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Multiply Self-Decomposable Probability Measures on $\mathbb{R}_+$ and $\mathbb{Z}_+$

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Summary. Self-decomposable probability measures μ on $\mathbb{R}_+$ are characterized in terms of minus the logarithm of the Laplace transform of μ, say f, by the requirement that $s \mapsto s f'(s)$ is again minus the logarithm of the Laplace transform of an infinitely divisible probability on $\mathbb{R}_+$. Iteration of this condition yields characterizations in the case of $\mathbb{R}_+$ of Urbanik's classes L_n of multiply self-decomposable probabilities. The analogous characterization for discrete (multiply) self-decomposable probabilities on $\mathbb{Z}_+$ is discussed and used to give a representation of the generating functions for discrete completely self-decomposable probabilities on $\mathbb{Z}_+$. Classes of generalized Γ-convolutions analogous to the multiply self-decomposable probabilities on $\mathbb{R}_+$ are studied as well as their discrete counterparts.

Introduction

In a study of limit laws for certain classes of normalized sums of real-valued random variables Urbanik [17] introduced classes L_n, $n=0,1,\ldots,\infty$, of multiply self-decomposable probability measures on $\mathbb{R}$ and characterized the probabilities in these classes by their Lévy measures. This led via the Lévy-Hinčin formula to integral representations of the (logarithm of) the Fourier transforms of these measures.

The classes L_n were further studied by Kumar and Schreiber [10] who gave characterizations of L_n by "monotonicity" properties of the associated Lévy measures, and used Choquet theory to obtain Urbanik's integral representations.

Multiply self-decomposable probabilities on more general spaces have been considered in [11]–[14].

The purpose of the present paper is to study decomposability properties of probabilities on $\mathbb{R}_+$ and $\mathbb{Z}_+$ directly in terms of (the logarithm of) their Laplace transform respectively their generating functions.

In §1 we start by giving a simple and useful characterization of self-decomposable probabilities on $\mathbb{R}_+$. The condition for self-decomposability of μ

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is that, with $f=-\log \mathscr{L}\mu$ ($\mathscr{L}\mu$ the Laplace transform of μ), the function $s \mapsto s f'(s)$ is again minus the logarithm of the Laplace transform of an infinitely divisible probability on $\mathbb{R}_+$. Iteration of this condition yields characterizations of L_n for $n=0,1,2,\ldots,\infty$. The results are of course consequences of Urbanik's integral representations, but being so simple we have preferred to establish them directly. Also the integral representations and the characterizations in terms of Lévy measures take very simple forms in the special case of $\mathbb{R}_+$.

In § 2 we study the analogues on $\mathbb{Z}_+$ of multiply self-decomposable probabilities. Our study is based on the notion of discrete self-decomposability due to Steutel and van Harn [15]. As in the case of $\mathbb{R}_+$ we obtain characterizations of the (logarithm of the) generating functions for these probabilities in terms of iterates of a certain differential operator, and we give their integral representations.

There is a strong similarity between the self-decomposability classes on $\mathbb{R}_+$ and on $\mathbb{Z}_+$ and one way of expressing this similarity is in terms of mixtures of Poisson distributions as discussed in § 3.

Finally in § 4 we consider self-decomposability properties of generalized Γ-convolutions on $\mathbb{R}_+$ and their discrete analogues on $\mathbb{Z}_+$, the generalized negative binomial convolutions.

§ 1. Multiply Self-Decomposable Probabilities on $\mathbb{R}_+$

Let $P(\mathbb{R}_+)$ be the set of probability measures on $\mathbb{R}_+=[0,\infty[$ and let $\mathscr{B}$ denote the set of C^∞-functions $f\colon]0,\infty[\to\mathbb{R}_+$ for which f' is completely monotone (in [2] called the set of *Bernstein functions*).

Denoting by $\mathscr{L}\mu$ the Laplace transform of $\mu\in P(\mathbb{R}_+)$, it is well known that the mapping $\mu\mapsto -\log\mathscr{L}\mu$ defines a one-to-one correspondence between the set $I(\mathbb{R}_+)$ of *infinitely divisible* probabilities on $\mathbb{R}_+$ and the set $\mathscr{B}_{-1}=\{f\in\mathscr{B}\mid f(0^+)=0\}$, cf. Feller [7] (see also [3]).

Furthermore a function f belongs to $\mathscr{B}_{-1}$ if and only if it has the form

$$f(s)=bs+\int_0^\infty (1-e^{-xs})\,d\nu(x) \qquad \text{for } s\geqq 0, \tag{1}$$

where $b\geqq 0$ and $\nu\geqq 0$ is a measure on $]0,\infty[$ such that the integral in (1) is finite for $s\geqq 0$ (i.e. such that $x\mapsto\min(x,1)$ is ν-integrable). The couple (b,ν) in (1) is uniquely determined by f and is called the *representing couple* for f; ν is the *Lévy measure* for $\mu\in I(\mathbb{R}_+)$, where $f=-\log\mathscr{L}\mu$.

A probability $\mu\in P(\mathbb{R}_+)$ is called *self-decomposable* if for each $c\in]0,1[$ there exists $\mu_c\in P(\mathbb{R}_+)$ such that

$$\mu=(T_c\mu)*\mu_c, \tag{2}$$

where $T_c\mu$ is the image measure of μ by the mapping $T_c\colon x\mapsto cx$ of $\mathbb{R}_+$ into $\mathbb{R}_+$. The measure μ_c in (2) is uniquely determined and is called the *c-component* of μ. The set of self-decomposable probabilities on $\mathbb{R}_+$ is denoted $L(\mathbb{R}_+)$.

It is well known that $L(\mathbb{R}_+)\subseteq I(\mathbb{R}_+)$ and that the c-components of any $\mu\in L(\mathbb{R}_+)$ are also infinitely divisible.

Let $\mu\in P(\mathbb{R}_+)$ with $f=-\log\mathscr{L}\mu$ and $c\in]0,1[$. Then $-\log\mathscr{L}(T_c\mu)=f\circ T_c$, and it follows that $\mu\in L(\mathbb{R}_+)$ if and only if $f-f\circ T_c\in\mathscr{B}$ for all $c\in]0,1[$ and that in this case the c-component μ_c of μ corresponds to the function $f-f\circ T_c$.

