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Compactness and Sequential Compactness in Spaces of Measures

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1. Introduction

This paper is concerned with several characterizations of conditionally compact and conditionally sequentially compact subsets of the space $\mathrm{ca}(X, \mathscr{F})$ endowed with the topology $\mathscr{T}_{\mathscr{F}}$ of set-wise convergence on the σ-field $\mathscr{F}$ in X. In contrast to the usual way of considering first compactness in the $\mathscr{T}_0 = \sigma(\mathrm{ca}(X, \mathscr{F}), \mathrm{ca}(X, \mathscr{F})^*)$-topology ([5], IV.9.1 and 9.2), we start proving directly and in a rather self-contained way compactness criterions in the $\mathscr{T}_{\mathscr{F}}$-topology (2.6). The main advantage of the approach presented in Section 2 is its simplicity, e.g. no resort on the Eberlein-Šmulian-Theorem is necessary in order to link compactness with sequential compactness. In the same simple way we establish the connection of $\mathscr{T}_{\mathscr{F}}$- with $\mathscr{T}_{\mathfrak{B}}$- resp. $\mathscr{T}_0$-compactness (2.11–2.19) which yields immediately the well known compactness criterions with respect to the latter topology. In Section 3 we consider the case where $X = (X, \mathscr{T})$ is a topological space and where $\mathscr{F}$ are the Borel sets in X. 3.7–3.13 are concerned with compactness results first studied by Grothendieck ([6], Théorème 2, (1)–(4), p. 146) in the case of Radon measures on locally compact basic spaces $(X, \mathscr{T})$. For $\mathscr{K}$-regular measures (Definition 3.1) 3.7 generalizes in particular Grothendieck's Criterion (4) to Hausdorff spaces and 3.11 shows that the equivalence of the statements (1), (3) and (4) holds true for regular Hausdorff spaces. Finally 3.12 proves the equivalence of (1) to (2) for completely regular Hausdorff basic spaces $(X, \mathscr{T})$. As a Corollary (3.14) we obtain for analytical spaces $(X, \mathscr{T})$ several characterizations for $\mathscr{T}_{\mathscr{F}}$-conditionally compact resp. $\mathscr{T}_{\mathscr{F}}$-conditionally sequentially compact subsets of $\mathrm{ca}(X, \mathscr{F})$, where the equivalence of $\mathscr{T}_{\mathscr{F}}$-conditional compactness to (iii) resp. (iv) may be considered as an analogon to Prohorov's well known criterion ([15], Theorem 1.12, p. 170).

A further application centers around Dieudonné's Theorem ([3], Propositions 8 and 9, p. 37) and its extension due to Wells, jr. ([17], Theorem 3, p. 125 and the corollary on p. 128). The results of Section 3 enable us to generalize Dieudonné's Theorem from compact metric spaces to regular Hausdorff spaces and to extend the theorem of Wells, jr. from compact spaces to normal Hausdorff spaces. Both is established in 4.5 together with 4.9 and yields the main Theorem 4.10 of Section 4.

Once 4.5 was proved it was first discovered by Pfanzagl that Dieudonné's Theorem could be even extended to arbitrary Hausdorff spaces (see the main

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theorem in [14c]). In Section 5 we derive this result using the compactness criterions of Section 3 together with Lemma 1 in [14c]. Finally we prove in 5.3 a boundedness result which generalizes [3], Proposition 9, from compact spaces to arbitrary Hausdorff spaces.

It should be emphasized that besides the techniques going back to Dieudonné and Grothendieck we resort to the concept of a sufficient sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$ for a given sequence of measures $(\mu_n|\mathscr{F})_{n\in\mathbb{N}}$ which is useful insofar as convergence of a bounded sequence $(\mu_n|\mathscr{F})_{n\in\mathbb{N}}$ on $\mathscr{F}_0$ implies its convergence on $\mathscr{F}$ (1.24). It turns out that for the present case this is the proper tool to link compactness with sequential compactness (see the proof of the equivalence of (i) to (ii) in 2.6).

Finally it should be remarked that the compactness criterions studied here play an essential role in statistical theory; the validity of important theorems in this field are limited by compactness assumptions on the underlying family of distributions (cf. [1], Section 6, [11], p. 146, [12], Lemma 4, [14a] and the literature cited there).

Let X be an arbitrary non-void set, $\mathscr{F}$ a σ-field of subsets of X and $\mathrm{ca}(X,\mathscr{F})$ be the family of all countably additive real-valued set functions defined on $\mathscr{F}$. It is well known (see [5], III.4.5) that every $\mu\in\mathrm{ca}(X,\mathscr{F})$ is a bounded set function and (see [5], III.4.7), given $\mu\in\mathrm{ca}(X,\mathscr{F})$, the set function $|\mu|$ defined by

$$|\mu|(A):=\sup\sum_{i=1}^{n}|\mu(A_i)|, \qquad A\in\mathscr{F},$$

(where the supremum is taken over all finite sequences of disjoint sets $A_i\in\mathscr{F}$ with $A_i\subset A$) is a non-negative bounded and countably additive set function on $\mathscr{F}$ which is equal to μ if μ itself is non-negative, i.e. if $\mu(A)\geqq 0$ for all $A\in\mathscr{F}$. With $\mathrm{ca}_+(X,\mathscr{F})$ we denote the space of all non-negative $\mu\in\mathrm{ca}(X,\mathscr{F})$. Besides $\mathrm{ca}(X,\mathscr{F})$ we occasionally consider the space $\mathrm{ba}(X,\mathscr{F})$ of all bounded additive real-valued set functions on $\mathscr{F}$. For $\mu\in\mathrm{ba}(X,\mathscr{F})$ the set function $|\mu|$ defined as above is a non-negative bounded and additive set function on $\mathscr{F}$ (see [5], III. 1.5 and 1.6). Hence $\|\mu\|:=|\mu|(X)$, $\mu\in\mathrm{ba}(X,\mathscr{F})$, defines a norm in $\mathrm{ba}(X,\mathscr{F})$ and endowed with this norm $(\mathrm{ba}(X,\mathscr{F}),\|\cdot\|)$ resp. $(\mathrm{ca}(X,\mathscr{F}),\|\cdot\|)$ become Banach spaces. Elements of $\mathrm{ca}(X,\mathscr{F})[\mathrm{ba}(X,\mathscr{F})]$ are called measures [contents] and if we write $\mu|\mathscr{F}$ it is to indicate that the measure [content] μ is to be considered as a set function on $\mathscr{F}$. Throughout $\mathfrak{M}=\mathfrak{M}|\mathscr{F}$ denotes subsets of $\mathrm{ca}(X,\mathscr{F})$.

$\mathbb{N}[\mathbb{R}]$ denotes the set of positive integers [real numbers], $\bar{A}$ the complement of a set A in X, i.e. $\bar{A}:=X-A$. We write A^c and $\operatorname{int} A$ for the closure resp. interior of a set A in a topological space.

Let us list some well known results which are basic for the following sections. If no special reference is given, the proofs may be found in [5].

1.1. Definition. $\mathfrak{M}|\mathscr{F}$ is said to be dominated by a non-negative measure $\lambda|\mathscr{F}$ ($\mathfrak{M}|\mathscr{F}\ll\lambda|\mathscr{F}$) iff for every $\mu\in\mathfrak{M}$ $\lambda(A)=0$ implies $\mu(A)=0$. If $\mathfrak{M}|\mathscr{F}\ll\lambda|\mathscr{F}$ and if $|\mu|(A)=0$ for all $\mu\in\mathfrak{M}$ implies $\lambda(A)=0$, then $\mathfrak{M}|\mathscr{F}$ and $\lambda|\mathscr{F}$ are called equivalent ($\mathfrak{M}|\mathscr{F}\sim\lambda|\mathscr{F}$).

1.2. Criterion. $\mathfrak{M}|\mathscr{F}\ll\lambda|\mathscr{F}$ *iff for every* $\mu\in\mathfrak{M}$ *and for each* $\varepsilon>0$ *there exists* $\delta(\mu,\varepsilon)>0$ *such that* $\lambda(A)<\delta(\mu,\varepsilon)$ *implies* $|\mu(A)|<\varepsilon$.

1.3. Definition. Given $\mathfrak{M}|\mathscr{F}$ we denote by $\mathfrak{M}^\Sigma|\mathscr{F}$ the set of all $\lambda \in \mathrm{ca}_+(X, \mathscr{F})$ of the form $\lambda = \sum_{n\in\mathbb{N}} 2^{-n} \frac{|\mu_n|}{1+\|\mu_n\|}$ with $\mu_n \in \mathfrak{M}$, $n \in \mathbb{N}$, and we write $|\mathfrak{M}|$ for the set $\{|\mu|: \mu \in \mathfrak{M}\}$.

1.4. Proposition. *If $\mathfrak{M}|\mathscr{F}$ is countable, there exists $\lambda \in \mathfrak{M}^\Sigma$ such that $|\mathfrak{M}| \sim \lambda|\mathscr{F}$.*

1.5. Definition. $\mathfrak{M}|\mathscr{F}$ is said to be uniformly dominated by $\lambda \in \mathrm{ca}_+(X, \mathscr{F})$ ($\mathfrak{M}|\mathscr{F} \ll \lambda|\mathscr{F}$) iff for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\lambda(A) < \delta(\varepsilon)$ implies $\sup_{\mu\in\mathfrak{M}} |\mu(A)| < \varepsilon$.

1.6. Criterion. $\mathfrak{M}|\mathscr{F} \ll \lambda|\mathscr{F}$ *iff* $|\mathfrak{M}| \ll \lambda|\mathscr{F}$.

1.7. Definition. $\mathfrak{M}|\mathscr{F}$ is called equicontinuous iff $\lim_{k\in\mathbb{N}} \sup_{\mu\in\mathfrak{M}} |\mu(A_k)| = 0$ for each sequence $(A_k)_{k\in\mathbb{N}} \downarrow \emptyset$, $A_k \in \mathscr{F}$, $k \in \mathbb{N}$.

1.8. *Remark.* If $\mathfrak{M}|\mathscr{F} \ll \lambda|\mathscr{F}$, then $\mathfrak{M}|\mathscr{F}$ is equicontinuous. Using 2.6 below together with a result of Brooks [2] we obtain a simple proof for the converse statement (see 2.7). This was first proved by Dubrovski ([4], p. 738, Theorem 2).

1.9. Theorem (Vitali-Hahn-Saks). *If for a sequence $\mu_n \in \mathrm{ca}(X, \mathscr{F})$, $n \in \mathbb{N}$, $(\mu_n(A))_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ for every $A \in \mathscr{F}$, then $\{\mu_n|\mathscr{F}: n \in \mathbb{N}\}$ is uniformly dominated by every dominating (non-negative) measure.*

1.10. (Nikodým). If for a sequence $\mu_n \in \mathrm{ca}(X, \mathscr{F})$, $n \in \mathbb{N}$, $(\mu_n(A))_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ for every $A \in \mathscr{F}$, then the set function $\mu_0|\mathscr{F}$ defined by $\mu_0(A) := \lim_{n\in\mathbb{N}} \mu_n(A)$, $A \in \mathscr{F}$, is a measure and the set $\{\mu_n|\mathscr{F}: n \in \mathbb{N} \cup \{0\}\}$ is uniformly dominated by every dominating (non-negative) measure.

1.11. Lemma. *Let $\mathfrak{M}|\mathscr{F} = \{\mu_n|\mathscr{F}: n \in \mathbb{N}\}$ be equicontinuous and suppose that $\mathscr{S} \subset \mathscr{F}$ is a subsystem of $\mathscr{F}$, containing X, which is closed under finite intersections and which generates $\mathscr{F}$. Then convergence of $(\mu_n)_{n\in\mathbb{N}}$ on $\mathscr{S}$ (i.e. $(\mu_n(S))_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ for every $S \in \mathscr{S}$) implies convergence of $(\mu_n)_{n\in\mathbb{N}}$ on $\mathscr{F}$. Conversely, if convergence of $(\mu_n)_{n\in\mathbb{N}}$ on $\mathscr{S}$ implies its convergence on $\sigma(\mathscr{S})$ (the σ-field generated by $\mathscr{S}$), it follows from* 1.9 *that $\{\mu_n|\sigma(\mathscr{S}): n \in \mathbb{N}\}$ is equicontinuous.*

Proof. Let $\mathscr{D} := \{A \in \mathscr{F}: (\mu_n(A))_{n\in\mathbb{N}}$ converges in $\mathbb{R}\}$. By assumption $\mathscr{S} \subset \mathscr{D}$. Obviously $\mathscr{D}$ is closed under disjoint unions and proper differences. As $\mathscr{S}$ is closed under finite intersections it follows that the field generated by $\mathscr{S}$ is contained in $\mathscr{D}$. We shall show below that $\mathscr{D}$ is, furthermore, a monotone system (i.e. $\uparrow$ and $\downarrow$ closed). Since a monotone system containing a field is a σ-field, we obtain $\mathscr{D} = \mathscr{F}$.

Let $(A_k)_{k\in\mathbb{N}} \uparrow A \in \mathscr{F}$ with $A_k \in \mathscr{D}$, $k \in \mathbb{N}$. Then $B_k := A - A_k \downarrow \emptyset$. By equicontinuity there exists for every $\varepsilon > 0$ a $k(\varepsilon) \in \mathbb{N}$ such that $|\mu_n(B_{k(\varepsilon)})| \leqq \varepsilon/3$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} |\mu_n(A) - \mu_m(A)| &= |\mu_n(B_{k(\varepsilon)}) + \mu_n(A_{k(\varepsilon)}) - \mu_m(B_{k(\varepsilon)}) - \mu_m(A_{k(\varepsilon)})| \\ &\leqq |\mu_n(A_{k(\varepsilon)}) - \mu_m(A_{k(\varepsilon)})| + |\mu_n(B_{k(\varepsilon)})| + |\mu_m(B_{k(\varepsilon)})| \leqq \varepsilon \end{aligned}$$

for all sufficiently large $n, m \in \mathbb{N}$. This implies $A \in \mathscr{D}$. As $\mathscr{D}$ is closed with respect to proper differences, it follows that it is also $\downarrow$ closed.

1.12. Theorem (Nikodým). *Let $\mathfrak{M} \subset \mathrm{ca}(X, \mathscr{F})$. If $\{\mu(A)\colon \mu \in \mathfrak{M}\}$ is bounded for every $A \in \mathscr{F}$, then $\{\mu(A)\colon \mu \in \mathfrak{M}, A \in \mathscr{F}\}$ is bounded, too. Therefore $\{|\mu|(A)\colon \mu \in \mathfrak{M}, A \in \mathscr{F}\}$ is bounded.*

Proof. See [5], IV.9.8. The second assertion follows from the Hahn-Decomposition-Theorem ([5], III.4.10).

1.13. Definition. $\mathfrak{M} \subset \mathrm{ca}(X, \mathscr{F})$ is called bounded iff $\mathfrak{M}$ is bounded as a subset of $(\mathrm{ca}(X, \mathscr{F}), \|\cdot\|)$, i.e. iff $b_{\mathfrak{M}} := \sup_{\mu \in \mathfrak{M}} \|\mu\| = \sup_{\mu \in \mathfrak{M}} |\mu|(X) < \infty$. A sequence $\mu_n \in \mathrm{ca}(X, \mathscr{F})$, $n \in \mathbb{N}$, is called bounded iff the set $\{\mu_n|\mathscr{F}\colon n \in \mathbb{N}\}$ is bounded in $(\mathrm{ca}(X, \mathscr{F}), \|\cdot\|)$.

As to the topological concepts and theorems used in later sections we should mention the following:

1.14. Definition. Let $(X, \mathscr{T})$ be a Hausdorff space and X_0 be a subset of X.

(i) X_0 is called conditionally sequentially compact (c.s.c.) in $(X, \mathscr{T})$ iff each sequence in X_0 contains a subsequence which converges in X.

(ii) X_0 is called sequentially compact (s.c.) in $(X, \mathscr{T})$ iff each sequence in X_0 contains a subsequence which converges in X_0.

(iii) X_0 is called conditionally compact (c.c.) in $(X, \mathscr{T})$ iff the closure of X_0 in $(X, \mathscr{T})$ is compact.

Relativized topologies on subsets X_0 of a topological space $(X, \mathscr{T})$ will be usually denoted by the same letter, i.e. we write $(X_0, \mathscr{T})$ (instead of $(X_0, \mathscr{T} \cap X_0)$) for the topological subspace X_0 of $(X, \mathscr{T})$.

