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Strong Invariance for Local Times

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Dedicated to Prof. I. Vincze on the occasion of his 70-th birthday

Summary. Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with distribution $P(Y_1=k)=p_k$ $(k=\pm 1, \pm 2, \ldots)$, $E(Y_1)=0$, $E(Y_1^2)=\sigma^2<\infty$. Put $T_n=Y_1+\ldots+Y_n$ and $N(x,n)=\#\{k\colon 0<k\leqq n, T_k=x\}$. Extending the result of Révész (1981) it is shown that for appropriate Skorohod construction we have

$$\sup_{x\in\mathbb{Z}}|L(x,n\sigma^2)-\sigma^2 N(x,n)|=o(n^{1/4+\varepsilon}) \qquad \text{a.s.}$$

provided all moments $E(|Y_1|^m)$, $m\geqq 0$ exists where L is the local time of a Wiener process. Certain rate of convergence is given also under weaker conditions and for $|L(x,n\sigma^2)-\sigma^2 N(x,n)|$ too, when x is fixed.

1. Introduction

Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables taking on integer values with $P(Y_i=k)=p_k$ $(k=\pm 1, \pm 2, \ldots)$. We assume that

$$E\,Y_1=\sum_k k p_k=0, \qquad E\,Y_1^2=\sum_k k^2 p_k=\sigma^2<\infty,$$
$$\text{g.c.d. } \{k\colon p_k>0\}=1. \tag{1.1}$$

Let furthermore $T_n=Y_1+\ldots+Y_n$ $(n=1,2,\ldots)$, $T_0=0$.

The occupation time of the recurrent random walk $T_1, T_2, \ldots$ is defined by

$$N(x,n)=\#\{k\colon 0<k\leqq n, T_k=x\},$$
$$n=1,2,\ldots;\ x=0, \pm 1, \pm 2, \ldots \tag{1.2}$$

i.e. $N(x,n)$ is the number of visits of the random walk to the point x in the time interval $(0,n]$. Set

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$$N(n)=\max_x N(x,n), \tag{1.3}$$

where $\max_x$ is taken over all integers. Kesten (1965) proved the following results:

Theorem A.

$$\limsup_{n\to\infty} \frac{N(n)}{(2n\log\log n)^{1/2}}=\frac{1}{\sigma} \quad a.s. \tag{1.4}$$

and

$$\liminf_{n\to\infty} \left(\frac{\log\log n}{n}\right)^{1/2} N(n)=\gamma \quad a.s. \tag{1.5}$$

where γ is a positive constant.

The exact value of γ is not known.

In the case of simple symmetric random walk, i.e. for the i.i.d. sequence $X_1, X_2, \ldots$ with

$$P(X_1=+1)=P(X_1=-1)=1/2, \tag{1.6}$$

Kesten (1965) has shown:

Theorem B. *Let $S_n=X_1+\ldots+X_n$ $(n=1,2,\ldots)$, $S_0=0$ and*

$$M(x,n)=\#\{k\colon 0<k\leqq n, S_k=x\}, \tag{1.7}$$

$$M(n)=\max_x M(x,n), \tag{1.8}$$

where $\max_x$ is taken over all integers. Then

$$\liminf_{n\to\infty} \left(\frac{\log\log n}{n}\right)^{1/2} M(n)=\gamma_1, \tag{1.9}$$

where

$$\frac{q_0}{2}\leqq\gamma_1\leqq\sqrt{3}\,q_0^2(2q_0^2-4)^{-1/2} \tag{1.10}$$

and $q_0=2, 405\ldots$ is the smallest root of the Bessel function $I_0(x)$.

For the case of symmetric random walk Chung and Hunt (1949) proved:

Theorem C. *Let $f(x)$ $(x\geqq 0)$ be an increasing function with $\lim_{x\to\infty} f(x)=\infty$ and define the functional*

$$I_1(f)=\int_1^\infty \frac{f(y)}{y}\exp\left(-\frac{f^2(y)}{2}\right)dy. \tag{1.11}$$

If $I_1(f)<\infty$, then

$$M(x,n)\leqq n^{1/2}f(n) \quad a.s. \tag{1.12}$$

for all but finitely many n and for any given x.

If $I_1(f)=\infty$, then for any given x there exists a random sequence $0<n_1<n_2<\ldots$ of integers such that for all i,

$$M(x,n_i)\geqq n_i^{1/2}f(n_i) \quad a.s. \tag{1.13}$$

Theorem D. *Let $f(x)$ $(x \geqq 0)$ be a function with $f(x) \downarrow 0$ and $x^{1/2} f(x) \uparrow \infty$ as $x \to \infty$ and define the functional*

$$I_2(f) = \int_1^\infty \frac{f(y)}{y} \, dy. \tag{1.14}$$

If $I_2(f) < \infty$, then

$$M(x,n) \geqq n^{1/2} f(n) \quad a.s. \tag{1.15}$$

for all but finitely many n and for any given x.

If $I_2(f) = \infty$, then for any given x there exists a random sequence $0 < n_1 < n_2 < \dots$ of integers such that for all i

$$M(x,n_i) \leqq n_i^{1/2} f(n_i) \quad a.s. \tag{1.16}$$

Kesten (1965) also studied the limiting behavior of the local time of a Wiener process $\{W(t), t \geqq 0\}$. He showed:

Theorem E. *Let $L(x,t)$ denote the local time at x of a Wiener process and put $L(t) = \sup_{-\infty < x < \infty} L(x,t)$. Then for any given x,*

$$\limsup_{t \to \infty} \frac{L(t)}{(2t \log\log t)^{1/2}} = \limsup_{t \to \infty} \frac{L(x,t)}{(2t \log\log t)^{1/2}} = 1 \quad a.s. \tag{1.17}$$

and

$$\liminf_{t \to \infty} \left(\frac{\log\log t}{t} \right)^{1/2} L(t) = \gamma_1 \quad a.s., \tag{1.18}$$

where γ_1 is the same constant as that of Theorem B.

