Maxima of Branching Random Walks

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Summary. In recent years several authors have obtained limit theorems for L_n, the location of the rightmost particle in a supercritical branching random walk but all of these results have been proved under the assumption that the offspring distribution has $\varphi(\theta) = \int \exp(\theta x) \, dF(x) < \infty$ for some $\theta > 0$. In this paper we investigate what happens when there is a slowly varying function K so that $1 - F(x) \sim x^{-q} K(x)$ as $x \to \infty$ and $\log(-x) F(x) \to 0$ as $x \to -\infty$. In this case we find that there is a sequence of constants a_n, which grow exponentially, so that L_n/a_n converges weakly to a nondegenerate distribution. This result is in sharp contrast to the linear growth of L_n observed in the case $\varphi(\theta) < \infty$.

Introduction

Consider a supercritical branching random walk in R^1 with an offspring distribution p_k which has $\sum k \, p_k = m < \infty$ and assume that the displacements of the offspring from the parent are independent and have a distribution F. Let L_n be the position of the right most particle alive at time n and set $L_n = -\infty$ if the nth generation is empty. In this paper we will prove a limit theorem for L_n under the assumption that there is a slowly varying function K so that

$$1 - F(x) \sim x^{-q} L(x) \quad \text{as } x \to \infty$$

and

$$\log(-x) F(x) \to 0 \quad \text{as } x \to -\infty.$$

To describe these results we need to introduce some notation. By a result of Seneta and Heyde (see Athreya and Ney (1972), p. 30) we can pick a sequence $c_n \to \infty$ so that $Z_n/c_n \to W$ where $P(W=0) = P(Z_n \to 0)$. Since $c_n \to \infty$ and $1 - F(x) \sim x^{-q} K(x)$ as $x \to \infty$ we can pick a_n so that $c_n(1 - F(a_n)) \to 1$. Having selected a_n in this way we obtain the following limit theorem:

Theorem 1. *For all $x > 0$ we have*

$$P(L_n \leqq a_n x) \to \int_0^\infty P(r W \in dy) \exp(-y x^{-q})$$

where

$$r = \sum_{j=0}^{\infty} m^{-j} P(Z_j > 0).$$

In the theorem above if $\sum (k \log k) p_k < \infty$ we can take $c_n = m^n$ (see Athreya and Ney, p. 24) and if $1 - F(x) \sim x^{-q}$ we can let $a_n = c_n^{1/q}$. If both conditions hold we have $a_n = m^{n/q}$ so the results in this case are much different from those obtained by Bramson (private communication, see [3] for some related results). He has shown that if the conditions of Bahadur and Rao (1960) are satisfied and F assigns a probability less than $1/m$ to $\sup\{x : F(x) < 1\}$ then there is a sequence of constants $b_n \sim c\,n$ so that $L_n - b_n$ is tight.

To see why our results are different from Bramson's consider M_n the maximum displacement experienced by a particle which has offspring alive at time n. When $1 - F(x) \sim x^{-q} K(x)$ we have that

$$P(M_n \leq a_n x) \to \int_0^{\infty} P(r\,W \in dy) \exp(-y x^{-q})$$

so the position of the rightmost particle is determined by the largest displacement. On the other hand if we have a bounded distribution (which is one of the possibilities in Bramson's theorem) then the rightmost particle gets to its location by a sequence of small jumps.

Section 2

In this section we will prove Theorem 1. The proof will be accomplished in four steps:

1. Let Z_k^n be the number of particles alive at time k which have offspring alive at time n and let $Y_n = Z_1^n + \ldots + Z_n^n$. As $n \to \infty$, $Y_n/c_n \to r\,W$ almost surely.