1.15. Lemma. *Let $(X, \mathscr{T})$ be a compact Hausdorff space and $X_0 \subset X_1 \subset X$. Then X_0 is c.c. in $(X_1, \mathscr{T})$ iff $X_0^c \subset X_1$ (where X_0^c denotes the closure of X_0 in $(X, \mathscr{T})$).*

Proof. We remark that the closure of X_0 in $(X_1, \mathscr{T})$ is $X_0^c \cap X_1$.

$\rhd$ If $X_0^c \cap X_1$ is compact in $(X_1, \mathscr{T})$, it is also compact and therefore closed in $(X, \mathscr{T})$. As $X_0 \subset X_0^c \cap X_1$, this implies $X_0^c \subset X_0^c \cap X_1$, whence $X_0^c \subset X_1$.

$\lhd$ $X_0^c \subset X_1$ implies $X_0^c \cap X_1 = X_0^c$ which is closed and therefore compact in $(X, \mathscr{T})$, hence $X_0^c \cap X_1$ is compact in $(X_1, \mathscr{T})$.

1.16. *Remark.* Any subset of a c.s.c. [c.c.] set is c.s.c. [c.c.] again.

1.17. Theorem ([9], Theorem 8, p. 141). *Let f be a continuous function carrying the compact Hausdorff space $(X, \mathscr{T})$ onto the Hausdorff space $(Y, \mathscr{T}')$. Then $(Y, \mathscr{T}')$ is compact. If f is 1-1, then it is a homeomorphism.*

1.18. Corollary. *Let $(X, \mathscr{T})$ be a compact Hausdorff space and $\mathscr{T}' \subset \mathscr{T}$ a Hausdorff topology. Then $\mathscr{T}' = \mathscr{T}$.*

1.19. Lemma. *Let $\mathscr{T}_1 \subset \mathscr{T}_2$ be Hausdorff topologies on X. Then any set $X_0 \subset X$ which is c.c. in $(X, \mathscr{T}_2)$ is also c.c. in $(X, \mathscr{T}_1)$ and $X_0^{c_1} = X_0^{c_2}$ (where $X_0^{c_i}$ denotes the $\mathscr{T}_i$-closure of X_0 in $(X, \mathscr{T}_i)$, $i = 1, 2$). Furthermore, $\mathscr{T}_1$ coincides with $\mathscr{T}_2$ on $X_0^{c_1}$.*

Proof. As $\mathscr{T}_1 \subset \mathscr{T}_2$, $\mathscr{T}_2$-compactness of $X_0^{c_2}$ implies $\mathscr{T}_1$-compactness. Hence $X_0^{c_2}$ is $\mathscr{T}_1$-closed and therefore $X_0^{c_1} \subset X_0^{c_2}$. Hence $X_0^{c_1}$ is $\mathscr{T}_1$-compact. As $\mathscr{T}_1 \subset \mathscr{T}_2$ implies $X_0^{c_2} \subset X_0^{c_1}$, we obtain $X_0^{c_1} = X_0^{c_2}$. By 1.18, $\mathscr{T}_1$ coincides with $\mathscr{T}_2$ on $X_0^{c_1}$.

Our main interest in the following sections centers around compactness and sequential compactness in $\mathrm{ca}(X, \mathscr{F})$ endowed with special topologies. One of them will be the topology of set-wise convergence on $\mathscr{F}$. In this connection it turns out that the concept of a sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$ which is sufficient for a family $\mathfrak{M}|\mathscr{F}$ in $\mathrm{ca}(X, \mathscr{F})$ is a very useful tool.

1.20. Definition. Let $\mathscr{T}_{\mathscr{F}}$ denote the topology in $\mathrm{ca}(X, \mathscr{F})$ of set-wise convergence on $\mathscr{F}$, i.e. $\mathscr{T}_{\mathscr{F}}$ is the weakest topology for which all mappings $\mu \to \mu(A)$, $A \in \mathscr{F}$, are continuous. In other words: If $\mu \in \mathrm{ca}(X, \mathscr{F})$ and if $(\mu_\beta)_{\beta \in B}$ is a net in $\mathrm{ca}(X, \mathscr{F})$, then $(\mu_\beta)_{\beta \in B}$ $\mathscr{T}_{\mathscr{F}}$-converges to $\mu\big((\mu_\beta)_{\beta \in B} \to \mu(\mathscr{T}_{\mathscr{F}})\big)$ iff $\big(\mu_\beta(A)\big)_{\beta \in B} \to \mu(A)$ for all $A \in \mathscr{F}$.

1.21. *Remark.* If $\mathfrak{B} = \mathfrak{B}(X, \mathscr{F})$ denotes the space of all bounded $\mathscr{F}$-measurable functions on X, then $\mathscr{T}_{\mathscr{F}}$ coincides on every bounded $\mathfrak{M} \subset \mathrm{ca}(X, \mathscr{F})$ with the topology $\mathscr{T}_{\mathfrak{B}}$, being the coarsest topology rendering all mappings $\mu \to \mu(f)$, $f \in \mathfrak{B}$, continuous. $\big(\mu(f)$ denotes the integral of f with respect to $\mu.\big)$ Furthermore, $\mathscr{T}_{\mathfrak{B}}$ coincides with $\mathscr{T}_{\mathscr{F}}$ on $\mathrm{ca}_+(X, \mathscr{F})$ but it can be shown by examples that on $\mathrm{ca}(X, \mathscr{F})$ $\mathscr{T}_{\mathscr{F}}$ is strictly coarser than $\mathscr{T}_{\mathfrak{B}}$, in general.

For every $\mu \in \mathrm{ca}(X, \mathscr{F})$ and sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$ we denote by $\mu^{\mathscr{F}_0}(1_A)$, $A \in \mathscr{F}$, the conditional expectation of the indicator function 1_A of the set $A \in \mathscr{F}$ with respect to μ, given $\mathscr{F}_0$, i.e. $\mu^{\mathscr{F}_0}(1_A)$ is defined to be the μ-equivalence class of all $\mathscr{F}_0$-measurable functions fulfilling $\mu(A \cap A_0) = \mu\big(\mu^{\mathscr{F}_0}(1_A)\, 1_{A_0}\big)$ for every $A_0 \in \mathscr{F}_0$. The existence of $\mu^{\mathscr{F}_0}(1_A)$ for every $\mu \in \mathrm{ca}(X, \mathscr{F})$ and $A \in \mathscr{F}$ follows from the Radon-Nikodým Theorem $\big([7]$, p. 128 together with remark (4) on p. 131$\big)$.

1.22. Definition. A sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$ is called sufficient for $\mathfrak{M}|\mathscr{F} \subset \mathrm{ca}(X, \mathscr{F})$ iff for every $A \in \mathscr{F}$ there exists $\varphi_A \in \bigcap_{\mu \in \mathfrak{M}} \mu^{\mathscr{F}_0}(1_A)$.

1.23. Lemma. *If $\mathfrak{M}|\mathscr{F}$ is countable, there exists a countably generated sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$ which is sufficient for $\mathfrak{M}|\mathscr{F}$.*

Proof. Let $\mathfrak{M}|\mathscr{F} = \{\mu_n|\mathscr{F} : n \in \mathbb{N}\}$. Then $\lambda := \sum_{n \in \mathbb{N}} 2^{-n} \dfrac{|\mu_n|}{1 + \|\mu_n\|}$ dominates $\mathfrak{M}|\mathscr{F}$. Let $p_n \in \dfrac{d\mu_n|\mathscr{F}}{d\lambda|\mathscr{F}}$ (the class of Radon-Nikodým derivatives of $\mu_n|\mathscr{F}$ with respect to $\lambda|\mathscr{F}$), $n \in \mathbb{N}$, and let $\mathscr{F}_0$ be the σ-field generated by $\{p_n : n \in \mathbb{N}\}$ (i.e. the smallest σ-field with respect to which all p_n, $n \in \mathbb{N}$, become measurable). $\mathscr{F}_0$ is countably generated and we have for every $n \in \mathbb{N}$ and $A \in \mathscr{F}$:

$$\mu_n(A \cap A_0) = \lambda(p_n 1_{A \cap A_0}) = \lambda\big(\lambda^{\mathscr{F}_0}(p_n 1_{A \cap A_0})\big) = \lambda\big(p_n 1_{A_0} \lambda^{\mathscr{F}_0}(1_A)\big) = \mu_n\big(\lambda^{\mathscr{F}_0}(1_A)\, 1_{A_0}\big) \quad \text{for all } A_0 \in \mathscr{F}_0.$$

This implies that for every $A \in \mathscr{F}$

$$\lambda^{\mathscr{F}_0}(1_A) \in \bigcap_{n \in \mathbb{N}} \mu_n^{\mathscr{F}_0}(1_A),$$

hence $\mathscr{F}_0$ is sufficient for $\mathfrak{M}|\mathscr{F}$.

1.24. Lemma. *Let $\mathfrak{M}|\mathscr{F} \subset \mathrm{ca}(X, \mathscr{F})$ be bounded and suppose that the sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$ is sufficient for $\mathfrak{M}|\mathscr{F}$. Then the topology $\mathscr{T}_{\mathscr{F}_0}$ (of set-wise convergence on $\mathscr{F}_0$) coincides with $\mathscr{T}_{\mathscr{F}}$ on $\mathfrak{M}$. Furthermore: Any sequence $\mu_n \in \mathfrak{M}$, $n \in \mathbb{N}$, which converges on $\mathscr{F}_0$ does also converge on $\mathscr{F}$. If $\mathfrak{M}|\mathscr{F} \subset \mathrm{ca}_+(X, \mathscr{F})$, the boundedness assumption on $\mathfrak{M}$ can be dropped.*

Proof. Let $A \in \mathscr{F}$ be arbitrary. As $\mathscr{F}_0$ is sufficient for $\mathfrak{M}|\mathscr{F}$, there exists $\varphi_A \in \bigcap_{\mu \in \mathfrak{M}} \mu^{\mathscr{F}_0}(1_A)$; in particular: $\mu(A) = \mu(\varphi_A)$ for all $\mu \in \mathfrak{M}$. Hence for arbitrary $r, s \in \mathbb{R}$: $\{\mu \in \mathfrak{M}: s < \mu(A) < r\} = \{\mu \in \mathfrak{M}: s < \mu(\varphi_A) < r\}$. This implies

$$\mathscr{T}_{\mathscr{F}} \cap \mathfrak{M} \subset \mathscr{T}_{\mathfrak{B}(X, \mathscr{F}_0)} \cap \mathfrak{M} = \mathscr{T}_{\mathscr{F}_0} \cap \mathfrak{M} \subset \mathscr{T}_{\mathscr{F}} \cap \mathfrak{M}$$

(cf. 1.21). The proof of the second assertion follows the same patterns.

2. Compactness and Sequential Compactness in Spaces of Measures Defined on an Arbitrary Measurable Space $(X, \mathscr{F})$

Let $(X, \mathscr{F})$ be a measurable space, i.e. X is a non-void set and $\mathscr{F}$ a σ-field of subsets of X. Our main aim is to give several characterizations of conditionally compact (c.c.) and conditionally sequentially compact (c.s.c.) subsets of $\mathrm{ca}(X, \mathscr{F})$ endowed with the topology $\mathscr{T}_{\mathscr{F}}$ of set-wise convergence on $\mathscr{F}$.

2.1. Lemma. *Let $\mathfrak{M}|\mathscr{F}$ be bounded. Then $\mathfrak{M}|\mathscr{F}$ is c.c. in $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$ iff each $\mathscr{T}_{\mathscr{F}}$-accumulation point of $\mathfrak{M}$ in $[-b_{\mathfrak{M}}, b_{\mathfrak{M}}]^{\mathscr{F}}$ belongs to $\mathrm{ca}(X, \mathscr{F})$.*

Proof. Follows from 1.15 applied with $X = [-b_{\mathfrak{M}}, b_{\mathfrak{M}}]^{\mathscr{F}}$, $\mathscr{T} = \mathscr{T}_{\mathscr{F}}$, $X_0 = \mathfrak{M}$ and $X_1 = \mathrm{ca}(X, \mathscr{F}) \cap [-b_{\mathfrak{M}}, b_{\mathfrak{M}}]^{\mathscr{F}}$ together with the fact that $\mathfrak{M}|\mathscr{F}$ is c.c. in $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$ iff $\mathfrak{M}|\mathscr{F}$ is c.c. in $(\mathrm{ca}(X, \mathscr{F}) \cap [-b_{\mathfrak{M}}, b_{\mathfrak{M}}]^{\mathscr{F}}, \mathscr{T}_{\mathscr{F}})$.

2.2. Lemma. *If $\mathfrak{M}|\mathscr{F}$ is c.s.c. [c.c.] in $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$, then $\mathfrak{M}|\mathscr{F}_0$ is c.s.c. [c.c.] in $(\mathrm{ca}(X, \mathscr{F}_0), \mathscr{T}_{\mathscr{F}_0})$ for every sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$.*

Proof. The c.s.c. case is obvious. As to the c.c. case we remark that the continuous image of a c.c. set into a Hausdorff space is again c.c. Applying this for the continuous map f: $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}}) \to (\mathrm{ca}(X, \mathscr{F}_0), \mathscr{T}_{\mathscr{F}_0})$ defined by $f(\mu|\mathscr{F}) := \mu|\mathscr{F}_0$, the assertion follows.

2.3. Lemma. *Let $\mathfrak{M}|\mathscr{F}$ be c.c. in $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$. If $\mathscr{S} \subset \mathscr{F}$ separates the elements of $\mathrm{ca}(X, \mathscr{F})$, then the topologies $\mathscr{T}_{\mathscr{S}}$ and $\mathscr{T}_{\mathscr{F}}$ coincide on the $\mathscr{T}_{\mathscr{F}}$-closure of $\mathfrak{M}|\mathscr{F}$ in $\mathrm{ca}(X, \mathscr{F})$. Furthermore: Any sequence $\mu_n \in \mathfrak{M}$, $n \in \mathbb{N}$, which converges on $\mathscr{S}$ also converges on $\mathscr{F}$.*

Proof. Let $\mathfrak{M}^c$ be the $\mathscr{T}_{\mathscr{F}}$-closure of $\mathfrak{M}|\mathscr{F}$ in $\mathrm{ca}(X, \mathscr{F})$. By assumption $\mathfrak{M}^c$ is $\mathscr{T}_{\mathscr{F}}$-compact. As $\mathscr{S}$ separates the elements of $\mathrm{ca}(X, \mathscr{F})$, $\mathscr{T}_{\mathscr{S}}$ is a Hausdorff topology. By 1.19 $\mathscr{T}_{\mathscr{S}}$ coincides with $\mathscr{T}_{\mathscr{F}}$ on $\mathfrak{M}^c$ and $\mathfrak{M}^c$ is also the $\mathscr{T}_{\mathscr{S}}$-closure of $\mathfrak{M}|\mathscr{F}$ in $\mathrm{ca}(X, \mathscr{F})$. By 2.1 $\mu_n \in \mathfrak{M}$, $n \in \mathbb{N}$, and $(\mu_n)_{n \in \mathbb{N}} \to \mu_0(\mathscr{T}_{\mathscr{S}})$ (with $\mu_0(A) := \limsup \mu_n(A)$, $A \in \mathscr{F}$) implies $\mu_0 \in \mathfrak{M}^c$ and therefore $(\mu_n)_{n \in \mathbb{N}} \to \mu_0(\mathscr{T}_{\mathscr{F}})$.

2.4. Corollary. *Let $\mathfrak{M}|\mathscr{F}$ be c.c. in $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$ and let $\mathscr{G} \subset \mathscr{F}$ be a field generating the σ-field $\mathscr{F}$. Then convergence of a sequence $\mu_n \in \mathfrak{M}$, $n \in \mathbb{N}$, on $\mathscr{G}$ implies its convergence on $\mathscr{F}$.*

Proof. Follows from 2.3 and [5], III.5.9.

2.5. *Example.* Convergence on a field does not imply convergence on the generated σ-field, in general: Let $X := [0, 1]$, $\mathscr{F}$ the Borel sets in $[0, 1]$. For each $n \in \mathbb{N}$ let μ_n be the probability measure concentrated in the point $1/n$ and μ_0 be the probability measure concentrated in 0. Let $\mathscr{F}_0$ be the field generated by the intervals $[a, b)$ in $[0, 1]$. Then $\mathscr{F}_0$ generates $\mathscr{F}$ and $(\mu_n)_{n \in \mathbb{N}} \to \mu_0(\mathscr{T}_{\mathscr{F}_0})$ but $(\mu_n(A))_{n \in \mathbb{N}}$ is not convergent for $A := \{2^{-k}: k \in \mathbb{N}\}$, i.e. $(\mu_n)_{n \in \mathbb{N}} \to \mu_0(\mathscr{T}_{\mathscr{F}})$ is false.