Consider now the differential operator S on $]0,\infty[$ given by

$$S\,h(s)=s\,h'(s) \qquad \text{for } s>0.$$

1.1. **Lemma.** *Let f: $]0,\infty[\to\mathbb{R}$ be a C^1-function. Then $Sf\in\mathscr{B}$ if and only if the function $f-f\circ T_c$ belongs to $\mathscr{B}$ for all $c\in]0,1[$.*

Proof. The set $\mathscr{B}$ is a convex cone of functions which is closed in the topology of pointwise convergence on $]0,\infty[$, cf. [2], and stable under composition with the family of mappings $\{T_c|c>0\}$. The assertion now follows from the formulas

$$Sf(s)=sf'(s)=\lim_{c\to 1^-}\frac{f(s)-f(c\,s)}{1-c},$$

$$f(s)-f(c\,s)=\int_c^1\frac{d}{du}f(u\,s)\,du=\int_c^1(Sf)(u\,s)\frac{1}{u}\,du,$$

valid for all $s>0$ and $c\in]0,1[$. □

1.2. **Theorem.** *Let $\mu\in P(\mathbb{R}_+)$ with $f=-\log\mathscr{L}\mu$. Then μ is self-decomposable if and only if $Sf\in\mathscr{B}$.*

Proof. This follows immediately from Lemma 1.1 and the preceding discussion. □

The c-components of a self-decomposable probability are not in general self-decomposable, and we will therefore consider the following classes of *multiply self-decomposable* probabilities on $\mathbb{R}_+$ (cf. Urbanik [17])

$$L_0(\mathbb{R}_+)\supseteq L_1(\mathbb{R}_+)\supseteq\dots \quad\text{and}\quad L_\infty(\mathbb{R}_+)=\bigcap_{n=0}^{\infty}L_n(\mathbb{R}_+),$$

where $L_0(\mathbb{R}_+)=L(\mathbb{R}_+)$ and $L_{n+1}(\mathbb{R}_+)$ for $n\geq 0$ is defined inductively by

$$L_{n+1}(\mathbb{R}_+)=\{\mu\in L_n(\mathbb{R}_+)|\forall c\in]0,1[:\ \mu_c\in L_n(\mathbb{R}_+)\}.$$

Theorem 1.2 can now be generalized.

1.3. **Theorem.** *Let $\mu\in P(\mathbb{R}_+)$ with $f=-\log\mathscr{L}\mu$. Then*

$$\mu\in L_n(\mathbb{R}_+)\Leftrightarrow S^kf\in\mathscr{B} \qquad \textit{for } k=1,2,\dots,n+1 \tag{3}$$

for all $n=0,1,2,\dots$, and

$$\mu\in L_\infty(\mathbb{R}_+)\Leftrightarrow S^kf\in\mathscr{B} \qquad \textit{for all } k\in\mathbb{N}, \tag{4}$$

where S^k denotes the k-th iterate of S.

Proof. Theorem 1.2 gives (3) for $n=0$, and we shall prove (3) by induction. Suppose that (3) holds for some $n \geqq 0$. It follows that $\mu \in L_{n+1}(\mathbb{R}_+)$ if and only if $S^k f \in \mathscr{B}$ for $k=1,2,\ldots,n+1$ and

$$S^k(f-f\circ T_c)=S^k f-(S^k f)\circ T_c \in \mathscr{B}$$

for $k=1,2,\ldots,n+1$ and all $c\in]0,1[$, and by Lemma 1.1 this holds if and only if $S^k f\in\mathscr{B}$ for $k=1,2,\ldots,n+1,n+2$. Clearly (4) follows from (3). □

1.4. *Remark.* Let $\mu\in L_0(\mathbb{R}_+)$ with $f=-\log\mathscr{L}\mu$. Using the representation (1) of f it is easy to see that $Sf(0^+)=0$, i.e. $Sf\in\mathscr{B}_{-1}$, and it follows that there exists $\mu^{(1)}\in I(\mathbb{R}_+)$ such that $Sf=-\log\mathscr{L}\mu^{(1)}$. Moreover $\mu\in L_1(\mathbb{R}_+)$ if and only if $\mu^{(1)}\in L_0(\mathbb{R}_+)$.

More generally, if $\mu\in L_n(\mathbb{R}_+)$ there exists $\mu^{(k)}\in L_{n-k}(\mathbb{R}_+)$ for $k=1,2,\ldots,n+1$ (with the convention $L_{-1}(\mathbb{R}_+)=\mathbf{I}(\mathbb{R}_+)$) such that $S^k f=-\log\mathscr{L}\mu^{(k)}$.

For comparison with the case of discrete multiply self-decomposable probabilities on $\mathbb{Z}_+$ treated in § 2, we will now give integral representations of the convex cones

$$\mathscr{B}_n=\{-\log\mathscr{L}\mu \mid \mu\in L_n(\mathbb{R}_+)\}, \qquad n=0,1,2,\ldots,\infty.$$

Let for $n=0,1,2,\ldots$ the function $F_n\colon \mathbb{R}_+\to\mathbb{R}_+$ be given by

$$F_n(s)=\frac{s}{(n+1)!}\int_0^1 (-\log t)^{n+1}e^{-ts}\,dt \qquad \text{for } s\geqq 0.$$

1.5. **Theorem.** *Let $f\colon \mathbb{R}_+\to\mathbb{R}$ and $n\in\mathbb{Z}_+$. Then $f\in\mathscr{B}_n$ if and only if f has the form*

$$f(s)=bs+\int_0^\infty F_n(sx)\,d\tau(x) \qquad \textit{for } s\geqq 0,$$

where $b\geqq 0$ and $\tau\geqq 0$ is a measure on $]0,\infty[$ such that

$$\int_0^1 x\,d\tau(x)<\infty \quad \textit{and} \quad \int_1^\infty (\log x)^{n+1}\,d\tau(x)<\infty.$$

Here (b,τ) is the representing couple for $S^{n+1}f\in\mathscr{B}$.