2. Let $X_{i,j}\, 1 \leq i, j < \infty$ be a collection of independent and identically distributed random variables with distribution F, which are defined so that $X_{n,1}, \ldots, X_{n,Z_n}$ were the steps taken by the Z_n offspring in the nth generation. Let $M_n = \max\{X_{i,j}, 1 \leq i < n, 1 \leq j \leq Z_i$, particle j has an offspring alive at time $n\}$

$$P(M_n < a_n x) = \int_0^{\infty} P\left(\frac{Y_n}{c_n} \in dy\right) F(a_n x)^{y c_n} \to \int_0^{\infty} P(r\,W \in dy) \exp(-y x^{-q}).$$

3. If $0 < \varepsilon < x$ then

$$P(L_n > a_n x) \leq P(M_n > a_n(x - \varepsilon)) + P(L_n > a_n x, M_n \leq a_n(x - \varepsilon))$$

and

$$P(L_n > a_n x, M_n \leq a_n(x - \varepsilon)) \to 0$$

4. If $\varepsilon > 0$ then

$$\liminf_{n \to \infty} \frac{P(L_n > a_n x)}{P(M_n \geq a_n(x + \varepsilon))} \geq 1.$$

Combining 3 and 4 with 2 shows that

$$\lim_{n \to \infty} P(L_n > a_n x)/P(M_n > a_n x) = 1$$

and proves the theorem. The rest of the paper is devoted to showing 1–4.

1. It is clear that for any fixed j we have

$$Z_{n-j}/c_{n-j} \to W \quad \text{and} \quad Z_{n-j}^n/Z_{n-j} \to P(Z_j > 0)$$

almost surely on $\{W > 0\}$ so for every positive integer N

$$\sum_{i=n-N}^{n} Z_i^n/c_n \to W \sum_{j=0}^{N} m^{-j} P(Z_j > 0).$$

To estimate the remainder we observe that from the proof of Seneta's result (see [1], p. 30) $c_{n+1}/c_n \uparrow m$ and $c_2/c_1 > 1$ so

$$0 \leq \sum_{i=1}^{n-N-1} Z_i^n/c_n \leq (\sup_{i \geq 1} Z_i/c_i) \sum_{n=N+1}^{\infty} \left(\frac{c_n}{c_1}\right)^{-1}.$$

Letting $N \to \infty$ now shows

$$Y_n/c_n \to W \sum_{j=0}^{\infty} m^{-j} P(Z_j > 0) = r W.$$

2. Since the particle displacements are independent of the branching process we have

$$P(M_n < a_n x) = E(F(a_n x)^{Y_n}) = \int_0^{\infty} P\left(\frac{Y_n}{c_n} \in dy\right) F(a_n x)^{y c_n}.$$

For fixed y

$$y\, c_n (1 - F(a_n x)) = y(c_n (1 - F(a_n))) \frac{1 - F(a_n x)}{1 - F(a_n)} \to y\, x^{-q}$$

so

$$F(a_n x)^{y c_n} \to \exp(-y x^{-q}).$$

Now $F(a_n x) < 1$ so $F(a_n x)^{y c_n}$ is a nonnegative decreasing function. From this it follows that

$$P(M_n < a_n x) \to \int_0^{\infty} P(r W \in dy) \exp(-y x^{-q}).$$

3. Let L_n be the position of the rightmost particle alive at time n. Let $0 < \varepsilon < x$

$$P(L_n > a_n x) \leq P(M_n > a_n(x - \varepsilon)) + P(L_n > a_n x, M_n \leq a_n(x - \varepsilon))$$

$$P(L_n > a_n x, M_n \leq a_n(x - \varepsilon)) \leq E(\eta_n(a_n x, \infty); M_n \leq a_n(x - \varepsilon))$$

where $\eta_n(a_n x, \infty)$ is the number of particles in $(a_n x, \infty)$ at time n.

We want to show that $P(L_n > a_n x, M_n \leq a_n(x-\varepsilon)) \to 0$. To do this we need to introduce the truncated distribution function. Let $F^y(x) = F(x) \wedge F(y)$ and let F_n^y be the nth convolution of F^y. It is easy to see that if S_n is a random walk which takes steps with distribution F then

$$F_n^y(x) = P(S_n \leq x, \sup_{1 \leq j \leq n} S_j - S_{j-1} < y)$$

and $F_n^{a_n(x-\varepsilon)}(a_n x)$ is the probability a given particle in the nth generation is in $(-\infty, a_n x]$ and all its ancestors took steps of size $\leq a_n(x-\varepsilon)$. From this it follows that

$$E(\eta_n(a_n x, \infty); M_n < a_n(x-\varepsilon)) \leq m^n(F_n^{a_n(x-\varepsilon)}(\infty) - F_n^{a_n(x-\varepsilon)}(a_n x)).$$

To estimate the right hand side we will use the following result which was proved by Durrett (1979), Sect. 3).