2.6. Theorem (cf. [5], IV.8.9, IV.9.1, 9.2 and [13], p. 119). *Let* $\mathfrak{M}|\mathscr{F} \subset \mathrm{ca}(X, \mathscr{F})$ *be bounded. Then the following assertions are equivalent:*

(i) $\mathfrak{M}|\mathscr{F}$ *is c.c. in* $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$.

(ii) $\mathfrak{M}|\mathscr{F}$ *is c.s.c in* $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$.

(iii) $\mathfrak{M}|\mathscr{F}$ *is uniformly dominated by some non-negative finite measure* $\lambda \in \mathfrak{M}^{\Sigma}$ *and every non-negative measure* $\nu|\mathscr{F}$ *dominating* $\mathfrak{M}|\mathscr{F}$ *dominates* $\mathfrak{M}|\mathscr{F}$ *uniformly.*

(iv) $\mathfrak{M}|\mathscr{F}$ *is equicontinuous.*

(v) *Every countable subset of* $\mathfrak{M}|\mathscr{F}$ *is equicontinuous.*

Proof. (i)$\rhd$(ii) Let $(\mu_n|\mathscr{F})_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{M}|\mathscr{F}$; then according to 1.23 there exists a countably generated sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$ which is sufficient for $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$. Let $\mathscr{G}_0$ be a countable field generating $\mathscr{F}_0$. As for every $A\in\mathscr{G}_0$ $\{\mu_n(A): n\in\mathbb{N}\}$ is bounded in $\mathbb{R}$, using the diagonal procedure, we obtain an infinite subset $\mathbb{N}_0\subset\mathbb{N}$ such that $(\mu_n)_{n\in\mathbb{N}_0}$ converges on $\mathscr{G}_0$. As by 2.2 $\mathfrak{M}|\mathscr{F}_0$ is c.c. in $(\mathrm{ca}(X, \mathscr{F}_0), \mathscr{T}_{\mathscr{F}_0})$ we obtain, according to 2.4, that $\lim_{n\in\mathbb{N}_0}\mu_n(A)$ exists in $\mathbb{R}$ for all $A\in\mathscr{F}_0$. Since $\mathscr{F}_0$ is sufficient for $\{\mu_n|\mathscr{F}: n\in\mathbb{N}_0\}$, 1.24 implies convergence of $(\mu_n)_{n\in\mathbb{N}_0}$ on $\mathscr{F}$, whence by 1.10 $\mu_0(A) := \lim_{n\in\mathbb{N}_0}\mu_n(A)$, $A\in\mathscr{F}$, defines a set function which belongs to $\mathrm{ca}(X, \mathscr{F})$.

(ii)$\rhd$(i) By 2.1 it suffices to show that each $\mathscr{T}_{\mathscr{F}}$-accumulation point of $\mathfrak{M}$ in $[-b_{\mathfrak{M}}, b_{\mathfrak{M}}]^{\mathscr{F}}$, say μ_0, belongs to $\mathrm{ca}(X, \mathscr{F})$. It is easy to verify that μ_0 is a content, hence it remains to show that μ_0 is countably additive on $\mathscr{F}$: Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathscr{F}$ and let $A_0 := \sum_{k\in\mathbb{N}} A_k$. As μ_0 is a $\mathscr{T}_{\mathscr{F}}$-accumulation point of $\mathfrak{M}$, for every $n\in\mathbb{N}$ there exists $\mu_n\in\mathfrak{M}$ such that $|\mu_0(A_k)-\mu_n(A_k)| \leqq 1/n$ for $k=0,1,\ldots,n$. We have $\lim_{n\in\mathbb{N}}\mu_n(A_k)=\mu_0(A_k)$ for all $k=0,1,\ldots$. By (ii) there exists an infinite subset $\mathbb{N}_0\subset\mathbb{N}$ and a $\tilde{\mu}_0\in\mathrm{ca}(X, \mathscr{F})$ such that $(\mu_n)_{n\in\mathbb{N}_0}\to\tilde{\mu}_0(\mathscr{T}_{\mathscr{F}})$. Hence $\mu_0(A_k)=\tilde{\mu}_0(A_k)$ for $k=0,1,\ldots$. As $\tilde{\mu}_0$ is countably additive, we obtain $\mu_0(A_0)=\tilde{\mu}_0(A_0)=\sum_{k\in\mathbb{N}}\tilde{\mu}_0(A_k)=\sum_{k\in\mathbb{N}}\mu_0(A_k)$.

(ii)$\rhd$(iii) (a) First we shall show that every c.s.c. subset in $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$ is dominated by a non-negative finite measure $\lambda\in\mathfrak{M}^{\Sigma}$. This follows from the following auxiliary proposition ([5], p. 307).

(P) For each $\varepsilon>0$ there exists $n\in\mathbb{N}$, an n-tuple $\mu_1, \mu_2, \ldots, \mu_n\in\mathfrak{M}$ and a $\delta>0$ such that $|\mu_i|(A)<\delta$, $i=1,\ldots,n$ implies $|\mu(A)|<\varepsilon$ for every $\mu\in\mathfrak{M}$.

According to (P) for every $n\in\mathbb{N}$ there exists a finite subset $\mathfrak{M}_n$ of $\mathfrak{M}$ and $\delta_n>0$ such that $|\mu|(A)<\delta_n$ for all $\mu\in\mathfrak{M}_n$ implies $|\mu(A)|<1/n$ for all $\mu\in\mathfrak{M}$. As $\mathfrak{M}_0 := \bigcup_{n\in\mathbb{N}}\mathfrak{M}_n$ is countable, there exists by 1.4 a non-negative finite measure $\lambda\in\mathfrak{M}_0^{\Sigma}$ which is equivalent to $|\mathfrak{M}_0|$. Hence $\lambda|\mathscr{F}$ dominates $\mathfrak{M}|\mathscr{F}$.

(b) We conclude the proof of (iii) by showing that a c.s.c. subset $\mathfrak{M}|\mathscr{F}$ in $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}}))$ is uniformly dominated by every non-negative dominating measure.

If this were not true, there would exist a non-negative dominating measure $\nu_0|\mathscr{F}$ for $\mathfrak{M}|\mathscr{F}$ and an $\varepsilon_0>0$ such that:

(+) For every $n\in\mathbb{N}$ there exists $A_n\in\mathscr{F}$ and $\mu_n\in\mathfrak{M}$ with $\nu_0(A_n)<1/n$ and $|\mu_n(A_n)|\geqq\varepsilon_0$.

By (ii) there exists an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that $(\mu_n)_{n\in\mathbb{N}_0}$ converges on $\mathscr{F}$ to some $\mu_0 \in \mathrm{ca}(X, \mathscr{F})$. According to 1.9 this implies that $\{\mu_n|\mathscr{F}: n\in\mathbb{N}_0\}$ is uniformly dominated by every non-negative dominating measure; this, however, contradicts $(+)$.

(iii)$\rhd$(iv) Follows by 1.8.

(iv)$\rhd$(v) Obvious.

(v)$\rhd$(ii) Let $(\mu_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathfrak{M}|\mathscr{F}$. According to 1.23 there exists a countably generated sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$ which is sufficient for $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$. Let $\mathscr{G}_0$ be a countable field generating $\mathscr{F}_0$. As for every $A\in\mathscr{G}_0$ the set $\{\mu_n(A): n\in\mathbb{N}\}$ is bounded in $\mathbb{R}$, using the diagonal procedure, we obtain an infinite subset $\mathbb{N}_0\subset\mathbb{N}$ such that $(\mu_n)_{n\in\mathbb{N}_0}$ converges on $\mathscr{G}_0$. As $\{\mu_n|\mathscr{F}: n\in\mathbb{N}_0\}$ is equicontinuous, $(\mu_n)_{n\in\mathbb{N}_0}$ converges on $\mathscr{F}_0$, by 1.11, and therefore on $\mathscr{F}$ by 1.24. Hence the set function $\mu_0|\mathscr{F}$ defined by $\mu_0(A) := \lim_{n\in\mathbb{N}_0} \mu_n(A)$, $A\in\mathscr{F}$, belongs to $\mathrm{ca}(X, \mathscr{F})$.

2.7. Corollary (Dubrovski [4], p. 738, Theorem 2). *Let* $\mathfrak{M}|\mathscr{F}\subset\mathrm{ca}(X,\mathscr{F})$ *be arbitrary. Then equicontinuity of* $\mathfrak{M}|\mathscr{F}$ *implies that there exists* $\lambda\in\mathrm{ca}_+(X,\mathscr{F})$ *such that* $\mathfrak{M}|\mathscr{F}\lll\lambda|\mathscr{F}$ *and every* $\nu\in\mathrm{ca}_+(X,\mathscr{F})$ *dominating* $\mathfrak{M}|\mathscr{F}$ *dominates* $\mathfrak{M}|\mathscr{F}$ *uniformly.*

Proof. (a) We shall show first that every equicontinuous $\mathfrak{M}|\mathscr{F}\subset\mathrm{ca}(X,\mathscr{F})$ is dominated by some $\lambda\in\mathrm{ca}_+(X,\mathscr{F})$. For every $k\in\mathbb{N}\cup\{0\}$ let $\mathfrak{M}_k|\mathscr{F} := \{\mu\in\mathfrak{M}: k\leqq\|\mu\|<k+1\}$. As $\|\mu\|<\infty$ for every $\mu\in\mathrm{ca}(X,\mathscr{F})$, it follows that $\mathfrak{M}|\mathscr{F} = \bigcup_{k=0}^{\infty}\mathfrak{M}_k|\mathscr{F}$ and the equicontinuity of $\mathfrak{M}|\mathscr{F}$ implies that each $\mathfrak{M}_k|\mathscr{F}$ is equicontinuous. As $\mathfrak{M}_k|\mathscr{F}$ is bounded, it follows from 2.6 that there exists $\lambda_k\in\mathrm{ca}_+(X,\mathscr{F})$ with $\lambda_k(X)\leqq 1$ such that $\mathfrak{M}_k|\mathscr{F}\lll\lambda_k|\mathscr{F}$, $k\in\mathbb{N}\cup\{0\}$. If we define $\lambda := \sum_{k=0}^{\infty} 2^{-k}\lambda_k$, then $\lambda\in\mathrm{ca}_+(X,\mathscr{F})$ and $\mathfrak{M}|\mathscr{F}\ll\lambda|\mathscr{F}$: For, given any $\mu\in\mathfrak{M}$, there exists $k=k(\mu)\in\mathbb{N}\cup\{0\}$ such that $\mu\in\mathfrak{M}_k$. As $\lambda(A)=0$ implies $\lambda_k(A)=0$, we obtain $\mu(A)=0$.

(b) Next we claim that, given $\mu_n\in\mathrm{ca}(X,\mathscr{F})$, $n\in\mathbb{N}$, with $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}\ll\lambda|\mathscr{F}$, equicontinuity of $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$ implies that $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}\lll\lambda|\mathscr{F}$. The proof of this claim can be found in a recent paper of Brooks ([2], p. 468, Theorem 3).

(c) We conclude the proof by showing that (a) and (b) imply the assertion. It follows from (a) that there exists $\lambda\in\mathrm{ca}_+(X,\mathscr{F})$ such that $\mathfrak{M}|\mathscr{F}\ll\lambda|\mathscr{F}$. We show that every $\nu_0\in\mathrm{ca}_+(X,\mathscr{F})$ dominating $\mathfrak{M}|\mathscr{F}$ dominates $\mathfrak{M}|\mathscr{F}$ uniformly. Assume that $\mathfrak{M}|\mathscr{F}\lll\lambda|\mathscr{F}$ does not hold. Then there exists $\varepsilon_0>0$ such that $(+)$ (see above p. 130) holds. Since $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}\ll\nu_0|\mathscr{F}$ and $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$ is equicontinuous, it follows from (b) that $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}\lll\nu_0|\mathscr{F}$ which contradicts $(+)$.

2.8. Corollary. *If* $\mathfrak{M}|\mathscr{F}\lll\lambda_0|\mathscr{F}$ *and if* $\mathfrak{M}|\mathscr{F}\ll\lambda|\mathscr{F}$, *then* $\mathfrak{M}|\mathscr{F}\lll\lambda|\mathscr{F}$.

Proof. $\mathfrak{M}|\mathscr{F}\lll\lambda_0|\mathscr{F}$ implies that $\mathfrak{M}|\mathscr{F}$ is equicontinuous, whence the assertion follows from 2.7.

2.9. Corollary. *Let* $\mathfrak{M}|\mathscr{F}\subset\mathrm{ca}(X,\mathscr{F})$ *be bounded. Then* $\mathfrak{M}|\mathscr{F}$ *is c.s.c. in* $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$ *iff the auxiliary proposition* (P) (see p. 130) *is fulfilled.*

Proof. c.s.c. of $\mathfrak{M}|\mathscr{F}$ implies (P) (see [5], p. 307). On the other hand (P) implies that $\mathfrak{M}|\mathscr{F}$ is equicontinuous, whencce the assertion follows from 2.6.

Finally we obtain the following compactness criterion:

2.10. Proposition. *Let $\mathfrak{M}|\mathscr{F}\subset \mathrm{ca}(X,\mathscr{F})$ be bounded. Then $\mathfrak{M}|\mathscr{F}$ is c.s.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$ iff for every countable subfield $\mathscr{G}$ of $\mathscr{F}$ convergence of a sequence in $\mathfrak{M}$ on $\mathscr{G}$ implies its convergence on the σ-field $\sigma(\mathscr{G})$ generated by $\mathscr{G}$.*

Proof. If $\mathfrak{M}|\mathscr{F}$ is c.s.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$, 2.2 implies that $\mathfrak{M}|\mathscr{F}_0$ is c.s.c. in $(\mathrm{ca}(X,\mathscr{F}_0),\mathscr{T}_{\mathscr{F}_0})$ for every sub-σ-field $\mathscr{F}_0$ of $\mathscr{F}$, hence, by 2.6, $\mathfrak{M}|\sigma(\mathscr{G})$ is c.c. in $(\mathrm{ca}(X,\sigma(\mathscr{G})),\mathscr{T}_{\sigma(\mathscr{G})})$, whence 2.4 (with $\mathscr{F}=\sigma(\mathscr{G})$) yields the assertion.

On the other hand, to any sequence $\mu_n\in\mathfrak{M}$, $n\in\mathbb{N}$, there exists, by 1.23, a countably generated σ-field $\mathscr{F}_0$ of $\mathscr{F}$ which is sufficient for $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$. Let $\mathscr{G}$ be a countable field generating $\mathscr{F}_0$. As for every $A\in\mathscr{G}$ the set $\{\mu_n(A): n\in\mathbb{N}\}$ is bounded in $\mathbb{R}$, using the diagonal procedure, we obtain an infinite subset $\mathbb{N}_0\subset\mathbb{N}$ such that $(\mu_n)_{n\in\mathbb{N}_0}$ converges on $\mathscr{G}$. By assumption this implies convergence on $\mathscr{F}_0$ and as $\mathscr{F}_0$ is sufficient for $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$, 1.24 implies convergence of $(\mu_n)_{n\in\mathbb{N}_0}$ on $\mathscr{F}$, whence, by 1.10, it follows that $\mathfrak{M}|\mathscr{F}$ is c.s.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$.

In the following we will briefly investigate the connection of compactness properties with respect to the topologies $\mathscr{T}_{\mathscr{F}}$, $\mathscr{T}_{\mathfrak{B}}$ and the topology $\mathscr{T}_0:=\sigma(\mathrm{ba}(X,\mathscr{F}),\mathrm{ba}(X,\mathscr{F})^*)$ [resp. $\sigma(\mathrm{ca}(X,\mathscr{F}),\mathrm{ca}(X,\mathscr{F})^*)$] usually used in analysis; i.e. $\mathscr{T}_0$ is defined as the coarsest topology on $\mathrm{ba}(X,\mathscr{F})$ [$\mathrm{ca}(X,\mathscr{F})$] rendering all continuous linear functionals on the Banach space $\mathrm{ba}(X,\mathscr{F})$ [$\mathrm{ca}(X,\mathscr{F})$] endowed with $\|\cdot\|$ continuous.