By comparing Theorems A, B, C, D with Theorem E, one can see the similarity in the limit behavior of $M(x,n)$ and $L(x,n)$ and that of $M(n)$ and $L(n)$. An explanation of this phenomenon was given by Révész (1981) who showed that for appropriate construction the processes $M(x,n)$ and $L(x,n)$ are close to each other. In fact he proved:

Theorem F. *Let $\{W(t);\ t \geqq 0\}$ be a Wiener process defined on a probability space $\{\Omega, \mathscr{S}, P\}$. Then on the same space one can define a sequence $X_1, X_2, \dots$ of i.i.d. random variables with*

$$P(X_1 = +1) = P(X_1 = -1) = 1/2 \quad (i = 1, 2, \dots) \tag{1.19}$$

such that for any $\delta > 0$ we have

$$\lim_{n \to \infty} n^{-1/4-\delta} \sup_x |M(x,n) - L(x,n)| = 0 \quad a.s. \tag{1.20}$$

and

$$\lim_{n \to \infty} n^{-1/4-\delta} |S_n - W(n)| = 0 \quad a.s. \tag{1.21}$$

where $S_n = X_1 + X_2 + \dots + X_n$, $M(x,n)$ is defined by (1.7) and $L(x,n)$ is the local time at x of $W(\cdot)$.

The aim of this paper is to generalize the above theorem for the case when the basic distribution given by (1.1) possesses some moments, but is arbitrary otherwise. We shall prove the following result:

Theorem. *On a rich enough probability space $\{\Omega, \mathcal{S}, P\}$ one can construct a Wiener process $\{W(t);\ t \geqq 0\}$ and a sequence $Y_1, Y_2, \ldots$ of i.i.d. random variables with distribution given in* (1.1) *such that for any* $\varepsilon > 0$

(i) $\mu_{2m} = \sum_k |k|^{2m} p_k < \infty$ *for some* $\frac{3}{2} \leqq m \leqq 2$ *implies*

$$\lim_{n\to\infty} n^{-1/(2m)-\varepsilon}(L(x, n\sigma^2) - \sigma^2 N(x,n)) = 0 \quad a.s. \tag{1.22}$$

(ii) $\mu_{m+1} = \sum_k |k|^{m+1} p_k < \infty$ *for some* $m > 6$ *implies*

$$\lim_{n\to\infty} n^{-1/4-3/(2m)-\varepsilon} \sup_x |L(x, n\sigma^2) - \sigma^2 N(x,n)| = 0 \quad a.s. \tag{1.23}$$

where $\sup_x$ *is taken for all integers,* $N(x,n)$ *is defined by* (1.2) *and* $L(x,t)$ *is the local time of the Wiener process* $W(\cdot)$. *Furthermore, for the same construction with* $\mu_4 < \infty$ *implies*

$$\lim_{n\to\infty} n^{-1/4-\varepsilon} |S_n - W(n)| = 0 \quad a.s. \tag{1.24}$$

The authors are indebted to the referee who suggested the application of the Burkholder inequality in Lemma 4. This suggestion led to the moment-condition (ii) of the Theorem instead of the original condition what was somewhat stronger.

2. Preliminaries

2.1 *The Skorohod Embedding.* In our construction in Sect. 3 we use the Skorohod embedding (Skorohod (1961)) as given in Breiman (1967). For a survey and further results in this field we refer to Sawyer (1974).

Assume that a Wiener process $\{W(t), t \geqq 0\}$ is defined on a probability space $\{\Omega, \mathcal{S}, P\}$ and on the same space a sequence $\tau_1, \tau_2, \ldots$ of randomized stopping times can be defined in the following way: Let $\{p_k, k = \pm 1, \pm 2, \ldots\}$ be a distribution satisfying (1.1) and put

$$p_{i,j} = \frac{2}{\mu_1}(i+j)\, p_i p_{-j}, \qquad i = 1, 2, \ldots \qquad j = 1, 2, \ldots \tag{2.1}$$

where $\mu_1 = \sum_k |k| p_k$. Let the random variables U_1, V_1 have the distribution

$$P(U_1 = i, V_1 = j) = p_{i,j}, \qquad i = 1, 2, \ldots \qquad j = 1, 2, \ldots$$

and be independent from $\{W(t), t \geqq 0\}$. Define the stopping time τ_1 by

$$\tau_1 = \inf\{t \geqq 0\colon W(t) = \text{either } U_1 \text{ or } -V_1\}. \tag{2.2}$$

Repeat this procedure by defining (U_2, V_2) to have distribution $p_{i,j}$ and to be independent from $\{W(t), t \geqq 0\}$ and from (U_1, V_1). Let τ_2 be defined by

$$\tau_2 = \inf\{t \geqq \tau_1 : W(t) - W(\tau_1) = \text{either } U_2 \text{ or } -V_2\}. \tag{2.3}$$

By repeating this procedure we can define a sequence $\{\tau_j, j=1,2,\ldots\}$ of stopping times for which the following theorem holds

Theorem G. *The random variables* $Y_1 = W(\tau_1)$, $Y_n = W(\tau_n) - W(\tau_{n-1})$, $n=2,3,\ldots$ *are independent and each has the distribution*

$$P(Y_n = k) = p_k, \qquad k = \pm 1, \pm 2, \ldots \tag{2.4}$$

Moreover $E(\tau_1) = \sigma^2$ *and the following inequality holds*

$$E(\tau_i^m) \leqq C_m \sum_k |k|^{2m} p_k, \qquad m \geqq 1. \tag{2.5}$$

2.2 *Wiener Local Time.* If $W(t)$ is a Wiener process on the probability space $\{\Omega, \mathscr{S}, P\}$, then (Trotter (1958)) there exists an $\Omega_0 \subseteq \Omega$ with $P(\Omega_0) = 1$ and a function $L(x,t) = L(x,t,\omega)$ $(x \in R, t \in R^+, \omega \in \Omega_0)$ jointly continuous in x and t, such that

$$L(x,t,\omega) = \frac{d}{dx} \int_0^t I_{(-\infty,x)}(W(s))\, ds, \tag{2.6}$$

where $I_{(-\infty,x)}(\cdot)$ denotes the indicator function of the interval $(-\infty, x)$. $L(x,t)$ is called the local time at x of the Wiener process $W(t)$. We also refer to the book of Knight (1981, p. 107) where the continuity of local time is proved using methods close to ours.