Proof. This is a special case of Proposition 2 in Urbanik [17] (and of Theorem 4.1 in Kumar and Schreiber [10]). A simple direct proof analogous to the proof of Theorem 2.6 below could of course be given. See also the proof of Theorem 4.3 below. □

1.6. **Theorem.** *A function $f\colon \mathbb{R}_+\to\mathbb{R}$ belongs to $\mathscr{B}_\infty$ if and only if it has the form*

$$f(s)=\int_0^1 s^\alpha\,d\sigma(\alpha) \qquad \textit{for } s\geqq 0,$$

where $\sigma\geqq 0$ is a finite measure on $]0,1]$.

Proof. This is a special case a Proposition 5 in Urbanik [17] (and of Theorem 4.2 in Kumar and Schreiber [10]). See also Remark b) in § 3 below. □

1.7. *Remark.* Let $\mu \in I(\mathbb{R}_+)$ with $f = -\log \mathscr{L}\mu$ and suppose f has representing couple (b, ν). It is well known that $\mu \in L_0(\mathbb{R}_+)$ if and only if ν is absolutely continuous with a density of the form $\frac{1}{x} h(x)$ where h is decreasing. In terms of the formal adjoint S^* of S ($S^* h(s) = -(s\,h(s))'$) this is the case if and only if $S^* \nu$ (derivative taken in the Schwartz distribution sense) is a non-negative measure. In the affirmative case $(b, S^* \nu)$ is the representing couple for Sf. By combination of this remark and Theorem 1.3 we get

$$\mu \in L_n(\mathbb{R}_+) \Leftrightarrow (S^*)^k \nu \geqq 0 \qquad \text{for } k = 1, \dots, n+1$$

where $n = 0, 1, \dots, \infty$.

1.8. *Remark.* For $0 < \alpha < 1$ we have

$$s^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-xs}) \frac{dx}{x^{1+\alpha}}, \qquad s \geqq 0,$$

(cf. [2]). It follows from 1.6 that the Lévy measures for the class $L_\infty(\mathbb{R}_+)$ have the densities

$$\int_0^1 \frac{d\tau(\alpha)}{x^{1+\alpha}},$$

where $\tau \geqq 0$ is a measure on $]0, 1[$ such that $\int_0^1 (\alpha(1-\alpha))^{-1}\, d\tau(\alpha) < \infty$.

§ 2. Multiply Self-Decomposable Probabilities on $\mathbb{Z}_+$

Let $P(\mathbb{Z}_+)$ be the set of probability measures $\mu = \sum_{k=0}^\infty p_k \varepsilon_k$ (ε_k degenerate probability at k) on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ for which $p_0 > 0$, and let $\tilde{\mathscr{B}}$ denote the set of C^∞-functions φ: $[0, 1[\to]-\infty, 0]$ with φ' absolutely monotone.

Denoting by M_μ the generating function of $\mu \in P(\mathbb{Z}_+)$, i.e. $M_\mu(z) = \sum_{k=0}^\infty p_k z^k$ for $z \in [0, 1]$, the mapping $\mu \mapsto \log M_\mu$ defines a one-to-one correspondence between the set $I(\mathbb{Z}_+) = P(\mathbb{Z}_+) \cap I(\mathbb{R}_+)$ of *infinitely divisible* probabilities on $\mathbb{Z}_+$ and the set $\tilde{\mathscr{B}}_{-1} = \{\varphi \in \tilde{\mathscr{B}} \mid \varphi(1^-) = 0\}$.

Furthermore a function φ belongs to $\tilde{\mathscr{B}}_{-1}$ if and only if it has the form

$$\varphi(z) = \sum_{k=1}^\infty h_k (z^k - 1) \qquad \text{for } z \in [0, 1], \tag{5}$$

where (h_k) is a sequence of non-negative numbers such that $\sum_{k=1}^\infty h_k < \infty$. The sequence (h_k) is uniquely determined by φ and is called the *representing* *sequence* for φ. In fact, $\sum_{k=1}^{\infty} h_k \varepsilon_k$ is the *Lévy measure* for $\mu \in I(\mathbb{Z}_+)$ where $\varphi = \log M_\mu$.

We shall now discuss a discrete version of the notion of self-decomposability which has been proposed and studied by Steutel and van Harn [15]. First some discrete analogues $\tilde{T}_c$ of the multiplications T_c on $P(\mathbb{R}_+)$ are introduced by

$$\tilde{T}_c\left(\sum_{k=0}^{\infty} p_k \varepsilon_k\right) = \sum_{k=0}^{\infty} p_k \left(\sum_{j=0}^{k} \binom{k}{j} (1-c)^{k-j} c^j \varepsilon_j\right) \quad \text{for } c \in]0,1[.$$

In terms of generating functions we find for $z \in [0,1]$

$$M_{\tilde{T}_c \mu}(z) = M_\mu(1-c+cz) = (M_\mu \circ \tau_c)(z)$$

where τ_c: $[0,1] \to [0,1]$ is given by $\tau_c(z) = 1-c+cz$.

Definition. A probability $\mu \in P(\mathbb{Z}_+)$ is called *discrete self-decomposable* if for all $c \in]0,1[$ there exists $\mu_c \in P(\mathbb{Z}_+)$ such that

$$\mu = (\tilde{T}_c \mu) * \mu_c. \tag{6}$$

Here the probability μ_c in (6) is uniquely determined and is called the *c-component* of μ. The set of discrete self-decomposable probabilities on $\mathbb{Z}_+$ will be denoted $L(\mathbb{Z}_+)$.

In Steutel and van Harn [15] it is shown that $L(\mathbb{Z}_+) \subseteq I(\mathbb{Z}_+)$ and that also the c-components of $\mu \in L(\mathbb{Z}_+)$ belong to $I(\mathbb{Z}_+)$.

Let $\mu \in P(\mathbb{Z}_+)$. In terms of the corresponding function $\varphi = \log M_\mu$ the condition for discrete self-decomposability is that $\varphi - \varphi \circ \tau_c \in \tilde{\mathscr{B}}$ for all $c \in]0,1[$, and in the affirmative case the function corresponding to the c-component of μ is $\varphi - \varphi \circ \tau_c$.