2.11. *Remark.* As $\mathrm{ba}(X,\mathscr{F})$ is isometrically isomorphic to the dual space $\mathfrak{B}(X,\mathscr{F})^*$ of $\mathfrak{B}(X,\mathscr{F})$ (see [5], IV. 5.1), the topology $\mathscr{T}_{\mathfrak{B}}=\mathscr{T}_{\mathfrak{B}(X,\mathscr{F})}$ (cf. 1.21) is the so-called weak*-topology (in the terminology of [5] (after identifying $\mathrm{ba}(X,\mathscr{F})$ with $\mathfrak{B}(X,\mathscr{F})^*$) $\mathscr{T}_{\mathfrak{B}}$ on $\mathrm{ba}(X,\mathscr{F})$ is the $\mathfrak{B}(X,\mathscr{F})$-topology). Hence it follows from [5], V.4.3 that every $\mathscr{T}_{\mathfrak{B}}$-compact subset of $\mathrm{ba}(X,\mathscr{F})$ is bounded in $(\mathrm{ba}(X,\mathscr{F}),\|\cdot\|)$, whence any $\mathscr{T}_{\mathfrak{B}}$-compact [c.c.] $\mathfrak{M}\subset\mathrm{ca}(X,\mathscr{F})$ is bounded, too. It follows easily from 1.12 that any $\mathscr{T}_{\mathscr{F}}$-compact [c.c.] resp. any $\mathscr{T}_{\mathscr{F}}$-s.c. [c.s.c.] $\mathfrak{M}\subset\mathrm{ca}(X,\mathscr{F})$ is also bounded. Furthermore, we remark that $\mathscr{T}_{\mathfrak{B}}$ is coarser than $\mathscr{T}_0$ ($\mathscr{T}_{\mathfrak{B}}\subset\mathscr{T}_0$) and that this relation is strict in general (see [1], 2.2, p. 42).

2.12. Lemma ([1], Lemma 2.4, p. 43). *For every $\nu\in\mathrm{ca}_+(X,\mathscr{F})$, $\mathrm{ca}(X,\mathscr{F};\nu):=\{\mu\in\mathrm{ca}(X,\mathscr{F}): \mu|\mathscr{F}\ll\nu|\mathscr{F}\}$ is a $\mathscr{T}_{\mathfrak{B}}$-closed linear subspace of $\mathrm{ca}(X,\mathscr{F})$ on which $\mathscr{T}_{\mathfrak{B}}$ coincides with $\mathscr{T}_0$.*

2.13. Proposition. *A subset $\mathfrak{M}|\mathscr{F}$ of $\mathrm{ca}(X,\mathscr{F})$ is $\mathscr{T}_0$-compact iff it is $\mathscr{T}_{\mathfrak{B}}$-compact or $\mathscr{T}_{\mathscr{F}}$-compact.*

Proof. If $\mathfrak{M}|\mathscr{F}$ is $\mathscr{T}_0$-compact, $\mathscr{T}_{\mathfrak{B}}\subset\mathscr{T}_0$ implies $\mathscr{T}_{\mathfrak{B}}$-compactness. On the other hand, if $\mathfrak{M}|\mathscr{F}$ is $\mathscr{T}_{\mathfrak{B}}$-compact it is bounded by 2.11, hence 2.6 implies that $\mathfrak{M}|\mathscr{F}$ is dominated by a non-negative finite measure; therefore, by 2.12, $\mathscr{T}_0$ and $\mathscr{T}_{\mathfrak{B}}$ coincide on $\mathfrak{M}|\mathscr{F}$. Hence $\mathfrak{M}|\mathscr{F}$ is $\mathscr{T}_0$-compact. In the same way one proves the equivalence of $\mathscr{T}_0$-compactness with $\mathscr{T}_{\mathscr{F}}$-compactness (cf. 1.21).

2.14. Corollary. *A subset $\mathfrak{M}|\mathscr{F}$ of $\mathrm{ca}(X,\mathscr{F})$ is c.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_0)$ iff it is c.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathfrak{B}})$ or c.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$ and subsets of this kind are necessarily bounded.*

Proof. We observe first that each of the conditions on $\mathfrak{M}|\mathscr{F}$ implies that $\mathfrak{M}|\mathscr{F}$ is bounded. This follows from [5], II.3.20 and 2.11.

$\rhd$ Follows from 1.19 applied with $X=\mathrm{ca}(X,\mathscr{F})$, $\mathscr{T}_1=\mathscr{T}_{\mathfrak{B}}$ or $\mathscr{T}_{\mathscr{F}}$, $\mathscr{T}_2=\mathscr{T}_0$ and $X_0=\mathfrak{M}|\mathscr{F}$.

$\lhd$ $\mathscr{T}_{\mathfrak{B}}\subset\mathscr{T}_0$ (see 2.11) implies $\mathfrak{M}^{co}\subset\mathfrak{M}^{c}$ (where $\mathfrak{M}^{co}$ [$\mathfrak{M}^{c}$] denotes the $\mathscr{T}_0-[\mathscr{T}_{\mathfrak{B}}-]$ closure of $\mathfrak{M}|\mathscr{F}$ in $\mathrm{ca}(X,\mathscr{F})$. If $\mathfrak{M}^{c}$ is $\mathscr{T}_{\mathfrak{B}}$-compact, it is $\mathscr{T}_0$-compact by 2.13. As $\mathfrak{M}^{co}$ is a $\mathscr{T}_0$-closed subset of a $\mathscr{T}_0$-compact set, it is $\mathscr{T}_0$-compact itself. In the same way one obtains the equivalence of $\mathscr{T}_0$-c.c. to $\mathscr{T}_{\mathscr{F}}$-c.c.

2.15. Proposition. *For any sequence* $\mu_n\in\mathrm{ca}(X,\mathscr{F})$, $n\in\mathbb{N}$, *$\mathscr{T}_0$-convergence is equivalent to $\mathscr{T}_{\mathfrak{B}}$-convergence and the latter is, in turn, equivalent to $\mathscr{T}_{\mathscr{F}}$-convergence* (cf. [5], IV.9.5.).

Proof. $\rhd$ Follows from the fact that $\mathscr{T}_{\mathfrak{B}}$ [$\mathscr{T}_{\mathscr{F}}$] is coarser than $\mathscr{T}_0$.

$\lhd$ Let $(\mu_n)_{n\in\mathbb{N}}\to\mu_0(\mathscr{T}_{\mathfrak{B}})$. As $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\cup\{0\}\}$ is dominated (see 1.4), $\mathscr{T}_0$ and $\mathscr{T}_{\mathfrak{B}}$ coincide on $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\cup\{0\}\}$, by 2.12, whence $(\mu_n)_{n\in\mathbb{N}}\to\mu_0(\mathscr{T}_0)$. If $(\mu_n)_{n\in\mathbb{N}}\to\mu_0(\mathscr{T}_{\mathscr{F}})$, it follows by 2.11 that $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$ is bounded, whence $(\mu_n)_{n\in\mathbb{N}}\to\mu_0(\mathscr{T}_{\mathfrak{B}})$ (see 1.21).

2.16. Corollary. *A subset* $\mathfrak{M}|\mathscr{F}$ *of* $\mathrm{ca}(X,\mathscr{F})$ *is s.c. [c.s.c.] in* $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_0)$ *iff it is s.c. [c.s.c.] in* $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathfrak{B}})$ *or s.c. [c.s.c.] in* $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$ *and subsets of this kind are necessarily bounded.*

Proof. Follows from [5], II.3.27 and 2.15.

2.17. Corollary ([1], Theorem 3.6, p. 43; cf. also [5], V.6.1). *For subsets* $\mathfrak{M}|\mathscr{F}$ *of* $\mathrm{ca}(X,\mathscr{F})$ *the following assertions are equivalent:*

(i) $\mathfrak{M}|\mathscr{F}$ *is c.c. in* $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathfrak{B}})$.

(ii) $\mathfrak{M}|\mathscr{F}$ *is c.s.c. in* $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathfrak{B}})$.

(iii) $\mathfrak{M}|\mathscr{F}$ *is c.s.c. in* $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_0)$.

(iv) $\mathfrak{M}|\mathscr{F}$ *is c.c. in* $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_0)$.

Proof. Follows immediately from 1.19, 2.6, 2.14 and 2.16.

Finally we obtain the following criterion:

2.18. Proposition. *A subset* $\mathfrak{M}|\mathscr{F}\subset\mathrm{ca}(X,\mathscr{F})$ *is $\mathscr{T}_{\mathscr{F}}$-compact iff it is $\mathscr{T}_{\mathscr{F}}$-sequentially compact.*

Proof. $\rhd$ If $\mathfrak{M}|\mathscr{F}$ is $\mathscr{T}_{\mathscr{F}}$-compact, it is c.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$ and as $\mathfrak{M}|\mathscr{F}$ is bounded, it follows from 2.6 that $\mathfrak{M}|\mathscr{F}$ is c.s.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$. Hence any sequence in $\mathfrak{M}$ contains a subsequence $\mathscr{T}_{\mathscr{F}}$-converging to some $\mu_0\in\mathrm{ca}(X,\mathscr{F})$. As $\mathfrak{M}|\mathscr{F}$ is $\mathscr{T}_{\mathscr{F}}$-closed, $\mu_0\in\mathfrak{M}$.

$\lhd$ If $\mathfrak{M}|\mathscr{F}$ is s.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$, it follows from 2.16 that $\mathfrak{M}|\mathscr{F}$ is s.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_0)$, hence $\mathscr{T}_0$-sequentially closed. Furthermore, it follows from 2.6 and 2.14 that $\mathfrak{M}|\mathscr{F}$ is c.c. in $(\mathrm{ca}(X,\mathscr{F}),\mathscr{T}_0)$. As in every metrizable locally convex space the class of all weakly conditionally compact and weakly sequentially closed subsets coincide with the class of all weakly compact subsets (see [10], §24, 1. (7)), it follows that $\mathfrak{M}|\mathscr{F}$ is $\mathscr{T}_0$-compact and hence by 2.13 $\mathscr{T}_{\mathscr{F}}$-compact.

2.19. Corollary. *A subset* $\mathfrak{M}|\mathscr{F}\subset\mathrm{ca}(X,\mathscr{F})$ *is $\mathscr{T}_0$-compact iff it is $\mathscr{T}_0$-sequentially compact.*

Proof. Follows from 2.13, 2.18 and 2.16.

3. Compactness and Sequential Compactness in Spaces of Measures Defined on the Borel Sets of a Topological Space $(X, \mathscr{T})$. Grothendieck's Theorem

In this section we shall give further compactness criterions under the additional assumption that the basic space X is endowed with a Hausdorff topology, say $\mathscr{T}$, and $\mathscr{F}$ is the σ-field of Borel sets in X. Let $\mathscr{K}$ denote the system of compact subsets of $(X, \mathscr{T})$. If not stated explicitely, all spaces $(X, \mathscr{T})$ in the following are supposed to be Hausdorff spaces.

3.1. Definition. (a) $\mu \in \mathrm{ca}(X, \mathscr{F})$ is called $\mathscr{K}$-regular iff for every $U \in \mathscr{T}$ and every $\varepsilon > 0$ there exists $K \in \mathscr{K}$, $K \subset U$, such that $|\mu(A)| < \varepsilon$ for all $A \in \mathscr{F}$ with $A \subset U - K$. The family of all [non-negative] $\mathscr{K}$-regular $\mu \in \mathrm{ca}(X, \mathscr{F})$ will be denoted by $\mathscr{K}\text{-rca}(X, \mathscr{F})$ [$\mathscr{K}\text{-rca}_+(X, \mathscr{F})$].

(b) Let $\nu \in \mathrm{ca}_+(X, \mathscr{F})$. We say that $\mathscr{K}$ ν-approximates $\mathscr{T}$ iff for every $U \in \mathscr{T}$ and every $\varepsilon > 0$ there exists $K \in \mathscr{K}$, $K \subset U$, such that $\nu(U - K) < \varepsilon$.

3.2. Criterion. $\mu \in \mathscr{K}\text{-rca}(X, \mathscr{F})$ *iff* $\mathscr{K}$ $|\mu|$*-approximates* $\mathscr{T}$.

Proof. Follows immediately from the fact that for every $\mu \in \mathrm{ca}(X, \mathscr{F})$

(*) $\sup_{A \in \mathscr{F}, A \subset U - K} |\mu(A)| \leqq |\mu|(U - K) \leqq 2 \cdot \sup_{A \in \mathscr{F}, A \subset U - K} |\mu(A)|$ (cf. [5], III. 1.5).

3.3. Lemma ([14b]). *Let* $\nu|\mathscr{F} \in \mathrm{ca}_+(X, \mathscr{F})$ *and assume that* $\mathscr{K}$ ν*-approximates* $\mathscr{T}$. *Then* $\mathscr{K}$ ν*-approximates also* $\mathscr{F}$, *i.e.: For every Borel set* A *and every* $\varepsilon > 0$ *there exists* $K \in \mathscr{K}$, $K \subset A$, *such that* $\nu(A - K) < \varepsilon$.

3.4. Corollary. $\mu \in \mathscr{K}\text{-rca}(X, \mathscr{F})$ *iff* $\mathscr{K}$ $|\mu|$*-approximates* $\mathscr{F}$.

Proof. Follows from 3.2 and 3.3.

3.5. Proposition. *Every* $\mu \in \mathscr{K}\text{-rca}(X, \mathscr{F})$ *is outer regular, i.e.: For every* $A \in \mathscr{F}$ *and every* $\varepsilon > 0$ *there exists* $U \in \mathscr{T}$, $A \subset U$, *such that* $|\mu(B)| < \varepsilon$ *for all* $B \in \mathscr{F}$ *with* $B \subset U - A$.

Proof. Follows immediately from 3.4.

3.6. Proposition. $\mathfrak{M}|\mathscr{F} \subset \mathscr{K}\text{-rca}(X, \mathscr{F})$ *implies that* $\mathscr{K}$ λ*-approximates* $\mathscr{T}$ *for every* $\lambda \in \mathfrak{M}^{\Sigma}$.

Proof. Let $\lambda \in \mathfrak{M}^{\Sigma}$, i.e.

$$\lambda = \sum_{n \in \mathbb{N}} 2^{-n} \frac{|\mu_n|}{1 + \|\mu_n\|}$$

with $\mu_n \in \mathfrak{M}$, $n \in \mathbb{N}$. As $\mu_n \in \mathscr{K}\text{-rca}(X, \mathscr{F})$, it follows from 3.2 that for every $U \in \mathscr{T}$ there exists an increasing sequence of compact sets whose union, say K_σ, is contained in U with $|\mu_n|(U - K_\sigma) = 0$ for all $n \in \mathbb{N}$. Therefore $\lambda(U - K_\sigma) = 0$ and this implies the assertion.

The next theorems are concerned with compactness results first studied by Grothendieck ([6], Théorème 2, (1)–(4), p. 146) in the case of Radon measures on locally compact basic spaces $(X, \mathscr{T})$. For $\mathscr{K}$-regular measures 3.7 generalizes in particular Grothendieck's criterion (4) to arbitrary Hausdorff spaces and 3.11 shows that the equivalence of the statements (1), (3) and (4) in [6], p. 146/147, holds true for regular Hausdorff spaces. Finally 3.12 proves the equivalence of

(1) to (2) in [6], p. 146, for completely regular Hausdorff basic spaces $(X, \mathscr{T})$. Necessary and sufficient conditions for $\mathscr{T}_{\mathscr{F}}$-compactness of the type (iii), (v) and (viii) below were obtained independently and with different methods by Topsøe ([16a], Theorems 8 and 9).

3.7. Theorem. *Let $(X, \mathscr{T})$ be a Hausdorff space and let $\mathfrak{M}|\mathscr{F}$ be a bounded subset of $\mathscr{K}$-rca$(X, \mathscr{F})$. Then the following assertions are equivalent:*

(i) *$\mathfrak{M}|\mathscr{F}$ is c.c. in $(\text{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$.*

(ii) *$\mathfrak{M}|\mathscr{F}$ is c.s.c. in $(\text{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$.*

(iii) *$\mathscr{K}$ $|\mu|$-approximates $\mathscr{T}$ uniformly with respect to $\mu \in \mathfrak{M}$.*

(iv) *$\mathscr{K}$ $|\mu|$-approximates $\mathscr{F}$ uniformly with respect to $\mu \in \mathfrak{M}$.*

(v) (a) *For every $K \in \mathscr{K}$ and every $\varepsilon > 0$ there exists $U \in \mathscr{T}$ such that $U \supset K$ and $|\mu|(U - K) < \varepsilon$ for all $\mu \in \mathfrak{M}$.*
(b) *$\{|\mu||\mathscr{F}: \mu \in \mathfrak{M}\}$ is uniformly tight, i.e.: For every $\varepsilon > 0$ there exists $K \in \mathscr{K}$ such that $|\mu|(\bar{K}) < \varepsilon$ for all $\mu \in \mathfrak{M}$.*

(vi) (a) *For every sequence of pairwise disjoint sets $U_j \in \mathscr{T}$, $j \in \mathbb{N}$, we have $\lim_{j \in \mathbb{N}} \mu(U_j) = 0$ uniformly with respect to $\mu \in \mathfrak{M}$.*
(b) *For every sequence of pairwise disjoint sets $K_j \in \mathscr{K}$, $j \in \mathbb{N}$, we have $\lim_{j \in \mathbb{N}} \mu(K_j) = 0$ uniformly with respect to $\mu \in \mathfrak{M}$.*

Proof. (i) ⇔ (ii) see 2.6.

