^j(n+\theta_j(n))} \{v_k^j/\varphi(n)\}\Big). \tag{5.1}$$

As demonstrated in the proof of Theorem 3.3 the right-hand side of (5.1) goes to zero a.s.

Lemma 5.2. *If* $\rho \leqq 1$, *then* $\rho(\mu_i^{-1} Q_n^i, \mu_j^{-1} Q_n^j) \to 0$ *a.s.*

Proof. First we relate $Q^i(t)$ to $L^i(t)$. Let $B^i(t)$ be the total number of customers who arrive in $(0, t]$ and are processed through the i^{th} service channel. Then

$$L^i(t) = \sum_{k=B^i(t)-[Q^i(t)-1]^+ +1}^{B^i(t)} v_k^i + r^i(t), \tag{5.2}$$

where $r^i(t)$ is the residual service time of the customer being served in the i^{th} service channel at time t, and $\{v_k^i\}$ are the actual service times in the i^{th} service channel. Since

$$\sup_{0 \leqq t \leqq 1} r^i(nt)/\varphi(n) \leqq \max_{1 \leqq k \leqq A(n)} v_k^i/\varphi(n) \to 0 \quad \text{a.s.},$$

we shall ignore the factor $r^i(t)$ in (5.1). Doing this we have

$$\frac{L^i(nt)}{\varphi(n)} = \frac{1}{\varphi(n)} \sum_{k=B^i(nt)-[Q^i(nt)-1]^+ +1}^{B^i(nt)} (v_k^i - \mu_i^{-1}) + \frac{[Q^i(nt)-1]^+ \mu_i^{-1}}{\varphi(n)}.$$

This leads us to

$$\rho(\mu_i^{-1} Q_n^i, \mu_j^{-1} Q_n^j) \leqq \rho(L_n^i, L_n^j) + \sup_{0 \leqq t \leqq 1} \left\{ \left| \frac{1}{\varphi(n)} \sum_{k=B^i(nt)-[Q^i(nt)-1]^+ +1}^{B^i(nt)} (v_k^i - \mu_i^{-1}) - \frac{1}{\varphi(n)} \sum_{k=B^j(nt)-[Q^j(nt)-1]^+ +1}^{B^j(nt)} (v_k^j - \mu_j^{-1}) \right| \right\}$$

after neglecting some terms of the order $1/\varphi(n)$ for all ω. Thus it will suffice to show that

$$\sup_{0 \leqq t \leqq 1} \left| \frac{1}{\varphi(n)} \sum_{k=B^i(nt)-[Q^i(nt)-1]^+ +1}^{B^i(nt)} (v_k^i - \mu_i^{-1}) \right| \to 0 \quad \text{a.s.} \tag{5.3}$$

since $\rho(L_n^i, L_n^j) \to 0$ a.s. by Lemma 5.1. We know that $\sup_{0 \leqq t \leqq 1} B^i(nt)/n \leqq A(n)/n \to \lambda$ a.s. and that $\sup_{0 \leqq t \leqq 1} Q^i(nt)/n \to 0$ a.s. since $\rho \leqq 1$. Thus with high probability $B^i(nt) \leqq 2\lambda n$ for all $t \in [0, 1]$ and all sufficiently large n. Similarly, for any $\delta > 0$, $Q^i(nt) < n\delta$ with high probability for all $t \in [0, 1]$ and sufficiently large n. So on a set of arbitrarily large probability for all sufficiently large n

$$\sup_{0 \leqq t \leqq 1} \left| \frac{1}{\varphi(n)} \sum_{k=B^i(nt)-[Q^i(nt)-1]^+ +1}^{B^i(nt)} (v_k^i - \mu_i^{-1}) \right| \leqq \sup_{\substack{0 \leqq s, t \leqq 2\lambda \\ |s-t| \leqq \delta}} \left| \frac{T_{[nt]}^i - T_{[ns]}^i}{\varphi(n)} \right| \leqq w'_{T_n^i}(\delta)$$

where $T_k^i = X_1^i + \cdots + X_k^i$, $X_k^i = v_k^i - \mu_i^{-1}$, $T_0^i = 0$, $T_n^i \equiv T_{[nt]}^i/\varphi(n)$, and w'_x is the modulus of continuity of x in the space $C[0, 2\lambda]$. But $\lim_{\delta \to 0} \overline{\lim}_{n \to \infty} w'_{T_n^i}(\delta) = 0$ a.s. by the same argument used in Theorem 3.3, since the $\{X_k^i: k \geqq 1\}$ are i.i.d. r.v.'s with mean zero and finite variance. This establishes (5.3) and completes the proof.

Lemma 5.3. *If* $\rho \leqq 1$, *then* $\rho[(\mu_j/\mu) Q_n, Q_n^j] \to 0$ *a.s.*

Proof.

$$\rho[Q_n, (\mu/\mu_j) Q_n^j] = \rho\left[Q_n^1 + \cdots + Q_n^s, \left(\frac{\mu_1}{\mu_j}\right) Q_n^j + \cdots + \left(\frac{\mu_s}{\mu_j}\right) Q_n^j\right]$$

$$\leqq \sum_{i=1}^{s} \rho\left[Q_n^i, \left(\frac{\mu_i}{\mu_j}\right) Q_n^j\right] \to 0 \quad \text{a.s.}$$

by Lemma 5.2.