Lemma. *Suppose* (a) $p \geq 1$, $E(X_1^+)^p < \infty$, $E(X_1^-)^2 < \infty$, *and* $EX_1 = 0$ *or* (b) $0 < p < 1$ $E(X_1^+)^p < \infty$. *If* $x_n/(n^{1/(p \wedge 2)} \log n) \to \infty$ *and* $y_n = r x_n$ *then there is a constant* K_p *so that*

$$F_n^{y_n}(\infty) - F_n^{y_n}(x_n) \leq 3(n K_p/y_n^p)^{1/r}.$$

for all n sufficiently large.

To apply the lemma we take $p < q$, $x_n = a_n x$ and $r = (x-\varepsilon)/x$. If $q \leq 1$ then $p < 1$ and we have

$$F_n^{a_n(x-\varepsilon)}(\infty) - F_n^{a_n(x-\varepsilon)}(a_n x) \leq 3 \left(\frac{n K_p}{a_n^p(x-\varepsilon)^p} \right)^{x/x-\varepsilon}$$

for all n sufficiently large. To apply the lemma when $q > 1$ we have to truncate the distribution. Let G be the distribution of $X_1^+ - EX_1^+$. It is easy to see that

$$G_n^{y_n}(\infty) - G_n^{y_n}(x_n - nEX_1^+) \geq G_n^{y_n - EX_1^+}(\infty) - G_n^{y_n - EX_1^+}(x_n - nEX_1^+)$$
$$\geq F_n^{y_n}(\infty) - F_n^{y_n}(x_n).$$

Applying the lemma now to G gives that if $\delta > 0$ we have that

$$G_n^{a_n(x-\varepsilon)}(\infty) - G_n^{a_n(x-\varepsilon)}(a_n x(1-\delta)) \leq 3 \left(\frac{n K_p}{a_n^p(x-\varepsilon)^p} \right)^{x(1-\delta)/(x-\varepsilon)}$$

for all n sufficiently large. Combining this with the inequalities above shows that if $q > 1$, $p < q$, and $\delta > 0$ then

$$F_n^{a_n(x-\varepsilon)}(\infty) - F_n^{a_n(x-\varepsilon)}(a_n x) \leq 3 \left(\frac{n K_p}{a_n^p(x-\varepsilon)^p} \right)^{x(1-\delta)/x-\varepsilon} \tag{1}$$

for all n sufficiently large. Since this result is weaker than the conclusion we have in the case $q \leq 1$, (1) also holds in that case.

To complete the proof at this point we want to show that there is a $p < q$ and $\delta > 0$ so that

$$3m^n \left(\frac{n K_p}{a_n^p(x-\varepsilon)^p} \right)^{x(1-\delta)/x-\varepsilon} \to 0.$$

To do this we need an expression for a_n. Now L is a slowly varying function so from Feller (1971, p. 277) we have that if $r>q$ then $(1-F(x))\geqq 2x^{-r}$ for all x sufficiently large

$$a_n \geqq \frac{2}{c_n(1-F(a_n))}^{1/r} c_n^{1/r} \geqq c_n^{1/r}.$$

Using this inequality gives that m^n times the right hand side of (1) is

$$\leqq \left(\frac{nK_p}{(x-\varepsilon)^p}\right)^{x(1-\delta)/x-\varepsilon} \cdot m^n/c_n^{\frac{px(1-\delta)}{r(x-\varepsilon)}}.$$

Now since $\varepsilon>0$ we can pick $\delta>0$ and $r>q>p$ so that $px(1-\delta)/r(x-\varepsilon)=\rho>1$. If we do this and take logarithms we have

$$\log E(\eta_n(a_n x, \infty); M_n < a_n(x-\varepsilon))$$
$$\leqq \frac{x(1-\delta)}{x-\varepsilon}\log\left(\frac{nK_p}{(x-\varepsilon)^p}\right) + n\log m - \rho\log c_n. \qquad (2)$$

Now $\log(c_n/c_{n-1})\to\log m$ and $\rho>1$ so the right hand side of (2) $\to-\infty$. This shows that

$$E(\eta_n(a_n x, \infty; M_n\leqq a_n(x-\varepsilon))\to 0$$

which proves the desired result.