For the distribution of $L(x,t)$ we have (see Lévy (1948))

$$P(L(x,t) \leqq u) = 2\Phi\left(\frac{|x|+u}{\sqrt{t}}\right) - 1, \qquad u \geqq 0. \tag{2.7}$$

We use also the following result of Knight (1969):

Theorem H. *Let the Wiener process* $W(t)$ *start from* $W(0) = z$ *and stopped at* τ *defined as the first point* $t \geqq 0$, *where* $W(t)$ *hits either* $z+i$ *or* $z-j$. $(i \geqq 0, j \geqq 0)$. *Then for* $0 \leqq z < j$, $L(0,\tau)$ *has the distribution*

$$L(0,\tau) = \begin{cases} 0 \text{ with probability } \dfrac{z}{z+i} \\ \operatorname{Exp}\left(\dfrac{i+j}{2(j-z)(z+i)}\right) \text{ with probability } \dfrac{i}{z+i}, \text{ where } \operatorname{Exp}(\lambda) \end{cases}$$

denotes a random variable having exponential distribution with parameter λ.

As a consequence of this result we have e.g.

$$E(L(0,\tau)) = \frac{2i(j-z)}{i+j}, \qquad 0 \leqq z \leqq j \tag{2.9}$$

and in general

$$E(L^m(0,\tau))=\frac{2^m\Gamma(m+1)\,i(j-z)^m(z+i)^{m-1}}{(i+j)^m},\qquad 0\leqq z\leqq j,\ m>0. \tag{2.10}$$

Similarly, for $-i\leqq z\leqq 0$ we obtain

$$E(L^m(0,\tau))=\frac{2^m\Gamma(m+1)\,j(i+z)^m(j-z)^{m-1}}{(i+j)^m}. \tag{2.11}$$

2.3 *Occupation Time.* We shall use certain properties of occupation times for random walk, given in Spitzer (1976) and in Kesten and Spitzer (1979).

Let $N(x,n)$, $x=0,\ \pm 1,\dots,n=1,2,\dots$ be the occupation time of the random walk $\{T_k, k=1,2,\dots\}$ as defined by (1.2).

Define the stopping time α by

$$\alpha=\min(j:\ T_j=0, j>0). \tag{2.12}$$

The following results hold (see Kesten and Spitzer (1979), Lemma 2).

Theorem I.

$$E(N(x,\alpha))=1,\qquad x=0,\ \pm 1,\dots \tag{2.13}$$

$$E(N^m(x,\alpha))\leqq K_m(1+a(x)+a(-x))^{m-1},\qquad m\geqq 1 \tag{2.14}$$

where K_m is a constant, independent from x and $a(\cdot)$ is the potential kernel of the random walk T_k. Moreover

$$a(x)+a(-x)\sim\frac{2}{\sigma^2}|x|\quad as\quad |x|\to\infty. \tag{2.15}$$

Hence the following corollary is true:

$$E(N^m(x,\alpha))\leqq K'_m|x|^{m-1},\qquad x=\pm 1,\ \pm 2,\dots \tag{2.16}$$

with some constant K'_m independent from x.

Note furthermore that (2.16) is true for arbitrary $m\geqq 1$, i.e. for non-integral value of m as well.

3. Proof of the Theorem

In this section we give the proof of our main result through several lemmas. Assume that on the probability space $\{\Omega,\mathcal{S},P\}$ we have a standard Wiener process $W(t)$, $t\geq 0$, $W(0)=0$ together with a sequence of stopping times τ_0, τ_1, $\tau_2,\dots$ such that $Y_n=W(\tau_n)-W(\tau_{n-1})$, $n=1,2,\dots$ are i.i.d. random variables with the distribution given by (1.1). The stopping times τ_i are constructed as described in Sect. 2.1.

Recall that $T_n=Y_1+\dots+Y_n=W(\tau_n)$, $n=1,2,\dots$ and

$$\alpha=\min(j:\ T_j=0, j>0).$$

In the sequel x always runs through the integers.

Lemma 1.

$$E(L(x,\tau_\alpha))=\sigma^2, \qquad x=0,\pm 1,\dots \tag{3.1}$$

and $\mu_3=\sum_k |k|^3 p_k<\infty$ *implies that*

$$E(L^2(x,\tau_\alpha))<\infty. \tag{3.2}$$

Proof. Obviously

$$L(x,\tau_\alpha)=\sum_{r=0}^{\alpha-1}(L(x,\tau_{r+1})-L(x,\tau_r)). \tag{3.3}$$