The c-components of $\mu \in L(\mathbb{Z}_+)$ are not in general discrete self-decomposable, and we therefore consider the following classes of *discrete multiply self-decomposable* probabilities on $\mathbb{Z}_+$

$$L_0(\mathbb{Z}_+) \supseteq L_1(\mathbb{Z}_+) \supseteq \ldots \quad \text{and} \quad L_\infty(\mathbb{Z}_+) = \bigcap_{n=0}^{\infty} L_n(\mathbb{Z}_+),$$

where $L_0(\mathbb{Z}_+) = L(\mathbb{Z}_+)$ and $L_{n+1}(\mathbb{Z}_+)$ for $n \geqq 0$ is defined inductively by

$$L_{n+1}(\mathbb{Z}_+) = \{\mu \in L_n(\mathbb{Z}_+) \mid \forall c \in]0,1[: \ \mu_c \in L_n(\mathbb{Z}_+)\}.$$

The differential operator $\tilde{S}$ on $[0,1[$ defined by

$$\tilde{S}\varphi(z) = (z-1)\varphi'(z) \quad \text{for } z \in [0,1[$$

is the "discrete" analogue of S.

2.1. **Lemma.** *Let* φ: $[0,1[\to \mathbb{R}$ *be a* C^1*-function. Then* $\tilde{S}\varphi \in \tilde{\mathscr{B}}$ *if and only if the function* $\varphi - \varphi \circ \tau_c$ *belongs to* $\tilde{\mathscr{B}}$ *for all* $c \in]0,1[$.

Proof. The set $\tilde{\mathscr{B}}$ is a convex cone which is closed in the topology of pointwise convergence on $[0,1[$ and $\tilde{\mathscr{B}}$ is stable under composition with the mappings τ_c, $0 < c < 1$, and as in the proof of Lemma 1.1 the conclusion follows. □

Using the observation that for any C^1-function φ

$$\tilde{S}(\varphi\circ\tau_c)=(\tilde{S}\varphi)\circ\tau_c \quad \text{for } c\in]0,1[,$$

a simple adaptation of the proof of Theorem 1.3 gives

2.2. **Theorem.** *Let* $\mu\in P(\mathbb{Z}_+)$ *with* $\varphi=\log M_\mu$. *Then*

$$\mu\in L_n(\mathbb{Z}_+)\Leftrightarrow \tilde{S}^k\varphi\in\tilde{\mathscr{B}} \quad \textit{for } k=1,2,\ldots,n+1$$

for all $n=0,1,2,\ldots$, *and*

$$\mu\in L_\infty(\mathbb{Z}_+)\Leftrightarrow \tilde{S}^k\varphi\in\tilde{\mathscr{B}} \quad \textit{for all } k\in\mathbb{N},$$

where $\tilde{S}^k$ *denotes the k-th iterate of* $\tilde{S}$.

2.3. *Remarks.* a) The above characterization gives for $n=0$ the Theorem 2.2 in Steutel and van Harn [15].

b) Let $\mu\in L_0(\mathbb{Z}_+)$ with $\varphi=\log M_\mu\in\tilde{\mathscr{B}}$. Using the representation (5) it can be seen that $\tilde{S}\varphi(1^-)=0$, and the situation on $\mathbb{Z}_+$ is as described in Remark 1.4 for $\mathbb{R}_+$.

We shall now give integral representations of the convex cones

$$\tilde{\mathscr{B}}_n=\{\log M_\mu \mid \mu\in L_n(\mathbb{Z}_+)\} \quad \text{for } n=0,1,2,\ldots,\infty.$$

Consider the family of functions $\Phi_{n,j}$: $[0,1]\to]-\infty,0]$, $n,j\in\mathbb{Z}_+$, defined by

$$\Phi_{n,j}(z)=\frac{(j+1)(z-1)}{(n+1)!}\int_0^1(-\log t)^{n+1}(1-t+tz)^j\,dt, \quad z\in[0,1]. \tag{7}$$

2.4. **Lemma.** *For* $n,j\in\mathbb{Z}_+$ *we have* $\Phi_{n,j}\in\tilde{\mathscr{B}}_n$.

Proof. The formula

$$\Phi_{n,j}(z)=\frac{1}{n!}\int_0^1(-\log t)^n[(1-t+tz)^{j+1}-1]\frac{dt}{t}$$

shows that $\Phi_{n,j}\in\tilde{\mathscr{B}}$ for $n,j\in\mathbb{Z}_+$ since $\tilde{\mathscr{B}}$ is closed in the topology of pointwise convergence and stable under composition with the mappings $\{\tau_t \mid t\in]0,1[\}$. Moreover, this formula gives by a simple computation that

$$\tilde{S}\Phi_{n,j}=\Phi_{n-1,j} \quad \text{for } n,j\in\mathbb{Z}_+$$

$\left(\text{putting } \Phi_{-1,j}(z)=(j+1)(z-1)\int_0^1(1-t+tz)^j\,dt=z^{j+1}-1\in\tilde{\mathscr{B}}\right)$, hence that $\tilde{S}^k\Phi_{n,j}=\Phi_{n-k,j}\in\tilde{\mathscr{B}}$ for $k=1,2,\ldots,n+1$, i.e. $\Phi_{n,j}\in\tilde{\mathscr{B}}_n$. □

2.5. **Lemma.** *For fixed* $n\in\mathbb{Z}_+$ *and* $z\in]0,1[$ *we have*

$$\Phi_{n,j}(z)\sim-\frac{1}{(n+1)!}(\log j)^{n+1} \quad \text{as } j\to\infty.$$

Proof. For $x>1$ we have

$$\frac{x}{(\log x)^{n+1}}\int_0^1(-\log t)^{n+1}(1-t+tz)^x\,dt$$
$$=\int_0^x\left(1-\frac{\log u}{\log x}\right)^{n+1}\left(1-\frac{u(1-z)}{x}\right)^x du.$$

Splitting the integral as $\int_0^1+\int_1^x$ we see that the limit for $x\to\infty$ is

$$\int_0^1 e^{-u(1-z)}\,du+\int_1^\infty e^{-u(1-z)}\,du=\frac{1}{1-z}$$

and the assertion follows. □

2.6. **Theorem.** *Let $\varphi\colon[0,1]\to\mathbb{R}$ and $n\in\mathbb{Z}_+$. Then $\varphi\in\tilde{\mathscr{B}}_n$ if and only if*

$$\varphi(z)=\sum_{j=0}^{\infty}a_j\,\Phi_{n,j}(z)\qquad \text{for}\ \ z\in[0,1],\tag{8}$$

where $(a_j)_{j\geqq 0}$ is a sequence of non-negative numbers satisfying

$$\sum_{j=1}^{\infty}a_j(\log j)^{n+1}<\infty.\tag{9}$$

In the affirmative case, $\sum_{j=1}^{\infty}a_{j-1}\,\varepsilon_j$ is the Lévy measure for $\tilde S^{n+1}\varphi$.