Lemma 5.4. *If* $\rho \leqq 1$, *then* $\rho(L_n^i, Q_n^i/\mu_i) \to 0$ *a.s.*

Proof. Use the method of Lemma 5.2.

Lemma 5.5. *If* $\rho \leqq 1$, *then* $\rho(L_n^i, Q_n/\mu) \to 0$ *a.s.*

Proof. Simply use Lemmas 5.3 and 5.4.

The total load for the entire system, $L(t)$ is just $L^1(t) + \cdots + L^s(t)$ and the virtual waiting time, $W(t)$, is just $\min_{1 \leqq i \leqq s} \{L^i(t)\}$. As usual let $L_n \equiv L(nt)/\varphi(n)$ and $W_n \equiv W(nt)/\varphi(n)$ with $t \in [0, 1]$. Then we have immediately from Lemma 5.5 the following result.

Lemma 5.6. *If* $\rho \leqq 1$, *then* $\rho\left(L_n, \frac{s}{\mu} Q_n\right) \to 0$ *and* $\rho\left(W_n, \frac{Q_n}{\mu}\right) \to 0$ *a.s.*

The results in Lemmas 5.2–5.5 are mainly of interest for the case $\rho=1$, since $\rho(Q_n, 0)\to 0$ a.s. when $\rho<1$. Combining Lemmas 5.3, 5.5, and 5.6 with Theorem 3.3 we obtain

Theorem 5.1. *If $\rho=1$, then with probability one the sequences $\{Q_n^i\colon n\geqq 3\}$, $\{L_n^i\colon n\geqq 3\}$, $\{L_n\colon n\geqq 3\}$, and $\{W_n\colon n\geqq 3\}$ are relatively compact in $(D, \mathscr{D})$ and their limit points coincide with the sets $f[K_1(\mu_j\gamma/\mu)]$, $f[K_1(\gamma/\mu)]$, $f[K_1(s\gamma/\mu)]$, and $f[K_1(\gamma/\mu)]$ respectively.*

As before, corollaries comparable to Corollaries 3.1 and 3.2 can easily be obtained.

Now we turn to the process W_n', the waiting time of the n^{th} customer in the standard system. Define the random function, Z_n, in D by

$$Z_n\equiv W'_{[nt]}/\varphi(n), \qquad t\in[0,1].$$

Then using the same methods already employed we can obtain

Theorem 5.2. *If $\rho=1$, then with probability one the sequence $\{Z_n\colon n\geqq 3\}$ is relatively compact and the set of its limit points coincides with $f[K_1(\gamma/\mu^{\frac{3}{2}})]$.*

We remark that for the classical queue $GI/G/1$ Theorem 5.2 can be immediately deduced from Strassen (1964, Theorem 3).

6. An Application

Strassen (1964) obtained a number of interesting results by applying his f.l.i.l. to particular functions. In this section we shall show how one of his results can be immediately taken over for the process $\{Q'(t)\colon t\geqq 0\}$ when $\rho=1$.

Following Strassen we define $\nu(t)=t^{-1}m\{\tau\colon e\leqq\tau\leqq t,\ Q'(t)>c\,\varphi(\tau)\}$ for $t\geqq e$, where $m\{\cdot\}$ is Lebesgue measure and $0\leqq c\leqq\gamma$. Hence $\nu(t)$ measures roughly the fraction of time in $[0,t]$ that the modified queue length process exceeds the function $c\,\varphi(\tau)$. In the case $\rho=1$, we can follow Strassen's argument and obtain

$$\overline{\lim_{t\to\infty}}\,\nu(t)=\sup_{x\in f[K_1(\gamma)]} m\{t\colon x(t)\geqq c\,\varphi(t)\} \qquad \text{a.s.}$$

But Strassen shows that

$$\sup_{x\in K_1(\gamma)} m\{t\colon x(t)\geqq c\,\varphi(t)\}=1-\exp\{-4((\gamma^2/c^2)-1)\}$$

and that the supremum is actually attained for the function

$$x_0(t)=\begin{cases}(c/s_0^{\frac{1}{2}})\cdot t, & 0\leqq t\leqq s_0\\ c\cdot t^{\frac{1}{2}}, & s_0\leqq t\leqq 1,\end{cases}$$

where $s_0=\exp\{-4((\gamma^2/c^2)-1)\}$. By definition of the function f it is clear that $f[K_1(\gamma)]\subset K_1(\gamma)$ and that $x_0\in f[K_1(\gamma)]$. Hence we have

$$\overline{\lim_{t\to\infty}}\,\nu(t)=1-\exp\{-4((\gamma^2/c^2)-1)\} \qquad \text{a.s.}$$

Comparable results could be obtained for processes other than $Q'(t)$.

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(Received January 7, 1970)