Rearrange the sum in (3.3) in the following manner: since the random walk $T_1,T_2,\dots$ visits all points $z=0,\pm 1,\dots$ infinitely many times with probability 1, for each z there is an infinite sequence $\beta_1^z<\beta_2^z<\dots$ such that $T_{\beta_i^z}=z$ and $T_j\neq z$ if $j\neq\beta_i^z$ $(i=1,2,\dots)$. Now define

$$L_i^z(x)=L(x,\tau_{\beta_i^z+1})-L(x,\tau_{\beta_i^z}),$$

i.e. $L_i^z(x)$ is the local time at x of $W(\cdot)$ between two consecutive stopping times τ_s and τ_{s+1}, where s is the i-th visit to z of the random walk $T_1,T_2,\dots$. For given x, the random variables $L_i^z(x)$, $i=1,2,\dots z=0\pm 1,\dots$ are independent and for given z, the variables $L_1^z(x),L_2^z(x),\dots$ are also identically distributed. For convenience we usually replace z by $z+x$ and put

$$H(z)=\sum_{s=1}^{\infty}L_s^{x+z}(x)\,I(\beta_s^{x+z}<\alpha) \tag{3.4}$$

where $I(A)$ is the indicator variable of the event A. We can write

$$L(x,\tau_\alpha)=\sum_{z=-\infty}^{\infty}H(z) \tag{3.5}$$

For brevity, in the sequel we put L_s^{x+z} for $L_s^{x+z}(x)$. From (2.9) and (2.11)

$$E(L_s^{x+z})=\sum_{i=1}^{\infty}\sum_{j=z}^{\infty}\frac{2i(j-z)}{i+j}p_{i,j}=2\sum_{j=z}^{\infty}(j-z)\,p_{-j} \qquad \text{if } z\geqq 0$$

and (3.6)

$$E(L_s^{x+z})=2\sum_{i=-z}^{\infty}(i+z)\,p_i, \qquad \text{if } z<0,\ s=1,2,\dots$$

Put $a_z=E(L_s^{x+z})$. Since L_s^{x+z} and $I(\beta_s^{x+z}<\alpha)$ are independent, we have from (2.13)

$$E(H(z))=a_z E\left(\sum_{s=0}^{\infty}I(\beta_s^{x+z}<\alpha)\right)=a_z E(N(x+z,\alpha))=\alpha_z. \tag{3.7}$$

Hence

$$E(L(x,\tau_\alpha))=\sum_{z=-\infty}^{\infty}a_z=\sum_{j=1}^{\infty}j(j+1)\,p_{-j}+\sum_{i=1}^{\infty}i(i-1)\,p_i=\sigma^2, \tag{3.8}$$

showing (3.1). To show (3.2), assume that $x \neq 0$.

$$H^2(z)=\sum_{s=1}^{\infty}(L_s^{x+z})^2\, I(\beta_s^{x+z}<\alpha)$$
$$+\sum_{s_1 \neq s_2}\sum L_{s_1}^{x+z}\, L_{s_2}^{x+z}\, I(\beta_{s_1}^{x+z}<\alpha, \beta_{s_2}^{x+z}<\alpha). \tag{3.9}$$

Therefore, by using (2.10), (2.11), (2.13) and (2.16), for $z \geqq 0$, we obtain

$$\begin{aligned} E(H^2(z)) &\leqq \sum_{i=1}^{\infty}\sum_{j=z}^{\infty}\frac{8i(j-z)^2(z+i)}{(i+j)^2}\, p_{i,j} \\ &\quad + a_z^2\, E(N^2(x+z,\alpha)) \\ &\leqq 8\sum_{j=z}^{\infty}(j-z)^2\, p_{-j}+K_2'(|x|+|z|)\, a_z^2 \\ &\leqq 8\sum_{|i|\geqq|z|} i^2\, p_i+K_2'(|x|+|z|)\, a_z^2. \end{aligned} \tag{3.10}$$

The same holds for $z<0$, i.e.

$$E(H^2(z)) \leqq 8\sum_{|i|\geqq|z|} i^2\, p_i+K_2'(|x|+|z|)\, a_z^2. \tag{3.11}$$

By using the inequality

$$E(N(x+z_1,\alpha)\, N(x+z_2,\alpha)) \leqq (E(N^2(x+z_1,\alpha))\, E(N^2(x+z_2,\alpha)))^{1/2}, \tag{3.12}$$

one can see that

$$E(H(z_1)\, H(z_2)) \leqq K_2'(|x|+|z_1|)^{1/2}\,(|x|+|z_2|)^{1/2}\, a_{z_1} a_{z_2}. \tag{3.13}$$

Therefore

$$\begin{aligned} E(L^2(x,\tau_\alpha)) &= \sum_{z=-\infty}^{\infty} E(H^2(z))+\sum_{z_1 \neq z_2}\sum E(H(z_1)\, H(z_2)) \\ &\leqq C\left(\sum_{i=-\infty}^{\infty}|i|^3\, p_i+\left(\sum_{z=-\infty}^{\infty}(|x|+|z|)^{1/2}\, a_z\right)^2\right) \\ &\leqq C(\mu_3+(|x|^{1/2}\,\sigma^2+\sum_k |k|^{5/2}\, p_k)^2) \end{aligned} \tag{3.14}$$

with some constant C. This shows (3.2) for $x \neq 0$. Similar procedure shows

$$E(L^2(0,\tau_\alpha)) \leqq C(\mu_3+(\sigma^2+\mu_{5/2})^2). \tag{3.15}$$

This completes the proof of Lemma 1.

Lemma 2. *Assume that* $\mu_3=\sum_k |k|^3\, p_k<\infty$. *Then for fixed* x *and any* $\varepsilon>0$ *we have*

$$\lim_{n\to\infty}\frac{L(x,\tau_n)-\sigma^2\, N(x,n)}{n^{1/4}(\log\log n)^{3/4+\varepsilon}}=0 \qquad a.s. \tag{3.16}$$

Proof. Put $L'_i = L(x, \tau_{\alpha_i}) - L(x, \tau_{\alpha_{i-1}})$, $i = 1, 2, \ldots$ where $\alpha_0 = 0$,

$$\alpha_1 = \alpha, \alpha_i = \min(j\colon T_j = 0, j > \alpha_{i-1}), \qquad i = 2, 3, \ldots.$$