Proof. Suppose first that $\varphi\in\tilde{\mathscr{B}}_n$. Since $\varphi(1^-)=0$ we find for $z\in[0,1]$

$$\varphi(z)=-\int_z^1\varphi'(u)\,du=\int_0^1\varphi'(1-t+tz)(z-1)\,dt=\int_0^1(\tilde S\varphi)(1-t+tz)\,\frac{dt}{t}.$$

More generally, for $k=0,1,2,\ldots,n+1$ we have by induction

$$\varphi(z)=\frac{1}{k!}\int_0^1(-\log t)^k(\tilde S^{k+1}\varphi)(1-t+tz)\,\frac{dt}{t},\qquad z\in[0,1].\tag{10}$$

In fact, suppose (10) holds for some $k\leqq n$. Then, since $(\tilde S^{k+1}\varphi)(1^-)=0$, we have

$$\tilde S^{k+1}\varphi(u)=-\int_u^1(\tilde S^{k+1}\varphi)'(v)\,dv=\int_0^1(\tilde S^{k+2}\varphi)(1-v+vu)\,\frac{dv}{v},$$

and therefore by interchange of integrations

$$\varphi(z)=\frac{1}{k!}\int_0^1(-\log t)^k\left(\int_0^1(\tilde S^{k+2}\varphi)(1-v+v(1-t+tz))\,\frac{dv}{v}\right)\frac{dt}{t}$$
$$=\frac{1}{k!}\int_0^1(-\log t)^k\left(\int_0^t(\tilde S^{k+2}\varphi)(1-w+wz)\,\frac{dw}{w}\right)\frac{dt}{t}$$
$$=\frac{1}{k!}\int_0^1\left(\int_w^1(-\log t)^k\,\frac{dt}{t}\right)(\tilde S^{k+2}\varphi)(1-w-wz)\,\frac{dw}{w}$$
$$=\frac{1}{(k+1)!}\int_0^1(-\log w)^{k+1}(\tilde S^{k+2}\varphi)(1-w+wz)\,\frac{dw}{w}.$$

For $k=n+1$ equation (10) can be written

$$\varphi(z)=\frac{1}{(n+1)!}\int_0^1(-\log t)^{n+1}(\tilde{S}^{n+1}\varphi)'(1-t+tz)(z-1)\,dt,$$

and since $\varphi\in\tilde{\mathscr{B}}_n$, we have

$$(\tilde{S}^{n+1}\varphi)'(z)=\sum_{j=0}^{\infty}(j+1)\,a_j z^j$$

where $\sum_{j=1}^{\infty} a_{j-1}\varepsilon_j$ is the Lévy measure for $\tilde{S}^{n+1}\varphi$, which gives

$$\begin{aligned}\varphi(z)&=\frac{1}{(n+1)!}\int_0^1(-\log t)^{n+1}\left(\sum_{j=0}^{\infty}(j+1)\,a_j(1-t+tz)^j\right)(z-1)\,dt\\&=\sum_{j=0}^{\infty}a_j\,\Phi_{n,j}(z)\end{aligned}$$

as desired. Since $\varphi(z)$ is finite for $z\in]0,1[$ we get in particular by Lemma 2.5 that (9) is satisfied.

If conversely $(a_j)_{j\geq 0}$ is a sequence of non-negative numbers satisfying (9) then by Lemma 2.4 and 2.5 the function φ defined by (8) belongs to $\tilde{\mathscr{B}}_n$. □

2.7. **Theorem.** *Let φ: $[0,1]\to\mathbb{R}$. Then $\varphi\in\tilde{\mathscr{B}}_\infty$ if and only if*

$$\varphi(z)=-\int_0^1(1-z)^\alpha\,d\sigma(\alpha),\qquad z\in[0,1],\tag{11}$$

for some non-negative finite measure σ on $]0,1]$.

Proof. Suppose first that $\varphi\in\tilde{\mathscr{B}}_\infty$. Defining $g(s)=-\varphi(1-e^{-s})$ for $s\geq 0$, we see that

$$g'(s)=-e^{-s}\varphi'(1-e^{-s})=(\tilde{S}\varphi)(1-e^{-s}),$$

and by induction for $k\geq 0$

$$(-1)^k g^{(k)}(s)=-(\tilde{S}^k\varphi)(1-e^{-s}).$$

Now $\tilde{S}^k\varphi\in\tilde{\mathscr{B}}$ and in particular $\tilde{S}^k\varphi\leq 0$, thus g is completely monotone, hence by Bernstein's theorem of the form

$$g(s)=\int_0^\infty e^{-\alpha s}\,d\sigma(\alpha)\qquad\text{for }\ s\geq 0,$$

for some non-negative finite measure $(g(0)<\infty)$ σ on $[0,\infty[$. For $z\in[0,1[$, $z=1-e^{-s}$ with $s\geq 0$ we may write

$$\varphi(z)=-g(s)=-\int_0^\infty(1-z)^\alpha\,d\sigma(\alpha),$$

and we shall show that σ is a measure on $]0,1]$.

For $k\in\mathbb{N}$ and $z\in]0,1[$ we have

$$\tilde{S}^k\varphi(z)=-\int_0^\infty \alpha^k(1-z)^\alpha\,d\sigma(\alpha)$$

and

$$(\tilde{S}^k\varphi)''(z)=\int_0^\infty \alpha^{k+1}(1-\alpha)(1-z)^{\alpha-2}\,d\sigma(\alpha).$$

Since $\varphi\in\tilde{\mathscr{B}}_\infty$ we have $\tilde{S}^k\varphi\in\tilde{\mathscr{B}}$ and therefore $(\tilde{S}^k\varphi)''\geqq 0$ for all $k\in\mathbb{N}$. If σ is not supported by $[0,1]$, then for any $z\in]0,1[$

$$\int_1^\infty \alpha^{k+1}(1-\alpha)(1-z)^{\alpha-2}\,d\sigma(\alpha)\to-\infty$$

for $k\to\infty$, but this is not possible since

$$\int_0^1 \alpha^{k+1}(1-\alpha)(1-z)^{\alpha-2}\,d\sigma(\alpha)\to 0$$

for $k\to\infty$, and the sum of these two terms is $\geqq 0$.