(ii) ⊳ (iv) If $\mathfrak{M}|\mathscr{F} \subset \mathscr{K}$-rca$(X, \mathscr{F})$ is c.s.c. in $(\text{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$, it follows from 2.6 that there exists $\lambda \in \mathfrak{M}^{\Sigma}$ which dominates $\mathfrak{M}|\mathscr{F}$ uniformly. By 1.6 $|\mathfrak{M}|$ is also uniformly dominated by $\lambda|\mathscr{F}$. 3.6 implies that $\mathscr{K}$ λ-approximates $\mathscr{T}$ and therefore also $\mathscr{F}$ (see 3.3), whence the assertion follows.

(iv) ⊳ (iii) Obvious, since $\mathscr{T} \subset \mathscr{F}$.

(iii) ⊳ (v) It is obvious that (iii) implies (v)(b). To see that (iii) implies (v)(a), let $K \in \mathscr{K}$ and $\varepsilon > 0$ be given. By assumption there exists for $U' := \bar{K} \in \mathscr{T}$ a $K' \in \mathscr{K}$ with $K' \subset U'$ and $|\mu|(U' - K') < \varepsilon$ for all $\mu \in \mathfrak{M}$. Therefore $U := \bar{K'} \supset \bar{U'} = K$ and $|\mu|(U - K) = |\mu|(U' - K') < \varepsilon$ for all $\mu \in \mathfrak{M}$.

(v) ⊳ (iii) Let $U \in \mathscr{T}$ and $\varepsilon > 0$ be given. According to (v)(b) there exists $K \in \mathscr{K}$ such that $|\mu|(\bar{K}) < \varepsilon/2$ for all $\mu \in \mathfrak{M}$. As $\bar{U} \cap K \in \mathscr{K}$, (v)(a) implies that there exists $U' \in \mathscr{T}$ with $U' \supset \bar{U} \cap K$ and $|\mu|(U' - (\bar{U} \cap K)) < \varepsilon/2$ for all $\mu \in \mathfrak{M}$. It follows that $K' := \bar{U'} \cap K \in \mathscr{K}$, $K' \subset U \cap K$ and $|\mu|((U \cap K) - K') < \varepsilon/2$, hence $K' \subset U$ and $|\mu|(U - K') < \varepsilon$ for all $\mu \in \mathfrak{M}$.

(iii) ⊳ (ii) By 2.6 and 1.8 it suffices to show that every countable subset of $\mathfrak{M}|\mathscr{F}$, say $\{\mu_n|\mathscr{F}: n \in \mathbb{N}\}$, is uniformly dominated by $\lambda := \sum_{n \in \mathbb{N}} 2^{-n}|\mu_n|$. As $\mathfrak{M}|\mathscr{F}$ is bounded, $\lambda|\mathscr{F}$ is finite and $\mathscr{K}$ λ-approximates $\mathscr{T}$ (cf. 3.6). Therefore it is sufficient (see 3.5) to show that $\{\mu_n|\mathscr{T}: n \in \mathbb{N}\}$ resp. $\{|\mu_n||\mathscr{T}: n \in \mathbb{N}\}$ (see 1.6) is uniformly dominated by $\lambda|\mathscr{T}$. If this were not true, then there would exist $\varepsilon_0 > 0$ such that for an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ there exists $U_n' \in \mathscr{T}$ such that $\lambda(U_n') < 2^{-n}$ and $|\mu_n|(U_n') \geqq \varepsilon_0$, $n \in \mathbb{N}_0$. For every $n \in \mathbb{N}_0$ let $U_n := \bigcup \{U_i': i \geqq n, i \in \mathbb{N}_0\}$. Then $U_n \in \mathscr{T}$, $n \in \mathbb{N}_0$, is non increasing with $\lim_{n \in \mathbb{N}_0} \lambda(U_n) = 0$ and $|\mu_n|(U_n) \geqq \varepsilon_0$ for all $n \in \mathbb{N}_0$. By assumption for every $n \in \mathbb{N}_0$ there exists $K_n' \in \mathscr{K}$ with $K_n' \subset U_n$ and $|\mu|(U_n - K_n') <$

$2^{-n-1}\varepsilon_0$ for all $\mu\in\mathfrak{M}$. For $n\in\mathbb{N}_0$ let $K_n := \bigcap\{K_i': 1\leq i\leq n, i\in\mathbb{N}_0\}$; then $K_n\in\mathcal{K}$, $K_n\subset K_n'\subset U_n$ and $|\mu_n|(K_n)=|\mu_n|(U_n)-|\mu_n|(U_n-K_n)$. We have

$$U_n-K_n=\bigcup\{U_n-K_i': 1\leq i\leq n, i\in\mathbb{N}_0\}\subset\bigcup\{U_i-K_i': 1\leq i\leq n, i\in\mathbb{N}_0\},$$

hence $|\mu_n|(U_n-K_n)\leq\sum_{\substack{1\leq i\leq n\\ i\in\mathbb{N}_0}}|\mu_n|(U_i-K_i')<\varepsilon_0/2$ and therefore $|\mu_n|(K_n)\geq\varepsilon_0/2$ for all $n\in\mathbb{N}_0$. Furthermore, $K := \bigcap_{n\in\mathbb{N}_0}K_n\in\mathcal{K}$ and $\lambda(K)=\lim_{n\in\mathbb{N}_0}\lambda(K_n)=0$ and therefore $|\mu_n|(K)=0$ for all $n\in\mathbb{N}$. As $\bar{K}\in\mathcal{F}$, there exists, by assumption, $K_0\in\mathcal{K}$ such that $K_0\subset\bar{K}$ and $|\mu_n|(\bar{K}-K_0)<\varepsilon_0/2$ for all $n\in\mathbb{N}$. Together with $|\mu_n|(K)=0$ for all $n\in\mathbb{N}$ this implies $|\mu_n|(\bar{K}_0)<\varepsilon_0/2$ for all $n\in\mathbb{N}$. As K_0 is compact, $K_0\subset\bar{K}=\bigcup_{n\in\mathbb{N}_0}\bar{K}_n$ implies that $K_0\subset\bar{K}_n$ for all sufficiently large $n\in\mathbb{N}_0$ and hence $|\mu_n|(\bar{K}_0)\geq|\mu_n|(K_n)\geq\varepsilon_0/2$ for all sufficiently large $n\in\mathbb{N}_0$ which contradicts $|\mu_n|(K_0)<\varepsilon_0/2$, $n\in\mathbb{N}$. This proves the equivalence of (i)–(v).

(v)$\rhd$(vi) We shall prove the following even stronger implication: (v) implies that for any sequence of pairwise disjoint sets $A_j\in\mathcal{F}$, $j\in\mathbb{N}$, we have $\lim_{j\in\mathbb{N}}\mu(A_j)=0$ uniformly with respect to $\mu\in\mathfrak{M}$. Suppose the latter to be wrong. Then there exists a sequence of pairwise disjoint sets $A_j\in\mathcal{F}$, $j\in\mathbb{N}$, so that there exists a $\varepsilon_0>0$, an infinite subset $\mathbb{N}_0\subset\mathbb{N}$ and measures $\mu_j\in\mathfrak{M}$, $j\in\mathbb{N}_0$, with $\inf_{j\in\mathbb{N}_0}|\mu_j(A_j)|>\varepsilon_0$. Without loss of generality we may assume $\mathbb{N}_0=\mathbb{N}$. As $\lambda := \sum_{j\in\mathbb{N}}2^{-j}|\mu_j|$ is a finite non-negative measure dominating $\{\mu_j|\mathcal{F}: j\in\mathbb{N}\}$, the equivalence of (v) to (ii) already obtained implies by 2.6 that $\{\mu_j|\mathcal{F}: j\in\mathbb{N}\}$ is uniformly dominated by $\lambda|\mathcal{F}$. Hence there exists $\delta(\varepsilon_0)>0$ such that $\lambda(A)<\delta(\varepsilon_0)$ implies $\sup_{j\in\mathbb{N}}|\mu_j(A)|<\varepsilon_0$. As $\lambda(\sum_{j\in\mathbb{N}}A_j)=\sum_{j\in\mathbb{N}}\lambda(A_j)<\infty$ implies $\lim_{j\in\mathbb{N}}\lambda(A_j)=0$, we obtain $|\mu_j(A_j)|<\varepsilon_0$ for all sufficiently large $j\in\mathbb{N}$ which contradicts $\inf_{j\in\mathbb{N}}|\mu_j(A_j)|>\varepsilon_0$.

(vi)(a)$\rhd$(v)(a) Assume (v)(a) to be wrong. Then there exists $K_0\in\mathcal{K}$ and $\varepsilon_0>0$ such that for every $U\in\mathcal{T}$ with $U\supset K_0$ there exists $\mu_U\in\mathfrak{M}$ with $|\mu_U|(U-K_0)>\varepsilon_0$. We shall construct inductively a decreasing sequence V_n of open neighborhoods of K_0, a sequence $U_n\in\mathcal{T}$ with $U_n\subset V_{n-1}\cap\bar{V}_n$, $n\in\mathbb{N}$, and a sequence $\mu_n\in\mathfrak{M}$ such that $\inf_{n\in\mathbb{N}}|\mu_n(U_n)|>\varepsilon_0/4$. (This obviously contradicts (vi)(a).) Assume, starting with $V_0 := X$, that this construction is already done up to $n-1$. Then, by assumption, there exists $\mu_n\in\mathfrak{M}$ such that $|\mu_n|(V_{n-1}-K_0)>\varepsilon_0$. As $\mu_n\in\mathcal{K}$-rca$(X,\mathcal{F})$, there exists $K_n\in\mathcal{K}$ with $K_n\subset V_{n-1}-K_0$ such that $|\mu_n(K_n)|>\varepsilon_0/2$ (cf. 3.2(*)). As K_0 and K_n are disjoint compact sets, there exists U_n', $V_n'\in\mathcal{T}$ such that $V_n'\supset K_0$, $U_n'\supset K_n$ and $V_n'\cap U_n'=\emptyset$, where, according to the outer regularity of μ_n (see 3.5), U_n' can be chosen so that $|\mu_n|(U_n'-K_n)<\varepsilon_0/4$. Let $U_n := U_n'\cap V_{n-1}$ and $V_n := V_n'\cap V_{n-1}$. Then $U_n\subset V_{n-1}\cap\bar{V}_n$, $K_0\subset V_n\subset V_{n-1}$ and $K_n\subset U_n\subset U_n'$, whence $|\mu_n|(U_n-K_n)<\varepsilon_0/4$. Therefore $|\mu_n(U_n)|\geq|\mu_n(K_n)|-|\mu_n|(U_n-K_n)>\varepsilon_0/4$.

(vi)(b)$\rhd$(v)(b) Assume (v)(b) to be wrong. Then there exists $\varepsilon_0>0$ such that for every $K\in\mathcal{K}$ there exists $\mu_K\in\mathfrak{M}$ with $|\mu_K|(\bar{K})>\varepsilon_0$. We shall construct inductively a sequence of disjoint sets $K_j\in\mathcal{K}$, $j\in\mathbb{N}$, and a sequence $\mu_j\in\mathfrak{M}$, $j\in\mathbb{N}$, such that $\sup_{j\in\mathbb{N}}|\mu_j(K_j)|>\varepsilon_0/2$. (This obviously contradicts (vi)(b).) Let $K_1' := \emptyset$. By assumption there exists $\mu_1 \in \mathfrak{M}$ with $|\mu_1|(\overline{K}_1') > \varepsilon_0$. As $\mu_1 \in \mathscr{K}$-rca$(X, \mathscr{F})$, there exists $K_1 \in \mathscr{K}$ such that $K_1 \subset \overline{K}_1'$ and $|\mu_1(K_1)| > \varepsilon_0/2$ (cf. 3,2(*)). This proves the inductive beginning. Now assume that K_i and μ_i with the property stated above are already obtained for $i=1, \dots, n-1$. As $K_n' := \sum_{i=1}^{n-1} K_i$ is compact, there exists $\mu_n \in \mathfrak{M}$ such that $|\mu_n|(\overline{K}_n') > \varepsilon_0$. Since $\mu_n \in \mathscr{K}$-rca$(X, \mathscr{F})$ there exists $K_n \in \mathscr{K}$ such that $K_n \subset \overline{K}_n'$ and $|\mu_n(K_n)| > \varepsilon_0/2$. This concludes the proof of 3.7.

3.8. Proposition. *Let $\mathfrak{M}|\mathscr{F} \subset \mathscr{K}$-rca$(X, \mathscr{F})$ be c.c. [c.s.c.] in $(\text{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$. Then every $\mathscr{T}_{\mathscr{F}}$-accumulation point of $\mathfrak{M}$ in ca$(X, \mathscr{F})$ belongs to $\mathscr{K}$-rca$(X, \mathscr{F})$. In particular: If $\mu_n \in \mathscr{K}$-rca$(X, \mathscr{F})$, $n \in \mathbb{N}$, and if $(\mu_n)_{n\in\mathbb{N}} \to \mu_0(\mathscr{T}_{\mathscr{F}})$, then $\mu_0 \in \mathscr{K}$-rca$(X, \mathscr{F})$.*

Proof. Let $\mu_0 \in$ ca$(X, \mathscr{F})$ be a $\mathscr{T}_{\mathscr{F}}$-accumulation point of $\mathfrak{M}|\mathscr{F}$ in ca$(X, \mathscr{F})$, i.e. $\mu_0 = \mathscr{T}_{\mathscr{F}}\text{-}\lim_{\beta \in B} \mu_\beta$ for some net $\{\mu_\beta\}_{\beta\in B} \subset \mathfrak{M}$. Since $\mathfrak{M}|\mathscr{F}$ is c.c. [c.s.c.] in $(\text{ca}(X,\mathscr{F}), \mathscr{T}_{\mathscr{F}})$, it is bounded (see 2.11) and 3.7(iv) implies that for every $A \in \mathscr{F}$ there exists an increasing sequence of compact sets whose union, say K_σ, is contained in A and so that $\sup_{\beta\in B} |\mu_\beta|(A-K_\sigma)=0$, i.e. $\sup_{\beta\in B} \mu_\beta(F)=0$ for all $F\in\mathscr{F}$ with $F \subset A-K_\sigma$. Hence $\mathscr{T}_{\mathscr{F}}$-convergence of $(\mu_\beta)_{\beta\in B}$ to μ_0 implies $\mu_0(F)=0$ for all $F\in\mathscr{F}$ with $F\subset A-K_\sigma$ which, in turn, implies $|\mu_0|(A-K_\sigma)=0$, i.e. $\mu_0 \in \mathscr{K}$-rca$(X, \mathscr{F})$.

3.9. Corollary. *$\mathfrak{M}|\mathscr{F} \subset \mathscr{K}$-rca$(X, \mathscr{F})$ is c.c. [c.s.c.] in $(\text{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$ iff $\mathfrak{M}|\mathscr{F}$ is c.c. [c.s.c.] in $(\mathscr{K}\text{-rca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$.*

Proof. The c.s.c. case is an immediate consequence of 3.8. To prove the equivalence in the c.c. case let $\mathfrak{M}^{c_0}$ and $\mathfrak{M}^{c_1}$ denote the $\mathscr{T}_{\mathscr{F}}$-closure of $\mathfrak{M}|\mathscr{F}$ in $\mathscr{K}$-rca$(X, \mathscr{F})$ and ca$(X, \mathscr{F})$ respectively. If $\mathfrak{M}|\mathscr{F}$ is c.c. in $(\text{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$, then it follows from 3.8 that $\mathfrak{M}^{c_0} = \mathfrak{M}^{c_1}$ is $\mathscr{T}_{\mathscr{F}}$-compact in $\mathscr{K}$-rca$(X, \mathscr{F})$. On the other hand, if $\mathfrak{M}|\mathscr{F}$ is c.c. in $(\mathscr{K}\text{-rca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$, it follows that $\mathfrak{M}^{c_0}$ is $\mathscr{T}_{\mathscr{F}}$-compact in ca$(X, \mathscr{F})$ and therefore, by 3.8, $\mathfrak{M}^{c_1} \subset (\mathfrak{M}^{c_0})^{c_1} = \mathfrak{M}^{c_0}$. This implies that $\mathfrak{M}^{c_1}$ is $\mathscr{T}_{\mathscr{F}}$-compact which was to be proved.