Then L'_i are i.i.d. random variables. From Lemma 1 it follows that $E(L'_i) = \sigma^2$, $E(L'^2_i) < \infty$. By the law of the iterated logarithm,

$$\lim_{k\to\infty} \frac{L'_1 + \ldots + L'_k - k\sigma^2}{k^{1/2}(\log\log k)^{1/2+\varepsilon}} = 0 \quad \text{a.s.} \tag{3.17}$$

Since $N_n = N(0, n) \to \infty$ with probability 1 as $n \to \infty$, we have also

$$\lim_{n\to\infty} \frac{L'_1 + \ldots + L'_{N_n} - N_n\sigma^2}{N_n^{1/2}(\log\log N_n)^{1/2+\varepsilon}} = 0 \quad \text{a.s.} \tag{3.18}$$

By Theorem A, $N_n = O((n \log\log n)^{1/2})$ a.s., therefore

$$\lim_{n\to\infty} \frac{L'_1 + \ldots + L'_{N_n} - \sigma^2 N_n}{n^{1/4}(\log\log n)^{3/4+\varepsilon}} = 0 \quad \text{a.s.} \tag{3.19}$$

Since $L'_1 + \ldots + L'_{N_n} \leqq L(x, \tau_n) \leqq L'_1 + \ldots + L'_{N_n+1}$ and (3.19) remains obviously true if $L'_1 + \ldots + L'_{N_n}$ is replaced by $L'_1 + \ldots + L'_{N_n} + L'_{N_n+1}$, we have also

$$\lim_{n\to\infty} \frac{L(x, \tau_n) - \sigma^2 N_n}{n^{1/4}(\log\log n)^{3/4+\varepsilon}} = 0 \quad \text{a.s.} \tag{3.20}$$

Similar argument shows that (2.13) and the law of the iterated logarithm imply

$$\lim_{n\to\infty} \frac{N(x, n) - N_n}{n^{1/4}(\log\log n)^{3/4+\varepsilon}} = 0 \quad \text{a.s.} \tag{3.21}$$

Now (3.16) follows from (3.20) and (3.21).

Lemma 3. *Assume that* $\mu_{2m} = \sum_k |k|^{2m} p_k < \infty$ *for some* m *with* $1 < m \leqq 2$. *Then for any* $\varepsilon > 0$ *we have*

$$\lim_{n\to\infty} \frac{\sup\limits_{|x| < (n\log n)^{1/2}} |L(x, \tau_n) - L(x, n\sigma^2)|}{n^{1/(2m)+\varepsilon}} = 0 \quad \text{a.s.} \tag{3.22}$$

Proof. Under the condition of Lemma 3, we have $E(\tau_1^m) < \infty$, therefore by the law of the iterated logarithm (if $m = 2$) or by a theorem in Loéve (1963, p. 243) (if $m < 2$),

$$|\tau_n - n\sigma^2| < \tfrac{1}{2} n^{1/m+\varepsilon} \quad \text{a.s.} \tag{3.23}$$

for n large enough. Hence for all x,

$$|L(x, \tau_n) - L(x, n\sigma^2)| \leqq L(x, n\sigma^2 + \tfrac{1}{2} n^{1/m+\varepsilon}) - L(x, n\sigma^2 - \tfrac{1}{2} n^{1/m+\varepsilon}). \tag{3.24}$$

But from (2.7)

$$\begin{aligned} &P(L(x, n\sigma^2 + \tfrac{1}{2} n^{1/m+\varepsilon}) - L(x, n\sigma^2 - \tfrac{1}{2} n^{1/m+\varepsilon}) > n^{1/(2m)+\varepsilon}) \\ &\leqq P(L(0, n^{1/m+\varepsilon}) > n^{1/(2m)+\varepsilon}) = 2(1 - \Phi(n^{\varepsilon/2})) \end{aligned}$$

and

$$P\Big(\sup_{|x|<(n\log n)^{1/2}}\big(L(x,n\sigma^2+\tfrac{1}{2}n^{1/m+\varepsilon})-L(x,n\sigma^2-\tfrac{1}{2}n^{1/m+\varepsilon})\big)$$
$$>n^{1/(2m)+\varepsilon}\Big)\leqq 2(n\log n)^{1/2}(1-\Phi(n^{\varepsilon/2})).$$

Since $\sum_n (n\log n)^{1/2}(1-\Phi(n^{\varepsilon/2}))<\infty$, Borel-Cantelli lemma implies that

$$\lim_{n\to\infty}\frac{\sup\limits_{|x|<(n\log n)^{1/2}}\big(L(x,n\sigma^2+\tfrac{1}{2}n^{1/m+\varepsilon})-L(x,n\sigma^2-\tfrac{1}{2}n^{1/m+\varepsilon})\big)}{n^{1/(2m)+\varepsilon}}=0, \qquad (3.25)$$

hence using (3.24), (3.22) follows.

Lemma 4. *Assume that* $\mu_{m+1}=\sum_k |k|^{m+1}p_k<\infty$ *for some* $m>6$. *Then for any* $\varepsilon>0$ *we have*

$$\lim_{n\to\infty}\frac{\sup\limits_{|x|<(n\log n)^{1/2}}\left|L(x,\tau_n)-\sum\limits_{z=-\infty}^{\infty}a_zN(x+z,n)\right|}{n^{1/4+3/(2m)+\varepsilon}}=0 \quad a.s. \qquad (3.26)$$