4. The last step in the proof is to show that if $\log(-x)F(x)\to 0$ as $x\to\infty$ then

$$\liminf_{n\to\infty}\frac{P(L_n > a_n x)}{P(M_n > a_n(x+\varepsilon))}\geqq 1. \qquad (3)$$

Let I_n, J_n be defined so that $X_{I_n, J_n}=M_n$. From the definition of M_n we know that this particle has at least one descendant alive at time n. Let $\bar{X}_1, \ldots, \bar{X}_n$ be the displacements experienced by his ancestors who were alive at times $1, \ldots, n$, and let $\bar{F}^y(x)=F^y(x)/F^y(\infty)$. Now $\bar{x}_{I_n}=M_n$ and the $\bar{X}_i, i\neq I_n$ are independent and have distribution $\bar{F}^{M_n}$ so to prove (3) it suffices to show

$$\bar{F}_n^{M_n}(-a_n\varepsilon)\to 0.$$

To do this we observe

$$\bar{F}_n^{M_n}(-a_n\varepsilon)\leqq\bar{F}_n^0(-a_n\varepsilon)+P(M_n\leqq 0)$$

and $P(M_n\leqq 0)\to 0$ so it suffices to show that $\bar{F}_n^0(-a_n\varepsilon)\to 0$ or that if S_n^0 is a random walk which takes steps with distribution $\bar{F}^0$ then $S_n/a_n\to 0$ in probability.

By the degenerate convergence criterion (see [8], p. 124) this occurs if and only if

(i) $n\bar{F}^0(-a_n\varepsilon)\to 0$ for all $\varepsilon>0$

(ii) $na_n^{-1}\int_{-a_n}^0 x^2\,d\bar{F}^0(x)\to 0$ and

(iii) $na_n^{-1}\int_{-a_n}^0 x\,d\bar{F}^0(x)\to 0.$

To check these conditions we begin by observing that

$$n\bar{F}^0(-a_n\varepsilon)=\frac{n}{\log(a_n\varepsilon)}\cdot\log(a_n\varepsilon)\,\bar{F}^0(-a_n\varepsilon).$$

Now from computations above we have that if $r>q$, $a_n>c_n^{1/r}$ so if we let $c_0=1$ then

$$\frac{\log(a_n\varepsilon)}{n}\gtrapprox\frac{\log\varepsilon+r^{-1}\log c_n}{n}=\frac{\log\varepsilon+r^{-1}\sum\limits_{k=1}^{n}\log(c_k/c_{k-1})}{n}\to\frac{\log m}{r}.$$

By our assumption we have $\log(a_n\varepsilon)\,\bar{F}^0(-a_n\varepsilon)\to 0$ so this shows condition (i) is satisfied.

To show that (ii) and (iii) are satisfied we observe that

$$n\,a_n^{-2}\int\limits_{-a_n}^{0}x^2\,d\bar{F}^0(x)\leqq-n\,a_n^{-2}\int\limits_{-a_n}^{0}2x\bar{F}^0(x)\,dx\leqq 2n\,a_n^{-1}\int\limits_{-a_n}^{0}\bar{F}^0(x)\,dx$$

and the last expression is also an upper bound for the negative of the expression in (iii) so it suffices to show that the right hand side of the last inequality converges to 0. To do this we write the integral as

$$2n\,a_n^{-1}\int\limits_{-a_n}^{-a_n^{1/2}}\bar{F}^0(x)\,dx+2n\,a_n^{-1}\int\limits_{-a_n^{1/2}}^{0}\bar{F}^0(x)\,dx\leqq 2n\bar{F}^0(-a_n^{1/2})+2n\,a_n^{-1/2}.$$

The first term $\to 0$ by the computation used to prove (i). The second term $\to 0$ since $a_n\to\infty$ exponentially rapidly.

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