3.10. *Remark.* Note that 3.7 ((iii)$\Rightarrow$(iv)) extends 3.3 from one-point subsets to bounded subsets of $\mathscr{K}$-rca$(X, \mathscr{F})$.

3.11. Theorem. *Let $(X, \mathscr{T})$ be a regular Hausdorff space and let $\mathfrak{M}|\mathscr{F}$ be a bounded subset of $\mathscr{K}$-rca$(X, \mathscr{F})$. Then each of the assertions in 3.7 is equivalent to one of the following assertions:*

(vii) *For every uniformly bounded sequence $f_j \in \mathfrak{B}(X, \mathscr{F})$, $j \in \mathbb{N}$, converging on every $K \in \mathscr{K}$ and for every $\mu \in \mathfrak{M}$ in μ-measure to a function f, we have $\lim_{j\in\mathbb{N}} \mu(f_j) = \mu(f)$ uniformly with respect to $\mu \in \mathfrak{M}$.*

(viii) *For every sequence of pairwise disjoint sets $U_j \in \mathscr{T}$, $j \in \mathbb{N}$, we have $\lim_{j\in\mathbb{N}} \mu(U_j) = 0$ uniformly with respect to $\mu \in \mathfrak{M}$.*

(Note that (viii) is nothing else than (vi)(a), i.e. for regular Hausdorff spaces c.c. [c.s.c.] is already implied by (vi)(a).)

Proof. (vii)$\Rightarrow$(viii) Let $U_j \in \mathscr{T}$, $j \in \mathbb{N}$, be a sequence of pairwise disjoint sets. Then $f_j := 1_{U_j}$, $j \in \mathbb{N}$, is a uniformly bounded sequence of $\mathscr{F}$-measurable functions converging pointwise to the zero function, whence by (vii) (with $f=0$) $\lim_{j\in\mathbb{N}} \mu(f_j) = \lim_{j\in\mathbb{N}} \mu(U_j) = 0$ uniformly with respect to $\mu \in \mathfrak{M}$.

(viii)$\rhd$(v)(b) Assume (v)(b) to be wrong. Then there exists $\varepsilon_0 > 0$ such that for every $K \in \mathscr{K}$ there exists $\mu_K \in \mathfrak{M}$ so that $|\mu_K|(\bar{K}) > \varepsilon_0$. This, together with (v)(a) (which is implied by (viii)=(vi)(a) as we have seen in 3.7), implies that one can construct inductively sequences $K_n \in \mathscr{K}$ and $U_n \in \mathscr{T}$, $n \in \mathbb{N}$, and a sequence $\mu_n \in \mathfrak{M}$, $n \in \mathbb{N}$, such that $K_n \subset U_n \subset U_n^c$, $n \in \mathbb{N}$, $U_n^c \cap U_m^c = \emptyset$ if $n \neq m$, $\sup_{\mu \in \mathfrak{M}} |\mu|(U_n^c - K_n) < 2^{-n} \varepsilon_0/4$, $n \in \mathbb{N}$, and $\inf_{n \in \mathbb{N}} |\mu_n(U_n)| > \varepsilon_0/4$. (This obviously contradicts (viii).)

Let $K_1' := \emptyset$. By assumption there exists $\mu_1 \in \mathfrak{M}$ with $|\mu_1|(\overline{K_1'}) > \varepsilon_0$. Since $\mu_1 \in \mathscr{K}$-rca$(X, \mathscr{F})$, there exists $K_1 \in \mathscr{K}$ such that $K_1 \subset \overline{K_1'}$ and $|\mu_1(K_1)| > \varepsilon_0/2$ (cf. 3.2(*)). By (v)(a) there exists $U_1' \in \mathscr{T}$ such that $U_1' \supset K_1$ and $\sup_{\mu \in \mathfrak{M}} |\mu|(U_1' - K_1) < \varepsilon_0/8$. Since $(X, \mathscr{T})$ is a regular Hausdorff space there exists $U_1 \in \mathscr{T}$ such that $K_1 \subset U_1 \subset U_1^c \subset U_1'$ and therefore

$$\sup_{\mu \in \mathfrak{M}} |\mu|(U_1^c - K_1) < \varepsilon_0/8 \quad \text{and} \quad |\mu_1(U_1)| \geqq |\mu_1(K_1)| - |\mu_1|(U_1^c - K_1) > \varepsilon_0/4.$$

This proves the inductive beginning. Now, assume that K_i, U_i and μ_i with the properties stated above are already constructed for $i = 1, \ldots, n$. Let $F_{n+1} := \bigcup_{i=1}^{n} U_i^c$ and $K_{n+1}' := \bigcup_{i=1}^{n} K_i$. Then $F_{n+1} \in \bar{\mathscr{T}}$ and $K_{n+1}' \in \mathscr{K}$. By assumption there exists $\mu_{n+1} \in \mathfrak{M}$ such that $|\mu_{n+1}|(\overline{K_{n+1}'}) > \varepsilon_0$. Hence $|\mu_{n+1}|(\overline{F_{n+1}}) > \varepsilon_0 - \sum_{i=1}^{n} 2^{-i} \varepsilon_0/4$. Since $\mu_{n+1} \in \mathscr{K}$-rca$(X, \mathscr{F})$, it follows that there exists $K_{n+1} \subset \overline{F_{n+1}}$ such that

$$|\mu_{n+1}(K_{n+1})| > \tfrac{1}{2}\left(\varepsilon_0 - \sum_{i=1}^{n} 2^{-i} \varepsilon_0/4\right).$$

As $K_{n+1} \cap F_{n+1} = \emptyset$ and as $(X, \mathscr{T})$ is a regular Hausdorff space, we can conclude using (v) (a) that there exists $U_{n+1}' \in \mathscr{T}$ such that $U_{n+1}' \supset K_{n+1}$, $U_{n+1}' \cap F_{n+1} = \emptyset$ and $\sup_{\mu \in \mathfrak{M}} |\mu|(U_{n+1}' - K_{n+1}) < 2^{-(n+1)} \varepsilon_0/4$. Using again that $(X, \mathscr{T})$ is a regular Hausdorff space it follows that there exists $U_{n+1} \in \mathscr{T}$ such that $K_{n+1} \subset U_{n+1} \subset U_{n+1}^c \subset U_{n+1}'$. In addition, we have

$$U_{n+1}^c \cap U_m^c = \emptyset \quad \text{for} \quad m = 1, 2, \ldots, n, \qquad \sup_{\mu \in \mathfrak{M}} |\mu|(U_{n+1}^c - K_{n+1}) < 2^{-(n+1)} \varepsilon_0/4$$

and

$$\begin{aligned} |\mu_{n+1}(U_{n+1})| &> |\mu_{n+1}(K_{n+1})| - |\mu_{n+1}|(U_{n+1}^c - K_{n+1}) \\ &> \tfrac{1}{2}\left(\varepsilon_0 - \sum_{i=1}^{n} 2^{-i} \varepsilon_0/4\right) - 2^{-(n+1)} \varepsilon_0/4 > \varepsilon_0/4. \end{aligned}$$

Since (v) (a), as we have already remarked, is implied by (viii), we have thus proved that (viii) implies (v).

According to the equivalence of (v) to (ii) (see 3.7), the proof of 3.11 will be concluded by showing that

(ii)$\rhd$(vii) Assume (vii) to be wrong. Then there exists a uniformly bounded sequence $f_j \in \mathfrak{B}$, $j \in \mathbb{N}$, so that there exists an $\varepsilon_0 > 0$, an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ and measures $\mu_n \in \mathfrak{M}$, $n \in \mathbb{N}_0$, with $\inf_{n \in \mathbb{N}_0} |\mu_n(f_n - f)| > \varepsilon_0$. W.l.o.g. we may assume $\mathbb{N}_0 = \mathbb{N}$.

By assumption $C:=\sup_{j\in\mathbb{N}}(\sup_{x\in X}|f_j(x)|)<\infty$. As (ii) implies (v)(b) (see 3.7). There exists $K_0\in\mathscr{K}$ such that $\sup_{n\in\mathbb{N}}|\mu_n|(\bar{K}_0)<\varepsilon_0/4C$, hence

$$|\mu_n(f_n-f)-\mu_n((f_n-f)\,1_{K_0})|\leqq|\mu_n|\,(|f_n-f|\,1_{\bar{K}_0})<\varepsilon_0/2 \qquad \text{for all } n\in\mathbb{N}$$

and therefore $\inf_{n\in\mathbb{N}}|\mu_n((f_n-f)\,1_{K_0})|>\varepsilon_0/2$.

On the other hand, if we define $\mu:=\sum_{m\in\mathbb{N}}2^{-m}|\mu_m|$, it follows that f_n convergences to f on K_0 in μ-measure. Now (ii) implies by 2.6 and 1.6 that $\mu|\mathscr{F}$ dominates $\{|\mu_m||\mathscr{F}:m\in\mathbb{N}\}$ uniformly, hence there exists $\delta(\varepsilon_0)>0$ such that $\mu(A)<\delta(\varepsilon_0)$ implies $|\mu_m|(A)<\varepsilon_0/8C$ for all $m\in\mathbb{N}$. Taking $\delta:=\min(\varepsilon_0/4b_{\mathfrak{M}},\delta(\varepsilon_0))$ ($b_{\mathfrak{M}}:=\sup_{\mu\in\mathfrak{M}}\|\mu\|<\infty$), we obtain for all $n\geqq n_0(K_0,\delta)$ that $|\mu_m|(K_0\cap\{|f_n-f|>\delta\})<\varepsilon_0/8C$ for every $m\in\mathbb{N}$, which implies for all $n\geqq n_0(K_0,\delta)$, $n\in\mathbb{N}$, that

$$\begin{aligned}|\mu_n((f_n-f)\,1_{K_0})|&\leqq|\mu_n|\,(|f_n-f|\,1_{K_0\cap\{|f_n-f|\leqq\delta\}})\\&\quad+|\mu_n|\,(|f_n-f|\,1_{K_0\cap\{|f_n-f|>\delta\}})<\delta b_{\mathfrak{M}}+2C\,\varepsilon_0/8\,C\leqq\varepsilon_0/2.\end{aligned}$$

This contradicts $|\mu_n((f_n-f)\,1_{K_0})|>\varepsilon_0/2$, $n\in\mathbb{N}$.

3.12. Theorem. *Let $(X,\mathscr{T})$ be a completely regular Hausdorff space and let $\mathfrak{M}|\mathscr{F}$ be a bounded subset of $\mathscr{K}$-rca$(X,\mathscr{F})$. Then each of the preceeding assertions* (i)–(viii) *is equivalent to the following assertion:*

(ix) *For every uniformly bounded sequence of continuous functions f_j, $j\in\mathbb{N}$, converging to zero at every point $x\in X$, we have $\lim_{j\in\mathbb{N}}\mu(f_j)=0$ uniformly with respect to $\mu\in\mathfrak{M}$.*

Proof. (vii)$\rhd$(ix) Obvious.

(ix)$\rhd$(viii) Suppose (viii) to be wrong. Then there exists a sequence of pairwise disjoint sets $U_j\in\mathscr{T}$, $j\in\mathbb{N}$, so that there exists $\varepsilon_0>0$, an infinite subset $\mathbb{N}_0\subset\mathbb{N}$ and $\mu_j\in\mathfrak{M}$ with $\inf_{j\in\mathbb{N}_0}|\mu_j(U_j)|>\varepsilon_0$. W.l.o.g. we may assume $\mathbb{N}_0=\mathbb{N}$. As $\mu_j\in\mathscr{K}$-rca$(X,\mathscr{F})$, there exists $K_j\subset U_j$ such that $|\mu_j|(U_j-K_j)\leqq\varepsilon_0/4$. Therefore $|\mu_j(K_j)|\geqq|\mu_j(U_j)|-|\mu_j|(U_j-K_j)>3\,\varepsilon_0/4$ for every $j\in\mathbb{N}$. Since $(X,\mathscr{T})$ is completely regular, for every $x\in K_j$ there exists a continuous function f_j^x with $0\leqq f_j^x\leqq1$, $f_j^x(x)>1-\varepsilon_j$ and $f_j^x|\bar{U}_j\equiv0$, where $0<\varepsilon_j<1$ is chosen so that $|\mu_j(K_j)(1-\varepsilon_j)|>3\,\varepsilon_0/4$ and $\varepsilon_j|\mu_j|(K_j)<\varepsilon_0/4$. As each f_j^x, $x\in K_j$, is continuous, there exist open neighborhoods V_j^x of x contained in U_j such that $f_j^x|V_j^x>1-\varepsilon_j$. Since K_j is compact, finitely many of them, say $V_j^{x_1},\dots,V_j^{x_n}$, will cover K_j. Put $f_j:=\max_{1\leqq i\leqq n}f_j^{x_i}$. Then f_j is continuous, $0\leqq f_j\leqq1$, $f_j|K_j>1-\varepsilon_j$ and $f_j|\bar{U}_j\equiv0$. It follows that $\lim_{j\in\mathbb{N}}f_j(x)=0$ for any $x\in X$ and $|\mu_j(f_j)|\geqq(1-\varepsilon_j)\,|\mu_j(K_j)|-|\mu_j|\,(U_j-K_j)-\varepsilon_j\,|\mu_j|\,(K_j)>\varepsilon_0/4$. This obviously contradicts (ix).

3.13. *Remark.* In order to give a most unified presentation of compactness criterions, 3.7, 3.11 and 3.12 are stated in the present form, although the reader will have realized that in some of the implications proved above there was made no use of regularity of $(X, \mathscr{T})$ resp. boundedness of $\mathfrak{M}|\mathscr{F}$ as it is summarized in the following table:

$(X, \mathscr{T})$	$\mathfrak{M}$ arbitrary 3.7	3.11	3.12	$\mathfrak{M}$ bounded 3.12
Hausdorff space	(iv)$\rhd$(iii) (iii)$\bowtie$(v) (viii)=(vi)(a)$\rhd$(v)(a) (vi)(b)$\rhd$(v)(b)	(vii)$\rhd$(viii)	(vii)$\rhd$(ix)	(ii)$\rhd$(vii)
Regular Hausdorff space		(viii)$\rhd$(v)(b)		
Completely regular Hausdorff space			(ix)$\rhd$(viii)	

If $(X, \mathscr{T})$ is an analytical space (i.e. regular Hausdorff space and continuous image of a Polish space), then every $\mu \in \mathrm{ca}(X, \mathscr{F})$ is $\mathscr{K}$-regular (see [8]). As in addition analytical spaces are normal (see [8]), we obtain from 3.7, 3.11 and 3.12 the following corollary:

3.14. Corollary. *Let* $(X, \mathscr{T})$ *be an analytical space. Then for bounded subsets* $\mathfrak{M}|\mathscr{F}$ *of* $\mathrm{ca}(X, \mathscr{F})$ *the assertions* (i)–(ix) *are all equivalent.*

Although we will not consider the weak topology $\mathscr{T}_c$, we want to mention one special case where, for non-negative $\mathscr{K}$-regular measures $\mathscr{T}_{\mathscr{F}}$-convergence is implied by $\mathscr{T}_c$-convergence. $\mathscr{T}_c$ (in the sense of [16b]) is defined as the coarsest topology in $\mathrm{ca}_+(X, \mathscr{F})$ for which every map $\mu \to \mu(f)$, where $f: X \to \mathbb{R}$ is bounded and upper semi-continuous, is upper semi-continuous. If $\mathscr{F}$ denotes as before the Borel sets in X, then $\mathscr{T}_c$ is clearly coarser than $\mathscr{T}_{\mathscr{F}}$ (cf. 1.21), hence $\mathscr{T}_{\mathscr{F}}$-convergence implies $\mathscr{T}_c$-convergence. The following Lemma goes into the converse direction:

3.15. Lemma. *Let* $(X, \mathscr{T})$ *be a Hausdorff space and let* $\mu_n \in \mathscr{K}\text{-}\mathrm{rca}_+(X, \mathscr{F})$, $n \in \mathbb{N}$, *be bounded and uniformly dominated by a finite non-negative measure. Then* $(\mu_n)_{n\in\mathbb{N}} \to \mu_0(\mathscr{T}_c)$ *implies* $(\mu_n)_{n\in\mathbb{N}} \to \mu_0(\mathscr{T}_{\mathscr{F}})$.