Proof. Recall that $a_z=E(L_s^{x+z})$ and

$$L(x,\tau_n)-\sum_{z=-\infty}^{\infty}a_zN(x+z,n)=\sum_{z=-\infty}^{\infty}\sum_{s=1}^{N(x+z,n)}(L_s^{x+z}-a_z), \qquad (3.27)$$

void sum being zero. Put

$$U_k^z=\sum_{s=1}^{k}(L_s^{x+z}-a_z). \qquad (3.28)$$

U_k^z is a partial sum of i.i.d. random variables with mean 0. From (2.10)

$$E((L_s^{x+z})^m)=\frac{2^{m+1}\Gamma(m+1)}{\mu_1}\sum_{i=1}^{\infty}\sum_{j=z}^{\infty}\frac{i(j-z)^m(z+i)^{m-1}}{(i+j)^{m-1}}p_ip_{-j}$$
$$\leqq 2^m\Gamma(m+1)\sum_{j=z}^{\infty}(j-z)^m p_{-j}\leqq 2^m\Gamma(m+1)\sum_{|i|\geqq|z|}|i|^m p_i \qquad (3.29)$$

for $z\geqq 0$ and similar inequality holds for $z<0$. From the inequality of Marcinkiewicz and Zygmund

$$E|U_k^z|^m\leqq C_m k^{m/2}E|L_s^{x+z}-a_z|^m$$
$$\leqq C_m k^{m/2}\Big(\sum_{|i|\geqq|z|}|i|^m p_i+a_z^m\Big)\leqq C_m k^{m/2}\sum_{|i|\geqq z}|i|^m p_i \qquad (3.30)$$

where C_m denotes a constant which may change from line to line. Let

$$V_z=\max_{1\leqq k\leqq n^{1/2+\varepsilon}}|U_k^z|,\ EV_z=v_z,$$
$$V_z'=V_z-v_z.$$

Then we also have

$$E V_z^m \leqq C_m n^{m/4+m\varepsilon/2} \sum_{|i| \geqq |z|} |i|^m p_i. \tag{3.31}$$

In case $1 \leqq m \leqq 2$ by (3.31) one gets

$$E V_z^m \leqq (E V_z^2)^{m/2} \leqq C_m n^{m/4+m\varepsilon/2} (\sum_{|i| \geqq |z|} i^2 p_i)^{m/2}. \tag{3.32}$$

The r.v.'s V_z $(z=0, \pm 1, \pm 2, \dots)$ are independent, hence by the inequality of Burkholder (1973, Theorem 21.1) we have

$$\begin{aligned} E(|\sum_z V_z'|^m) &\leqq C_m [(\sum_z E V_z'^2)^{m/2} + E(\max_z |V_z'|^m)] \\ &\leqq C_m [(\sum_z E V_z^2)^{m/2} + \sum_z E(V_z^m)]. \end{aligned}$$

Consequently, applying (3.31) and (3.32) in case $m=1,2$ we obtain

$$\begin{aligned} E[(\sum_z V_z)^m] &\leqq C_m [(\sum_z E V_z^2)^{m/2} + \sum_z E|V_z|^m + (\sum_z v_z)^m] \\ &\leqq C_m [(n^{1/2+\varepsilon} \sum_z \sum_{|i| \geqq |z|} i^2 p_i)^{m/2} + n^{m/4+m\varepsilon/2} \sum_z \sum_{|i| \geqq |z|} |i|^m p_i] \\ &\leqq C_m n^{m/4+m\varepsilon/2} [(\sum_i |i|^3 p_i)^{m/2} + \sum_i |i|^{m+1} p_i] \leqq C_m n^{m/4+m\varepsilon/2} \end{aligned} \tag{3.33}$$

providing $\mu_{m+1} < \infty$.

The Markov inequality and (3.33) imply

$$\begin{aligned} &P\left\{ \sum_{z=-\infty}^{\infty} \max_{1 \leqq k \leqq n} |U_k^z| > n^{1/4+3/2m+\varepsilon} \right\} \\ &= P\{(\sum_z V_z)^m > n^{m/4+3/2+\varepsilon}\} \leqq C_m n^{-3/2-m\varepsilon/2} \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} &P\left\{ \sup_{|x| < (n \log n)^{1/2}} \sum_{z=-\infty}^{+\infty} \max_{1 \leqq k \leqq n^{1/2+\varepsilon}} |U_k^z| > n^{1/4+3/2m+\varepsilon} \right\} \\ &\leqq C_m \frac{(\log n)^{1/2}}{n^{1+m\varepsilon/2}} \end{aligned} \tag{3.35}$$

therefore by Borel-Cantelli lemma

$$\lim_{n \to \infty} \frac{\sup_{|x| < (n \log n)^{1/2}} \sum_{z=-\infty}^{+\infty} \max_{1 \leqq k \leqq n^{1/2+\varepsilon}} |U_k^z|}{n^{1/4+3/2m+\varepsilon}} = 0 \quad \text{a.s.} \tag{3.36}$$

By Theorem A for sufficiently large n we have $N(n) < n^{1/2+\varepsilon}$ with probability 1 and hence

$$\left|\sum_{z=-\infty}^{\infty}\sum_{s=1}^{N(x+z,n)}(L_s^{x+z}-a_z)\right|\leqq\sum_{z=-\infty}^{\infty}\max_{1\leqq k\leqq n^{1/2+\varepsilon}}|U_k^z|. \tag{3.37}$$

(3.27), (3.28), (3.36) and (3.37) together imply (3.26).

Lemma 5. *For any $\varepsilon>0$ we have*

$$\lim_{n\to\infty}\frac{\sup_x|N(x+1,n)-N(x,n)|}{n^{1/4+\varepsilon}}=0 \quad a.s. \tag{3.38}$$

where $\sup_x$ *is taken over all integers.*

Proof. First we show that for all $m\geqq 2$ and all $\delta>0$ we have the inequality

$$E(|N(1,n)-N(0,n)|^m)\leqq Cn^{\frac{m}{4}+\delta} \tag{3.39}$$

with some constant C (which may depend on m but not on n). Put

$$S_k=\sum_{i=1}^{k}(N(1,\alpha_i)-N(1,\alpha_{i-1})-1), \tag{3.40}$$

where α_i are defined in the proof of Lemma 2. It follows from Theorem I, (see Sect. 2.3) that S_k is a sum of i.i.d. random variables with mean 0 and possessing all moments. Therefore

$$E(|S_k|^m)\leqq C_1k^{\frac{m}{2}} \tag{3.41}$$

and also

$$E(\max_{k\leqq n^{1/2+\delta_1}}|S_k|^m)\leqq C_2n^{\frac{m}{4}+\frac{m\delta_1}{2}}. \tag{3.42}$$