Finally σ has no mass at 0, since $\varphi(1)=0$.

If conversely φ has the form (11) then for $k\in\mathbb{Z}_+$

$$\tilde{S}^k\varphi(z)=-\int_0^1 \alpha^k(1-z)^\alpha\,d\sigma(\alpha),$$

and since for every $\alpha\in]0,1]$ the function $z\mapsto-(1-z)^\alpha$ belongs to $\tilde{\mathscr{B}}$, it follows that $\tilde{S}^k\varphi\in\tilde{\mathscr{B}}$ for $k\in\mathbb{Z}_+$, hence that $\varphi\in\tilde{\mathscr{B}}_\infty$. □

2.8. *Remarks.* a) The c-components of $\mu\in L_\infty(\mathbb{Z}_+)$ belong to $L_\infty(\mathbb{Z}_+)$ and if $\varphi=\log M_\mu\in\tilde{\mathscr{B}}_\infty$ has the representation (11) we find for $c\in]0,1[$ and $z\in[0,1]$

$$\log M_{\mu_c}(z)=\varphi(z)-\varphi(1-c+cz)=-\int_0^1(1-z)^\alpha(1-c^\alpha)\,d\sigma(\alpha).$$

b) Theorem 2.7 can be reformulated: $L_\infty(\mathbb{Z}_+)$ is the set of weak limits of finite convolutions of *discrete stable* (cf. Steutel and van Harn [15]) probabilities on $\mathbb{Z}_+$, i.e. the set of *discrete mixed stable* probabilities on $\mathbb{Z}_+$.

2.9. *Remark.* Let $\mu\in I(\mathbb{Z}_+)$ with $\varphi=\log M_\mu\in\tilde{\mathscr{B}}$, and suppose that φ has the representation (5). It is easy to see that $\tilde{S}\varphi\in\tilde{\mathscr{B}}$ if and only if the sequence $(jh_j)_{j\geqq 1}$ is decreasing and that in the affirmative case

$$\tilde{S}\varphi(z)=(z-1)\,\varphi'(z)=\sum_{j=1}^\infty(jh_j-(j+1)\,h_{j+1})(z^j-1).$$

By means of the operator $\tilde{S}^*$ acting on sequences $\underline{h}=(h_j)_{j\geqq 1}$ by

$$(\tilde{S}^*\underline{h})_j=jh_j-(j+1)\,h_{j+1}$$

we thus get the following: Let $\mu \in I(\mathbb{Z}_+)$ with Lévy measure $\sum_{j=1}^{\infty} h_j \varepsilon_j$ and put $\underline{h} = (h_j)_{j \geqq 1}$. Then

$$\mu \in L_n(\mathbb{Z}_+) \Leftrightarrow (\tilde{S}^*)^k \underline{h} \geqq 0 \quad \text{for } k = 1, 2, \ldots, n+1,$$

for $n \in \mathbb{Z}_+ \cup \{\infty\}$.

For "$n = \infty$" we get from Theorem 2.7 that $\mu \in I(\mathbb{Z}_+)$ belongs to $L_\infty(\mathbb{Z}_+)$ if and only if the Lévy measure $\sum_{j=1}^{\infty} h_j \varepsilon_j$ for μ satisfies

$$h_j = \int_0^1 (-1)^{j-1} \binom{\alpha}{j} \, d\tau(\alpha), \quad j \geqq 1,$$

for some finite measure $\tau \geqq 0$ on $]0, 1]$.

§3. Poisson Mixtures

With the purpose of relating the self-decomposability classes on $\mathbb{R}_+$ and $\mathbb{Z}_+$ consider for $\theta > 0$ the mapping $\mu \mapsto \pi_\mu(\theta)$ of $P(\mathbb{R}_+)$ into $P(\mathbb{Z}_+)$ defined by

$$M_{\pi_\mu(\theta)}(z) = \mathscr{L}\mu(\theta(1-z)) \quad \text{for } z \in [0, 1].$$

This formula means that $\pi_\mu(\theta)$ is the *Poisson mixture* (or *Poisson subordinate*, cf. [8], [9])

$$\pi_\mu(\theta) = \int_0^\infty \pi_{t\theta} \, d\mu(t),$$

where $(\pi_t)_{t>0}$ is the convolution semigroup of Poisson distributions $\pi_t = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \varepsilon_k$.

It is well known that many properties of $\mu \in P(\mathbb{R}_+)$ are reflected into "similar" properties of $\pi_\mu(\theta) \in P(\mathbb{Z}_+)$, $\theta > 0$. For self-decomposability properties this is based on the fact that the mapping $\mu \mapsto \pi_\mu(\theta)$ for each $\theta > 0$ is a convolution homomorphism (this is clear) which transforms the multiplication T_c on $P(\mathbb{R}_+)$ into the multiplication $\tilde{T}_c$ on $P(\mathbb{Z}_+)$ in the sense that

$$\pi_{T_c \mu}(\theta) = \tilde{T}_c(\pi_\mu(\theta)) \quad \text{for } c \in]0, 1[.$$

In fact, for $z \in [0, 1]$ we have

$$\begin{aligned} M_{\pi_{T_c \mu}(\theta)}(z) &= \mathscr{L}(T_c \mu)(\theta(1-z)) = \mathscr{L}\mu(\theta c(1-z)) \\ &= \mathscr{L}\mu(\theta(1-(1-c+cz))) = M_{\tilde{T}_c(\pi_\mu(\theta))}(z). \end{aligned}$$

Moreover, many properties of $\mu \in P(\mathbb{R}_+)$ can be characterized by "similar" properties of $\pi_\mu(\theta) \in P(\mathbb{Z}_+)$, $\theta > 0$, due to the following "inversion formula", cf. [8],

$$\mu = \lim_{\theta \to \infty} \sum_{k=0}^{\infty} \pi_\mu(\theta)_k \, \varepsilon_{k/\theta} \quad \text{weakly},$$

where $\pi_\mu(\theta)_k$ denotes the mass at k of $\pi_\mu(\theta)$.