Proof. As $\{\mu_n|\mathscr{F}: n \in \mathbb{N}\}$ is bounded and uniformly dominated, 2.6 implies that $\{\mu_n|\mathscr{F}: n \in \mathbb{N}\}$ is c.s.c. in $(\mathrm{ca}(X, \mathscr{F}), \mathscr{T}_{\mathscr{F}})$. Hence to every subsequence $\mathbb{N}_0 \subset \mathbb{N}$ there exists a further subsequence $\mathbb{N}_1 \subset \mathbb{N}_0$ such that $(\mu_n)_{n\in\mathbb{N}_1} \to \mu_{\mathbb{N}_1}(\mathscr{T}_{\mathscr{F}})$ with $\mu_{\mathbb{N}_1} \in \mathscr{K}\text{-}\mathrm{rca}(X, \mathscr{F})$ (see 3.8), whence $(\mu_n)_{n\in\mathbb{N}_1} \to \mu_{\mathbb{N}_1}(\mathscr{T}_c)$. Since $(\mathscr{K}\text{-}\mathrm{rca}_+(X, \mathscr{F}), \mathscr{T}_c)$ is a Hausdorff space (see [16b], Theorem 11.2), we obtain $\mu_{\mathbb{N}_1} = \mu_0$. This implies that the whole sequence $(\mu_n)_{n\in\mathbb{N}}$ is $\mathscr{T}_{\mathscr{F}}$-convergent to μ_0.

(If $(X, \mathscr{T})$ is a regular Hausdorff space, 3.15 holds for τ-smooth rather than tight measures applying [16b], Theorem 11.2.)

4. On the Theorems of Dieudonné and Wells, Jr.

In this section we will give an application of our previous results to the problem of giving sufficient conditions on a subfamily $\mathscr{C}$ of the family $\mathscr{T}$ of open sets in a Hausdorff space $(X, \mathscr{T})$ to ensure that convergence of a sequence $\mu_n \in \mathscr{K}\text{-}\mathrm{rca}(X, \mathscr{F})$, $n \in \mathbb{N}$, on $\mathscr{C}$ implies its convergence on the σ-field $\mathscr{F}$ of Borel sets in X. Sufficient conditions of this type were studied by Dieudonné and Grothendieck with $\mathscr{C}=\mathscr{T}$ ([3], Proposition 8, p. 37 and [6], p. 150) for compact metric resp. locally compact spaces $(X,\mathscr{T})$ and recently by Wells, jr. ([17], Theorem 3, p. 125) for compact spaces and families $\mathscr{C}$ which are strictly smaller than $\mathscr{T}$, in general. Topsøe obtained in [16a], Corollary 3, p. 28, independently and with different methods a result which extends, for non-negative tight measures, Dieudonné's Theorem to regular Hausdorff spaces.

As we shall see below the techniques presented by Wells, jr. can be extended to yield together with an application of our compactness criterions the main Theorem 4.10 which covers the results just cited.

4.1. Definition. Let $\mathscr{C}$ be a family of open sets in a Hausdorff space $(X,\mathscr{T})$. We call $\mathscr{C}$ a $\mathscr{T}_{\mathscr{F}}[\mathscr{T}_0]$-converging class (for $\mathscr{K}$-rca$(X,\mathscr{F})$) provided every sequence $\mu_n\in\mathscr{K}$-rca$(X,\mathscr{F})$, $n\in\mathbb{N}$, which converges on $\mathscr{C}$ (i.e. $\lim_{n\in\mathbb{N}}\mu_n(C)$ exists in $\mathbb{R}$ for every $C\in\mathscr{C}$) converges for the topology $\mathscr{T}_{\mathscr{F}}[\mathscr{T}_0]$.

$\mathscr{C}$ is called a bounding class (for $\mathscr{K}$-rca$(X,\mathscr{F})$) provided every sequence $\mu_n\in\mathscr{K}$-rca$(X,\mathscr{F})$, $n\in\mathbb{N}$, for which $\sup_{n\in\mathbb{N}}|\mu_n(C)|<\infty$ for every $C\in\mathscr{C}$, is bounded.

4.2. *Remark.* It follows from 2.15 that $\mathscr{C}$ is a $\mathscr{T}_0$-converging class iff $\mathscr{C}$ is a $\mathscr{T}_{\mathscr{F}}$-converging class.

If $\mathscr{C}$ is a $\mathscr{T}_{\mathscr{F}}$-converging class, it follows from 1.10 and 3.8 that for every sequence $\mu_n\in\mathscr{K}$-rca$(X,\mathscr{F})$ which converges on $\mathscr{C}$ there exists a measure $\mu_0\in\mathscr{K}$-rca$(X,\mathscr{F})$ such that $\lim_{n\in\mathbb{N}}\mu_n(A)=\mu_0(A)$ for every Borel set A; i.e.: If $\mathscr{C}$ is a $\mathscr{T}_{\mathscr{F}}$-converging class, then for every sequence $\mu_n\in\mathscr{K}$-rca$(X,\mathscr{F})$, $n\in\mathbb{N}$, which converges on $\mathscr{C}$, $\mu_0(C):=\lim_{n\in\mathbb{N}}\mu_n(C)$, $C\in\mathscr{C}$, is the trace on $\mathscr{C}$ of a $\mathscr{K}$-regular measure $\mu_0|\mathscr{F}$ to which $(\mu_n)_{n\in\mathbb{N}}$ converges $(\mathscr{T}_{\mathscr{F}})$.

4.3. Lemma. *Let $(X,\mathscr{T})$ be a regular Hausdorff space and let $\mathscr{C}$ be a family of open sets in X satisfying*

(S) *If $K\in\mathscr{K}$, $U\in\mathscr{T}$, and $K\subset U$, there exists $C\in\mathscr{C}$ such that $K\subset C\subset U$.*

Then every bounded sequence $\mu_n\in\mathscr{K}$-rca$(X,\mathscr{F})$, $n\in\mathbb{N}$, which converges on $\mathscr{C}$ is $\mathscr{T}_{\mathscr{F}}$-convergent if (viii) *of* 3.11 *holds with* $\mathfrak{M}|\mathscr{F}=\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$.

Proof. We remark first that any two $\mathscr{K}$-regular measures which coincide on $\mathscr{T}$ (or $\mathscr{K}$) are identical (cf. 3.3 and 3.5). By (S) it follows that $\mathscr{C}$ separates the elements of $\mathscr{K}$-rca$(X,\mathscr{F})$, hence by 2.3 (applied with $\mathscr{K}$-rca$(X,\mathscr{F})$ instead of ca$(X,\mathscr{F})$) $\mathscr{T}_{\mathscr{F}}$ coincides with $\mathscr{T}_{\mathscr{C}}$ on every c.c. subset $\mathfrak{M}|\mathscr{F}$ of $(\mathscr{K}$-rca$(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$ and any sequence $\mu_n\in\mathfrak{M}$ which converges on $\mathscr{C}$ does also converge on $\mathscr{F}$. Therefore it remains to show that $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$ is c.c. in $(\mathscr{K}$-rca$(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$. As by 3.11 (viii) with $\mathfrak{M}|\mathscr{F}=\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$ implies that $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$ is c.c. in $(\text{ca}(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$, it follows from 3.9 that $\{\mu_n|\mathscr{F}: n\in\mathbb{N}\}$ is c.c. in $(\mathscr{K}$-rca$(X,\mathscr{F}),\mathscr{T}_{\mathscr{F}})$.

The following lemma generalizes Lemma 1 in [17] from compact spaces to arbitrary Hausdorff spaces.

4.4. Lemma (cf. [17], Lemma 1, p. 125). *Let $(X,\mathscr{T})$ be a Hausdorff space and suppose that $\mathscr{C}$ is a family of open sets in X fulfilling the following conditions:*

(1) *$\mathscr{C}$ is closed with respect to finite intersections.*

(2) *$C_1, C_2\in\mathscr{C}$ and $C_1^c\cap C_2^c=\emptyset$ implies $C_1\cup C_2\in\mathscr{C}$.*

(3) *If* $F \in \bar{\mathscr{T}}$, $U \in \mathscr{T}$, *and* $F \subset U$, *there exists* $C \in \mathscr{C}$ *such that* $F \subset C \subset U$.

(4) *If* C'_n *and* C''_n, $n \in \mathbb{N}$, *are sequences from* $\mathscr{C}$ *such that*

$$C''_1 \subset C''_2 \subset \cdots \subset C''_n \subset \cdots \subset C'_n \subset C'_{n-1} \subset \cdots \subset C'_2 \subset C'_1,$$

then there exists $C_0 \in \mathscr{C}$ *interpolating the given sequence, i.e.* $C''_n \subset C_0 \subset C'_n$ *for every* $n \in \mathbb{N}$.

Let $C_n \in \mathscr{C}$, $n \in \mathbb{N}$, *and suppose that* $(\bigcup_{i \neq n} C_i)^c \cap C_n^c = \emptyset$ *for each* $n \in \mathbb{N}$.

Then for any $\lambda \in \mathscr{K}\text{-rca}_+(X, \mathscr{F})$ *and every* $\delta > 0$ *there exists an infinite subset* $\mathbb{N}_0 \subset \mathbb{N}$ *and a* $D_{\mathbb{N}_0} \in \mathscr{C}$ *such that* $\bigcup_{n \in \mathbb{N}_0} C_n \subset D_{\mathbb{N}_0}$ *and* $\lambda(D_{\mathbb{N}_0}) < \delta$.

Proof. Let $\delta > 0$ be given and let $F := \left(\bigcup_{i=1}^{\infty} C_{2i-1}\right)^c$. Then $F \in \bar{\mathscr{T}}$, $F \subset C_{2n}^c$, hence (3) applied with $U := \overline{C_{2n}^c}$ allows us to pick for every $n \in \mathbb{N}$ a set $D_{2n} \in \mathscr{C}$ such that $F \subset D_{2n} \subset \overline{C_{2n}^c}$ for every $n \in \mathbb{N}$. By this we obtain the following sequence:

$$C_1 \subset (C_1 \cup C_3) \subset (C_1 \cup C_3 \cup C_5) \subset \cdots \subset (D_6 \cap D_4 \cap D_2) \subset (D_4 \cap D_2) \subset D_2,$$

where, according to (1) and (2), each member occuring within the brackets belongs to $\mathscr{C}$. From now on the proof follows exactly the patterns as in [17], p. 126, and we will not repeat it here.

4.5. Theorem. (I) *Let* $(X, \mathscr{T})$ *be a regular Hausdorff space and suppose that* $\mathscr{C}$ *is a family of open sets in X fulfilling the conditions* (1), (3) *and* (4) *of* 4.4 *and*

(2′) $C_1, C_2 \in \mathscr{C}$ *and* $C_1 \cap C_2^c = \emptyset$ *implies* $C_1 \cup C_2 \in \mathscr{C}$.

Then every bounded sequence $\mu_n \in \mathscr{K}\text{-rca}(X, \mathscr{F})$, $n \in \mathbb{N}$, *which converges on* $\mathscr{C}$ *converges for the topology* $\mathscr{T}_{\mathscr{F}}$.

(II) *Let* $(X, \mathscr{T})$ *be a normal Hausdorff space and suppose that* $\mathscr{C}$ *is a family of open sets in X fulfilling the conditions* (1), (2) *and* (4) *of* 4.4 *and*

(3′) *If* $F \in \bar{\mathscr{T}}$, $U \in \mathscr{T}$, *and* $F \subset U$, *there exists* $C \in \mathscr{C}$ *such that* $F \subset C \subset C^c \subset U$. *Then every bounded sequence* $\mu_n \in \mathscr{K}$*-rca*$(X, \mathscr{F})$, $n \in \mathbb{N}$, *which converges on* $\mathscr{C}$ *converges for the topology* $\mathscr{T}_{\mathscr{F}}$.

Proof. (a) Let $\mu_n \in \mathscr{K}\text{-rca}(X, \mathscr{F})$, $n \in \mathbb{N}$, be a bounded sequence which converges on $\mathscr{C}$. Assume that, for some $A_0 \in \mathscr{F}$, $(\mu_n(A_0))_{n \in \mathbb{N}}$ does not converge. Then there exists $\varepsilon_0 > 0$ and for every $n \in \mathbb{N}$ a $m_n \in \mathbb{N}$ such that $|\mu_n(A_0) - \mu_{n+m_n}(A_0)| > \varepsilon_0$. Therefore $\nu_n := \mu_n - \mu_{n+m_n}$, $n \in \mathbb{N}$, defines a bounded sequence of $\mathscr{K}$-regular measures which converges to zero on $\mathscr{C}$, but does not converge to zero on $\mathscr{F}$. The proof of (I) and (II) will be concluded by showing that every bounded sequence of $\mathscr{K}$-regular measures, say $(\nu_n)_{n \in \mathbb{N}}$, which converges to zero on $\mathscr{C}$ converges to zero on $\mathscr{F}$. We remark that, if $(\nu_n)_{n \in \mathbb{N}}$ is $\mathscr{T}_{\mathscr{F}}$-convergent, it follows already from (S) in 4.3 that the limiting measure, being $\mathscr{K}$-regular by 3.8, must be identically zero.

(b) Assume, on the contrary, that $(\nu_n)_{n \in \mathbb{N}}$ is not $\mathscr{T}_{\mathscr{F}}$-convergent. Then, by 4.3, (viii) with $\mathfrak{M}|\mathscr{F} = \{\nu_n|\mathscr{F} : n \in \mathbb{N}\}$ must be wrong, i.e. there exists a sequence of pairwise disjoint sets $U_j \in \mathscr{T}$, $j \in \mathbb{N}$, so that there exists $\varepsilon_0 > 0$, an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ and a subsequence $(\nu_{n_j})_{j \in \mathbb{N}_0}$ of $(\nu_n)_{n \in \mathbb{N}}$ with $\inf_{j \in \mathbb{N}_0} |\nu_{n_j}(U_j)| > \varepsilon_0$. W.l.o.g. we may assume that $\inf_{j \in \mathbb{N}} |\nu_j(U_j)| > \varepsilon_0$. Since $\nu_j \in \mathscr{K}\text{-rca}(X, \mathscr{F})$, there exists $K_j \in \mathscr{K}$, $K_j \subset U_j$, such that $|\nu_j|(U_j-K_j)<|\nu_j(U_j)|-\varepsilon_0$. As $(X,\mathscr{T})$ is a regular Hausdorff space, there exist $U_j', U_j''\in\mathscr{T}$ with $K_j\subset U_j'\subset\overline{U_j''}\subset U_j$, $j\in\mathbb{N}$. Applying (3) with $F:=K_j$, $U:=U_j'$, there exists $C_j\in\mathscr{C}$ such that $K_j\subset C_j\subset U_j'$. It follows that $C_j\subset C_j^c\subset\overline{U_j''}\subset U_j$, hence $(\bigcup_{i\neq n} C_i)^c\cap C_n^c=\emptyset$ for each $n\in\mathbb{N}$; furthermore, $|\nu_j(C_j)|\geqq|\nu_j(U_j)|-|\nu_j|(U_j-K_j)>\varepsilon_0$ for every $j\in\mathbb{N}$.

If we now apply 4.4 with $\lambda:=|\nu_1|$, $\mathbb{N}-\{1\}$ instead of $\mathbb{N}$, and $\delta=\varepsilon_0/3$, we obtain an infinite set $\mathbb{N}_1\subset\mathbb{N}-\{1\}$ and a $D_{\mathbb{N}_1}\in\mathscr{C}$ such that $D_{\mathbb{N}_1}\supset\bigcup_{i\in\mathbb{N}_1} C_i$ and $|\nu_1|(D_{\mathbb{N}_1})<\varepsilon_0/3$. Applying (3) [(3')] with $F:=(\bigcup_{i\in\mathbb{N}_1} C_i)^c$ and $U:=\overline{C_1^c}$, we obtain $D_1\in\mathscr{C}$ such that $F\subset D_1\subset U$ $[F\subset D_1\subset D_1^c\subset U]$. Put $C_{\mathbb{N}_1}:=D_{\mathbb{N}_1}\cap D_1$; then, by (1), $C_{\mathbb{N}_1}\in\mathscr{C}$, $C_{\mathbb{N}_1}\supset\bigcup_{i\in\mathbb{N}_1} C_i$ and $|\nu_1|(C_{\mathbb{N}_1})<\varepsilon_0/3$. Furthermore, $C_{\mathbb{N}_1}\cap C_1^c=\emptyset$ $[C_{\mathbb{N}_1}^c\cap C_1^c=\emptyset]$ and therefore it follows by (2') [(2)] that $C_{\mathbb{N}_1}\cup C_1\in\mathscr{C}$. Following the inductive process described in [17], p. 127, one obtains a subsequence $(C_{n_i})_{i\in\mathbb{N}}$ of $(C_n)_{n\in\mathbb{N}}$ such that $(\nu_n)_{n\in\mathbb{N}}$ does not converge to zero on some $C_0\in\mathscr{C}$ which contradicts our hypothesis that $(\nu_n)_{n\in\mathbb{N}}$ converges to zero on every member of $\mathscr{C}$.