Since

$$\begin{aligned}N(1,\alpha_{N_n})-N(0,n)&\leqq N(1,n)-N(0,n)\\&\leqq N(1,\alpha_{N_n+1})-(N(0,n)+1)+1,\end{aligned} \tag{3.43}$$

where $N_n=N(0,n)$, on the event $\{N_n+1\leqq n^{1/2+\delta_1}\}$ we have

$$|N(1,n)-N(0,n)|\leqq\max_{1\leqq k\leqq n^{1/2+\delta_1}}|S_k| \tag{3.44}$$

and hence

$$E(|N(1,n)-N(0,n)|^m)\leqq E(\max_{1\leqq k\leqq n^{1/2+\delta_1}}|S_k|^m)+n^m P(N_n+1\geqq n^{1/2+\delta_1}). \tag{3.45}$$

Using the fact that $E(N_n^m)=O(n^{m/2})$ (see Kesten and Spitzer (1979)), from (3.42) and (3.45) we obtain (3.39). The same inequality holds with $N(1,n)$ replaced by $N(-1,n)$ and it is easily seen that for all integral x,

$$E(|N(x+1,n)-N(x,n)|^m)\leqq Cn^{m/4+\delta}, \tag{3.46}$$

where the constant C does not depend on x, because $N(x+1,n)-N(x,n)$ is stochastically smaller than $\max(|N(1,n)-N(0,n)|,\ |N(-1,n)-N(0,n)|)$.

On choosing $m=(2+\delta)/\varepsilon$ we obtain by Markov's inequality

$$P(|N(x+1,n)-N(x,n)|>n^{1/4+\varepsilon})\leqq\frac{C}{n^2} \tag{3.47}$$

and hence

$$P(\sup_{|x|<(n\log n)^{1/2}}|N(x+1,n)-N(x,n)|>n^{1/4+\varepsilon}) \leqq\frac{2C(\log n)^{1/2}}{n^{3/2}} \tag{3.48}$$

which by Borel-Cantelli lemma, implies that

$$\lim_{n\to\infty}\frac{\sup_{|x|\leqq(n\log n)^{1/2}}|N(x+1,n)-N(x,n)|}{n^{1/4+\varepsilon}}=0 \quad \text{a.s.} \tag{3.49}$$

The law of the iterated logarithm for T_n implies that $N(x,n)=0$ a.s. for $|x|\geqq(n\log n)^{1/2}$ and n sufficiently large, hence (3.38) follows.

Now we are ready to prove our Theorem. Since

$$L(x,n\sigma^2)-\sigma^2 N(x,n)=(L(x,n\sigma^2)-L(x,\tau_n))+(L(x,\tau_n)-\sigma^2 N(x,n)), \tag{3.50}$$

(1.22) follows from Lemma 2 and Lemma 3.

To show (1.23) we use the identity

$$\begin{aligned} L(x,n\sigma^2)-\sigma^2 N(x,n)&=(L(x,n\sigma^2)-L(x,\tau_n))\\ &\quad+\left((L(x,\tau_n))-\sum_{z=-\infty}^{\infty}a_z N(x+z,n)\right)\\ &\quad+\left(\sum_{z=-\infty}^{\infty}a_z(N(x+z,n)-N(x,n))\right) \end{aligned} \tag{3.51}$$

and the estimation

$$\begin{aligned} &\left|\sum_{z=-\infty}^{\infty}a_z(N(x+z,n)-N(x,n))\right|\\ &\leqq\sum_{z=-\infty}^{\infty}|z|\,a_z\sup_x|N(x+1,n)-N(x,n)|\\ &\leqq\mu_3\sup_x|N(x+1,n)-N(x,n)|. \end{aligned}$$

From Lemma 3, Lemma 4 and Lemma 5 it follows that

$$\lim_{n\to\infty}\frac{\sup_{|x|<(n\log n)^{1/2}}|L(x,n\sigma^2)-\sigma^2 N(x,n)|}{n^{1/4+3/(2m)+\varepsilon}}=0 \quad \text{a.s.} \tag{3.52}$$

The law of iterated logarithm for W and T_n implies that a.s. $L(x,n\sigma^2)=N(x,n)=0$ for $|x|\geqq(n\log n)^{1/2}$ and n sufficiently large, hence (1.23) follows. (1.24) follows from Strassen's theorem.

4. Remarks and Consequences

For fixed x we have the rate $n^{1/4+\varepsilon}$ provided $\mu_4<\infty$ and for $\sup\limits_x$ we have the same rate provided all moments exist. We do not know whether n^ε can be replaced by some logarithmic factor, and whether our moment conditions can be weakened. The rate $n^{1/4+\varepsilon}$ however, can not be replaced by $o(n^{1/4})$ for any construction, because by a result of Dobrushin (1955) for the simple symmetric random walk,

$$\frac{M(1,n)-M(0,n)}{2^{1/2}n^{1/4}}\xrightarrow{\mathcal{D}} Z_1\,|Z_2|^{1/2} \quad \text{as } n\to\infty \tag{4.1}$$

where $\xrightarrow{\mathcal{D}}$ means convergence in distribution, Z_1 and Z_2 are independent standard normal random variables. It can be shown similarly that

$$\frac{L(1,n)-L(0,n)}{2n^{1/4}}\xrightarrow{\mathcal{D}} Z_1\,|Z_2|^{1/2} \quad \text{as } n\to\infty \tag{4.2}$$

Our Theorem implies that all statements of the Introduction formulated to any of the sequences $M(x,n)$, $\sigma^2 N(x,n)$, $L(x,n\sigma^2)$ (resp. $M(n)$, $\sigma^2 N(n)$, $L(n\sigma^2)$) will be true to the other two sequences under the conditions of our Theorem. In particular the statements of Theorems C and D remain true if $M(x,n)$ is replaced by $L(x,n)$ or $\sigma N(x,n)$. Furthermore

$$\liminf_{n\to\infty}\left(\frac{\log\log n}{n}\right)^{1/2}\sigma N(n)=\gamma_1 \quad \text{a.s.,} \tag{4.3}$$

where γ_1 is the constant of Theorem B.