It follows that for $\mu \in P(\mathbb{R}_+)$

$$\mu \in I(\mathbb{R}_+) \Leftrightarrow \forall \theta > 0: \pi_\mu(\theta) \in I(\mathbb{Z}_+),$$

and for $n \in \mathbb{Z}_+ \cup \{\infty\}$

$$\mu \in L_n(\mathbb{R}_+) \Leftrightarrow \forall \theta > 0: \pi_\mu(\theta) \in L_n(\mathbb{Z}_+). \tag{12}$$

Remarks. a) The characterization (12) above for $n=0$ was given in [8].

b) It follows from the integral representations of $\mathscr{B}_\infty$ and $\tilde{\mathscr{B}}_\infty$ that the mapping $\mu \mapsto \pi_\mu(\theta)$ for each fixed $\theta > 0$ is a bijection of $L_\infty(\mathbb{R}_+)$ onto $L_\infty(\mathbb{Z}_+)$.

On the other hand, the integral representation of $\mathscr{B}_\infty$ is a simple consequence of the integral representation of $\tilde{\mathscr{B}}_\infty$ and the characterization (12). Also (12) in the case of $n=\infty$ can be sharpened to the following: $\mu \in P(\mathbb{R}_+)$ belongs to $L_\infty(\mathbb{R}_+)$ if there exists one $\theta > 0$ for which $\pi_\mu(\theta) \in L_\infty(\mathbb{Z}_+)$.

c) It is clear that the support of $\pi_\mu(\theta)$ is $\mathbb{Z}_+$ for any $\theta > 0$ and $\mu \in P(\mathbb{R}_+)$, and it follows that not every element of $I(\mathbb{Z}_+)$ is of the form $\pi_\mu(\theta)$ with $\mu \in P(\mathbb{R}_+)$ $(I(\mathbb{R}_+))$ and $\theta > 0$.

There also exist elements of $L_0(\mathbb{Z}_+)$ which are not Poisson mixtures of any $\mu \in L_0(\mathbb{R}_+)$ (and not even of any $\mu \in P(\mathbb{R}_+)$). In fact, the function

$$\varphi(z) = z - 1 + \tfrac{1}{2}(z^2 - 1) \quad \text{for} \quad z \in [0, 1],$$

belongs to $\tilde{\mathscr{B}}_0$, and if $\mu \in P(\mathbb{R}_+)$ and $\theta > 0$ where such that $\exp(\varphi(z)) = \mathscr{L}\mu(\theta(1-z))$, then the function

$$g(s) = \exp\left(\varphi\left(1 - \frac{s}{\theta}\right)\right) = \exp\left(\frac{1}{2}\frac{s^2}{\theta^2} - 2\frac{s}{\theta}\right), \quad s \in [0, \theta]$$

would be the restriction of a completely monotone function, and this is not the case.

§4. Self-Decomposability Classes of Generalized Γ-Convolutions and Negative Binomial Convolutions

The set of so-called generalized Γ-convolutions, introduced by Thorin, is an important subset of $L(\mathbb{R}_+)$.

We consider the set $\mathscr{T}$ of C^∞-functions $f:]0, \infty[\to \mathbb{R}_+$ for which f' is a *Stieltjes transform*, i.e. of the form

$$f'(s) = b + \int_0^\infty \frac{d\mu(x)}{s+x}, \quad s > 0$$

where $b \geqq 0$ and $\mu \geqq 0$ is a measure on $[0, \infty[$. It follows that $\mathscr{T}$ is a convex cone contained in $\mathscr{B}$, and it is easy to establish the following integral representation of the functions $f \in \mathscr{T}$:

$$f(s) = a + bs + \int_0^\infty \log(1 + sx)\, d\kappa(x), \quad s > 0 \tag{13}$$

where $a, b \geqq 0$ and $\kappa \geqq 0$ is a measure on $]0, \infty[$ such that the integral is finite for $s>0$. From (13) it can be seen that $\mathscr{T}$ is closed under pointwise convergence and stable under composition with T_c, $c>0$.

A *generalized Γ-convolution*, cf. Thorin [16] (see also [1] or Bondesson [5]) is a probability measure $\mu \in P(\mathbb{R}_+)$ for which $f=-\log \mathscr{L}\mu \in \mathscr{T}$. (Note that $f(0^+)=a=0$ in (13)). The set $\Gamma(\mathbb{R}_+)$ of generalized Γ-convolutions can be described as the set of weak limits of finite convolutions of Γ-distributions.

Clearly $\Gamma(\mathbb{R}_+) \subseteq I(\mathbb{R}_+)$, but we even have $\Gamma(\mathbb{R}_+) \subseteq L(\mathbb{R}_+)$. This result of Thorin can be seen from Theorem 1.2 because $f \in \mathscr{T}$ implies $Sf \in \mathscr{B}$. In fact, if f has the representation (13), then

$$sf'(s)=bs+\int_0^\infty \frac{sx}{1+sx}\, d\kappa(x) \tag{14}$$

which belongs to $\mathscr{B}$ because $s \mapsto s(1+sx)^{-1}$ does so for every $x>0$. For $0<\beta \leqq 1$ the fucntion $s^{-\beta}$ is a Stieltjes transform, cf. [2], so by Theorem 1.6 we get $L_\infty(\mathbb{R}_+) \subseteq \Gamma(\mathbb{R}_+)$.