4.6. *Remark.* If $(X,\mathscr{T})$ is a normal Hausdorff space, (1), (2), (3') and (4) in 4.5 are fulfilled with $\mathscr{C}=\mathscr{T}_r:=\{U\in\mathscr{T}: U=\operatorname{int}(U^c)\}$, the family of the so-called regular open sets in X. It is easy to see that $\mathscr{T}_r=\{\operatorname{int} F: F\in\overline{\mathscr{T}}\}$ and that $\mathscr{T}_r$ is strictly smaller than $\mathscr{T}$, in general.

Proof. We remark first that for any two sets A and B we have

$$(*)\qquad \operatorname{int}(A\cup B)\subset(\operatorname{int}A\cup B^c)\cap(A^c\cup\operatorname{int}B).$$

(1) $U_1, U_2\in\mathscr{T}_r$ implies $U_1\cap U_2\in\mathscr{T}$ and $U_1\cap U_2\subset\operatorname{int}(U_1\cap U_2)^c$; conversely: $\operatorname{int}(U_1\cap U_2)^c\subset\operatorname{int}(U_1^c\cap U_2^c)=\operatorname{int}(U_1^c)\cap\operatorname{int}(U_2^c)=U_1\cap U_2$.

(2) Let $U_1, U_2\in\mathscr{T}_r$, $U_1^c\cap U_2^c=\emptyset$. Then $U_1\cup U_2\in\mathscr{T}$, $U_1\cup U_2\subset\operatorname{int}(U_1\cup U_2)^c$; conversely by (*): $\operatorname{int}(U_1\cup U_2)^c=\operatorname{int}(U_1^c\cup U_2^c)\subset(\operatorname{int}(U_1^c)\cup U_2^c)\cap(U_1^c\cup\operatorname{int}(U_2^c))=(U_1\cup U_2^c)\cap(U_1^c\cup U_2)=U_1\cup U_2$.

(3') Since $(X,\mathscr{T})$ is a normal Hausdorff space, $F\in\overline{\mathscr{T}}$, $U\in\mathscr{T}$, and $F\subset U$, implies that there exists $V\in\mathscr{T}$ with $F\subset V\subset V^c\subset U$. From this it follows that $F\subset V\subset\operatorname{int}(V^c)\subset V^c\subset U$ and, furthermore, $F\subset V\subset\operatorname{int}(V^c)\subset(\operatorname{int}(V^c))^c\subset V^c\subset U$, where $\operatorname{int}(V^c)\in\mathscr{T}_r$.

(4) $U_1''\subset U_2''\subset\cdots\subset U_n''\subset\cdots\subset U_n'\subset U_{n-1}'\subset\cdots\subset U_2'\subset U_1'$ with $U_i', U_i''\in\mathscr{T}_r$ $i\in\mathbb{N}$, is interpolated by $U:=\operatorname{int}(\bigcup_{i\in\mathbb{N}} U_i'')^c\in\mathscr{T}_r$ (or by $U:=\operatorname{int}(\bigcap_{i\in\mathbb{N}} U_i')^c\in\mathscr{T}_r$).

So far we have not considered the boundedness problem, i.e., given any sequence $\mu_n\in\mathscr{K}$-rca$(X,\mathscr{F})$, $n\in\mathbb{N}$, for which $\sup_{n\in\mathbb{N}}|\mu_n(C)|<\infty$ for every $C\in\mathscr{C}$, under what conditions on $\mathscr{C}$ does it follow that $(\mu_n)_{n\in\mathbb{N}}$ is bounded.

Our aim is to prove that the families $\mathscr{C}$ considered in 4.5 are also bounding classes.

Let us start with the following basic result due to Dieudonné:

4.7. Lemma ([3], Proposition 9, p. 37). *If $(X,\mathscr{T})$ is a compact Hausdorff space, then $\mathscr{T}$ is a bounding class (for $\mathscr{K}$-rca$(X,\mathscr{F})$).*

4.8. Corollary. *Let* $(X, \mathcal{T})$ *be a Hausdorff space,* $\mu_n \in \mathcal{K}$-rca$(X, \mathcal{F})$, $n \in \mathbb{N}$, *a sequence of measures for which* $\sup_{n \in \mathbb{N}} |\mu_n(U)| < \infty$ *for every* $U \in \mathcal{T}$. *If* $\{|\mu_n| | \mathcal{F} : n \in \mathbb{N}\}$ *is uniformly tight, then* $(\mu_n)_{n \in \mathbb{N}}$ *is bounded.*

Proof. Let $K \in \mathcal{K}$ be arbitrary. Then

$$\sup_{n \in \mathbb{N}} |\mu_n(U \cap K)| \leqq \sup_{n \in \mathbb{N}} |\mu_n(U)| + \sup_{n \in \mathbb{N}} |\mu_n(U \cap \bar{K})| < \infty$$

for every $U \in \mathcal{T}$. Let $\mathcal{F}_K$ denote the Borel sets in $(K, \mathcal{T})$; then $\mu_n | \mathcal{F}_K \in \mathcal{K}$-rca$(K, \mathcal{F}_K)$ and, by 4.7, it follows that $\sup_{n \in \mathbb{N}} |\mu_n|(K) < \infty$, whence the uniform tightness of $\{|\mu_n| | \mathcal{F} : n \in \mathbb{N}\}$ implies the assertion.

4.9. Lemma (cf. [17], Corollary, p. 128). *Under the conditions of* 4.5 *the families* $\mathcal{C}$ *considered there are also bounding classes, i.e. any sequence* $\mu_n \in \mathcal{K}$-rca$(X, \mathcal{F})$, $n \in \mathbb{N}$, *with* $\sup_{n \in \mathbb{N}} |\mu_n(C)| < \infty$ *for every* $C \in \mathcal{C}$, *is bounded.*

Proof. If $(\mu_n)_{n \in \mathbb{N}}$ is not bounded, there exists an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that $\lim_{n \in \mathbb{N}_0} \|\mu_n\| = \infty$. W.l.o.g. we may assume $\mathbb{N}_0 = \mathbb{N}$. Let $\tilde{\mu}_n := \mu_n / \|\mu_n\|^{\frac{1}{2}}$; then $\lim_{n \in \mathbb{N}} \tilde{\mu}_n(C) = 0$ for every $C \in \mathcal{C}$ while maintaining $\sup_{n \in \mathbb{N}} \|\tilde{\mu}_n\| = \infty$. We shall show that this is impossible. It follows from the proof of 4.5 (part (b)) that $(\tilde{\mu}_n)_{n \in \mathbb{N}}$ fulfills (viii) and hence (v) which is equivalent to (iii) (see 3.13). According to (iii) there exists for every $U \in \mathcal{T}$ and $\varepsilon > 0$ a $K \in \mathcal{K}$ such that $\sup_{n \in \mathbb{N}} |\tilde{\mu}_n|(U - K) < \varepsilon$. Applying (3′) [(3)] there exists $C \in \mathcal{C}$ with $K \subset C \subset U$, whence $\sup_{n \in \mathbb{N}} |\tilde{\mu}_n(U)| \leqq \sup_{n \in \mathbb{N}} |\tilde{\mu}_n(C)| + \sup_{n \in \mathbb{N}} |\tilde{\mu}_n|(U - K) < \infty$ for every $U \in \mathcal{T}$. Since by (v)(b) $\{|\tilde{\mu}_n| | \mathcal{F} : n \in \mathbb{N}\}$ is uniformly tight, 4.8 yields the desired contradiction.

With 4.9 we obtain

4.10. Theorem. *Under the conditions of* 4.5, *the families* $\mathcal{C}$ *considered there are both* $\mathcal{T}_{\mathcal{F}}$*-converging classes and bounding classes.*

5. Extensions

After I had obtained 4.5 it was first discovered by Pfanzagl that for bounded sequences $\mu_n \in \mathcal{K}$-rca$(X, \mathcal{F})$, $n \in \mathbb{N}$, Dieudonné's Theorem (with $\mathcal{C} = \mathcal{T}$) could be extended even to arbitrary Hausdorff spaces by use of the following lemma:

5.1. Lemma ([14c], Lemma 1). *Let* $(X, \mathcal{F})$ *be a measurable space,* $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, *a sequence of pairwise disjoint sets and let* $\mu_n \in$ ca$(X, \mathcal{F})$, $n \in \mathbb{N}$, *be a sequence of measures such that* $\lim_{n \in \mathbb{N}} \mu_n(A_m) = 0$ *for every* $m \in \mathbb{N}$ *and* $\inf_{n \in \mathbb{N}} |\mu_n(A_n)| > 0$. *Then there exists an infinite subset* $\mathbb{N}_0 \subset \mathbb{N}$ *such that* $\inf_{n \in \mathbb{N}_0} |\mu_n(\sum_{m \in \mathbb{N}_0} A_m)| > 0$.

Once becoming aware of 5.1 the main result in [14c] can be proved applying 5.1 and the compactness criterions of Section 3 in rather the same way as 4.5 and 4.9. The final proof given in [14c] for 5.2 below is more comprehensive insofar as the case of bounded sequences is not handled separately.

5.2. Theorem. *Let* $(X, \mathcal{T})$ *be a Hausdorff space and suppose that* $\mathcal{C}$ *is a family of open sets in X fulfilling* (S) *of* 4.3 *and*

(4′) $C_i \in \mathscr{C}$, $i \in \mathbb{N}$, *implies* $\bigcup_{i\in\mathbb{N}} C_i \in \mathscr{C}$.

Then $\mathscr{C}$ *is a* $\mathscr{T}_{\mathscr{F}}$*-converging class.*

Proof. As in the proof of 4.5 it suffices to show that every sequence of $\mathscr{K}$-regular measures which converges to zero on $\mathscr{C}$ converges to zero on $\mathscr{F}$.

(a) We show first that every *bounded* sequence of $\mathscr{K}$-regular measures, say $(\nu_n)_{n\in\mathbb{N}}$, which converges to zero on $\mathscr{C}$ is c.s.c. in $(\mathscr{K}\text{-rca}(X,\mathscr{F}), \mathscr{T}_{\mathscr{F}})$. We remark that, if this holds true, then every subsequence of $(\nu_n)_{n\in\mathbb{N}}$ contains a further subsequence converging to the ($\mathscr{K}$-regular) measure identically zero on $\mathscr{F}$, whence the whole sequence $(\nu_n)_{n\in\mathbb{N}}$ converges to zero on $\mathscr{F}$.

(b) If a bounded sequence of $\mathscr{K}$-regular measures fails to be c.s.c., then, by 3.7, either (vi)(a) is wrong or, if (vi)(a) holds true, then (vi)(b) is wrong. Applying 5.1 we will show that both assumptions contradict the assumed convergence of $(\nu_n)_{n\in\mathbb{N}}$ to zero on $\mathscr{C}$.

(c) If (vi)(a) is wrong, there exists a sequence of pairwise disjoint sets $U_j \in \mathscr{T}$, $j \in \mathbb{N}$, and a subsequence of $(\nu_n)_{n\in\mathbb{N}}$, w.l.o.g. still called $(\nu_n)_{n\in\mathbb{N}}$, such that $\inf_{n\in\mathbb{N}} |\nu_n(U_n)| > \varepsilon_0 > 0$. Since $\nu_n \in \mathscr{K}\text{-rca}(X,\mathscr{F})$, $n\in\mathbb{N}$, there exists $K_n \in \mathscr{K}$, $K_n \subset U_n$, such that $|\nu_n|(U_n - K_n) < \varepsilon_0/4$; hence $|\nu_n(K_n)| \geqq |\nu_n(U_n)| - |\nu_n|(U_n - K_n) > 3\varepsilon_0/4$ for all $n\in\mathbb{N}$. Applying (S) there exists $C_n \in \mathscr{C}$ such that $K_n \subset C_n \subset U_n$, $n \in \mathbb{N}$, the C_n being disjoint with $|\nu_n(C_n)| \geqq |\nu_n(K_n)| - |\nu_n|(U_n - K_n) > \varepsilon_0/2$ for all $n\in\mathbb{N}$. Therefore 5.1 yields an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that $\inf_{n\in\mathbb{N}_0} |\nu_n(\sum_{i\in\mathbb{N}_0} C_i)| > 0$. Hence $(\nu_n(\sum_{i\in\mathbb{N}_0} C_i))_{n\in\mathbb{N}}$ does not converge to zero which is a contradiction since, by (4′), $\sum_{i\in\mathbb{N}_0} C_i \in \mathscr{C}$. Therefore (vi)(a) holds true.

(d) If (vi)(b) would be wrong, there would exist a sequence of pairwise disjoint sets $K_j \in \mathscr{K}$, $j\in\mathbb{N}$, and a subsequence of $(\nu_n)_{n\in\mathbb{N}}$, w.l.o.g. still called $(\nu_n)_{n\in\mathbb{N}}$, such that $\inf_{n\in\mathbb{N}} |\nu_n(K_n)| > \varepsilon_0 > 0$. As (vi)(a) implies (v)(a) (see 3.13) and as $(\nu_n)_{n\in\mathbb{N}}$ converges to zero on $\mathscr{C}$, it follows from (S) that $(\nu_n)_{n\in\mathbb{N}}$ converges to zero on $\mathscr{K}$. Hence again 5.1 can be applied to obtain an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that $\delta := \inf_{n\in\mathbb{N}_0} |\nu_n(\sum_{j\in\mathbb{N}_0} K_j)| > 0$. Applying (vi)(a) and (S) we obtain $C_j \in \mathscr{C}$, $C_j \supset K_j$, such that $\sup_{n\in\mathbb{N}} |\nu_n|(C_j - K_j) < 2^{-j-1}\delta$ and therefore $\sup_{n\in\mathbb{N}} |\nu_n|(\bigcup_{j\in\mathbb{N}_0} C_j - \sum_{j\in\mathbb{N}_0} K_j) \leqq \delta/2$, i.e. $\inf_{n\in\mathbb{N}_0} |\nu_n(\bigcup_{j\in\mathbb{N}_0} C_j)| \geqq \delta/2$, which again contradicts the assumed convergence of $(\nu_n)_{n\in\mathbb{N}}$ to zero on $\mathscr{C}$. This concludes the proof of 5.2 for bounded sequences $\nu_n \in \mathscr{K}\text{-rca}(X,\mathscr{F})$, $n\in\mathbb{N}$.

(e) It remains to show that any sequence of $\mathscr{K}$-regular measures which converges to zero on $\mathscr{C}$ is bounded.

This follows from the following analogon to 4.9, generalizing [3], Proposition 9, p. 37, from compact spaces to arbitrary Hausdorff spaces, and by this the proof of 5.2 will be concluded.

5.3. Lemma. *Let* $(X, \mathscr{T})$ *be a Hausdorff space and suppose that* $\mathscr{C}$ *is a family of open sets in* X *fulfilling* (S) *of* 4.3 *and* (4′) *of* 5.2. *Then* $\mathscr{C}$ *is a bounding class.*

Proof. Let $\mu_n \in \mathscr{K}\text{-rca}(X,\mathscr{F})$, $n\in\mathbb{N}$, be a sequence of measures for which $\sup_{n\in\mathbb{N}} |\mu_n(C)| < \infty$ for every $C \in \mathscr{C}$. If $(\mu_n)_{n\in\mathbb{N}}$ is not bounded, we arrive at $\tilde{\mu}_n \in \mathscr{K}\text{-rca}(X,\mathscr{F})$ with $\lim_{n\in\mathbb{N}} \tilde{\mu}_n(C) = 0$ for every $C \in \mathscr{C}$ while maintaining $\sup_{n\in\mathbb{N}} \|\tilde{\mu}_n\| = \infty$

(cf. 4.9). We shall show that this leads to a contradiction. As pointed out in proving 5.2(c) and (d), it follows from (S) and (4′) that (vi) holds true for $\{\tilde{\mu}_n|\mathscr{F}: n\in\mathbb{N}\}$. Since (vi) implies (v) which is equivalent to (iii) (see 3.13), we obtain for every $U\in\mathscr{T}$ and $\varepsilon>0$ a $K\in\mathscr{K}$ such that $\sup_{n\in\mathbb{N}}|\tilde{\mu}|(U-K)<\varepsilon$. Applying (S) there exists $C\in\mathscr{C}$ with $K\subset C\subset U$, whence $\sup_{n\in\mathbb{N}}|\tilde{\mu}_n(U)|\leqq\sup_{n\in\mathbb{N}}|\tilde{\mu}_n(C)|+\sup_{n\in\mathbb{N}}|\tilde{\mu}_n|(U-K)<\infty$. As $\{|\mu_n||\mathscr{F}: n\in\mathbb{N}\}$ is uniformly tight, 4.8 yields the desired contradiction.

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