Moreover our Theorem implies also that the functional law of the iterated logarithm for $L(x,t)$ due to Donsker and Varadhan (1977) is inherited by $N(x,n)$.

Corollary. *Assume* (1.1) *and* $\mu_m=\sum\limits_k |k|^m p_k<\infty$, *for some* $m>7$. *Then the set of limit points* (*in the topology of* $C(-\infty,\infty)$) *of the functions*

$$(n\log\log n)^{-1/2}\,\sigma N\left(\left[x\sigma\left(\frac{n}{\log\log n}\right)^{1/2}\right],n\right)\quad -\infty<x<\infty \tag{4.4}$$

consists of those and only those subprobability density functions $f(x)$ *for which*

$$\frac{1}{8}\int\frac{(f'(x))^2}{f(x)}\,dx\leqq 1. \tag{4.5}$$

The occupation time $N(x,n)$ of the random walk $T_1, T_2, \ldots$ was defined by (1.2). In order to characterize the time spent at x by the random walk we may also introduce the random variables

$$N_1(x,n)=\#\{t\colon 0<t\leqq n,\ T(t)=x\}, \tag{4.6}$$

$$N_2(x,n)=\lim_{\varepsilon\to 0}(2\varepsilon)^{-1}\,\lambda\{t\colon 0\leqq t\leqq n,\ |T(t)-x|\leqq\varepsilon\}, \tag{4.7}$$

where λ is the Lebesgue measure and $T(t)$ is the stochastic process defined by

$$T(t)=T_k+(t-k)\,Y_{k+1}, \quad \text{if } k\leqq t<k+1,\ (k=0,1,2,\ldots). \tag{4.8}$$

The connection between N_1, N_2 and N is given in the following

Lemma 6. *Assume* (1.1). *Then for any* $\varepsilon>0$

$$\lim_{n\to\infty} n^{-1/4-\varepsilon} \sup_x |N_1(x,n)-\mu_1 N(x,n)|=0 \quad a.s. \tag{4.9}$$

and

$$\lim_{n\to\infty} n^{-1/4-\varepsilon} \sup_x |N_2(x,n)-N(x,n)|=0 \quad a.s. \tag{4.10}$$

where $\mu_1=\sum_k |k|\,p_k$ *and* $\sup_x$ *is taken for all integers.*

The proof runs along the same lines as the proof of our theorem. Define the following variables:

$$Q_s^{x+z}(x)=\begin{cases} I(Y_{\beta_s^{x+z}+1}<-z) & \text{if } z>0\\ 1 & \text{if } z=0\\ I(Y_{\beta_s^{x+z}+1}>-z) & \text{if } z<0\end{cases} \tag{4.11}$$

and

$$R_s^{x+z}(x)=\begin{cases} |Y_{\beta_s^{x+z}+1}|^{-1}\, I(Y_{\beta_s^{x+z}+1}<-z) & \text{if } z>0\\ \frac{1}{2}(Y_{\beta_s^x}^{-1}+Y_{\beta_s^x+1}^{-1}) & \text{if } z=0\\ Y_{\beta_s^{x+z}+1}^{-1}\, I(Y_{\beta_s^{x+z}+1}>-z) & \text{if } z<0.\end{cases} \tag{4.12}$$

Then the following identities hold:

$$N_1(x,n)=\sum_{z=-\infty}^{\infty}\ \sum_{s=1}^{N(x+z,n)} Q_s^{x+z}(x), \tag{4.13}$$

$$N_2(x,n)=\sum_{z=-\infty}^{\infty}\ \sum_{s=1}^{N(x+z,n)} R_s^{x+z}(x). \tag{4.14}$$

Moreover,

$$A_z=E(Q_s^{x+z}(x))=\begin{cases} \sum_{i=z+1}^{\infty} p_{-i} & \text{if } z>0\\ 1 & \text{if } z=0\\ \sum_{i=-z+1}^{\infty} p_i & \text{if } z<0\end{cases} \tag{4.15}$$

$$B_z=E(R_s^{x+z}(x))=\begin{cases} \sum_{i=z+1}^{\infty} \frac{p_{-i}}{i} & \text{if } z>0\\ \sum_{i=-\infty}^{\infty} \frac{p_i}{|i|} & \text{if } z=0\\ \sum_{i=-z+1}^{\infty} \frac{p_i}{i} & \text{if } z<0.\end{cases} \tag{4.16}$$

It is easy to see that $\sum_{z=-\infty}^{\infty} A_z = \mu_1$ and $\sum_{z=-\infty}^{\infty} B_z = 1$, hence

$$N_1(x,n) - \mu_1 N(x,n) = \sum_{z=-\infty}^{\infty} \sum_{s=1}^{N(x+z,n)} (Q_s^{x+z}(x) - A_z) + \sum_{z=-\infty}^{\infty} A_z (N(x+z,n) - N(x,n)) \tag{4.17}$$

and

$$N_2(x,n) - N(x,n) = \sum_{z=-\infty}^{\infty} \sum_{s=1}^{N(x+z,n)} (R_s^{x+z}(x) - B_z) + \sum_{z=-\infty}^{\infty} B_z (N(x+z,n) - N(x,n)). \tag{4.18}$$

The first sums in (4.17) and (4.18) can be treated similarly to the proof of Lemma 4, while for the second sums we refer to Lemma 5.

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