In analogy with the classes $L_n(\mathbb{R}_+)$ we now consider

$$\Gamma_0(\mathbb{R}_+) \supseteq \Gamma_1(\mathbb{R}_+) \supseteq \ldots \supseteq \Gamma_n(\mathbb{R}_+) \supseteq \ldots \quad \text{and} \quad \Gamma_\infty(\mathbb{R}_+)=\bigcap_{n=0}^\infty \Gamma_n(\mathbb{R}_+)$$

where $\Gamma_0(\mathbb{R}_+)=\Gamma(\mathbb{R}_+)$ and $\Gamma_n(\mathbb{R}_+)$ for $n \geqq 0$ is defined inductively by

$$\Gamma_{n+1}(\mathbb{R}_+)=\{\mu \in \Gamma_n(\mathbb{R}_+) \mid \forall c \in]0,1[: \mu_c \in \Gamma_n(\mathrm{R}_+)\}.$$

By induction we find $L_\infty(\mathbb{R}_+) \subseteq \Gamma_n(\mathbb{R}_+) \subseteq L_n(\mathbb{R}_+)$ for $n \geqq 0$, and consequently

$$L_\infty(\mathbb{R}_+)=\Gamma_\infty(\mathbb{R}_+).$$

The classes $\Gamma_n(\mathbb{R}_+)$ can be described in terms of S and $\mathscr{T}$ like $L_n(\mathbb{R}_+)$ could be in terms of S and $\mathscr{B}$ (Theorem 1.3). In fact, for a C^1-function $f:]0, \infty[\to \mathbb{R}$ we have $Sf \in \mathscr{T}$ if and only if $f-f \circ T_c \in \mathscr{T}$ for all $c \in]0,1[$, because $\mathscr{T}$ is a convex cone of functions which is closed in the topology of pointwise convergence and stable under composition with the family $\{T_c, c>0\}$, cf. Lemma 1.1. We therefore have:

4.1. **Theorem.** *Let $\mu \in P(\mathbb{R}_+)$ with $f=-\log \mathscr{L}\mu$.* Then for $n \in \mathbb{Z}_+$

$$\mu \in \Gamma_n(\mathbb{R}_+) \Leftrightarrow S^k f \in \mathscr{T} \qquad \text{for } k=0,1,\ldots,n.$$

In analogy with the classes $\mathscr{B}_n$ we will find the integral representation of the convex cones

$$\mathscr{T}_n=\{-\log \mathscr{L}\mu \mid \mu \in \Gamma_n(\mathbb{R}_+)\}, \qquad n=0,1,\ldots.$$

Clearly $\mathscr{T}_0=\{f \in \mathscr{T} \mid f(0^+)=0\}$. Let for $n \in \mathbb{Z}_+$ the function $G_n: \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$G_n(s)=\frac{s}{(n+1)!}\int_0^1 (-\log t)^{n+1}(1+st)^{-2}\, dt. \tag{15}$$

4.2. **Lemma.** *For* $n \in \mathbb{Z}_+$ *we have* $G_n \in \mathcal{T}_n$ *and*

$$G_n(s) \sim s \qquad \textit{for } s \to 0,$$

$$G_n(s) \sim \frac{1}{(n+1)!} (\log s)^{n+1} \qquad \textit{for } s \to \infty.$$

Proof. By two partial integrations we find (for $n \geqq 1$)

$$G_n(s) = \frac{s}{n!} \int_0^1 (-\log t)^n (1+st)^{-1}\, dt$$
$$= \frac{1}{(n-1)!} \int_0^1 \log(1+st)\,(-\log t)^{n-1} \frac{dt}{t},$$

and the last formula shows $G_n \in \mathcal{T}$. Furthermore we see that $SG_n = G_{n-1}$ which combined with Theorem 4.1 shows that $G_n \in \mathcal{T}_n$. The asymptotic behaviour is established as in 2.5. □

4.3. **Theorem.** *Let* $n \in \mathbb{Z}_+$. *A function* $f: \mathbb{R}_+ \to \mathbb{R}$ *belongs to* $\mathcal{T}_n$ *if and only if it has the form*

$$f(s) = bs + \int_0^\infty G_n(sx)\, d\kappa(x), \qquad s \geqq 0 \tag{16}$$

where $b \geqq 0$ *and* $\kappa \geqq 0$ *is a measure on* $]0, \infty[$ *such that*

$$\int_0^1 x\, d\kappa(x) < \infty \quad \textit{and} \quad \int_1^\infty (\log x)^{n+1}\, d\kappa(x) < \infty. \tag{17}$$

Proof. Suppose first that $\mu \in \Gamma_n(\mathbb{R}_+)$ with $f = -\log \mathcal{L}\mu$, so $S^k f \in \mathcal{T}$ for $k = 0, 1, \ldots, n$. Clearly $f(0) = 0$ and (14) implies that $Sf(0) = 0$. It follows that $S^k f(0) = 0$ for $k = 0, 1, \ldots, n+1$. We have

$$f(s) = \int_0^s f'(u)\, du = \int_0^1 Sf(ts) \frac{dt}{t}$$

and in general

$$f(s) = \frac{1}{k!} \int_0^1 (-\log t)^k S^{k+1} f(ts) \frac{dt}{t} \qquad \text{for } k = 0, 1, \ldots, n+1. \tag{18}$$

In fact, suppose (18) holds for $k \leqq n$. Then since $S^{k+1} f(0) = 0$ we have

$$S^{k+1} f(s) = \int_0^1 S^{k+2} f(us) \frac{du}{u},$$

and inserting this in (18) yields

$$f(s) = \frac{1}{k!} \int_0^1 \left((-\log t)^k \int_0^1 S^{k+2} f(uts) \frac{du}{u} \right) \frac{dt}{t}$$
$$= \frac{1}{k!} \int_0^1 \left(\int_x^1 (-\log t)^k \frac{dt}{t} \right) S^{k+2} f(xs) \frac{dx}{x}$$
$$= \frac{1}{(k+1)!} \int_0^1 (-\log x)^{k+1} S^{k+2} f(xs) \frac{dx}{x}.$$

If

$$S^n f(s) = b s + \int_0^\infty \log(1+sx)\, d\kappa(x)$$

and hence

$$S^{n+2} f(s) = b s + \int_0^\infty \frac{sx}{(1+sx)^2}\, d\kappa(x),$$

we get from (18) with $k=n+1$

$$\begin{aligned} f(s) &= \frac{1}{(n+1)!} \int_0^1 (-\log t)^{n+1} \left(bts + \int_0^\infty \frac{tsx}{(1+tsx)^2}\, d\kappa(x)\right) \frac{dt}{t} \\ &= bs + \int_0^\infty G_n(sx)\, d\kappa(x). \end{aligned}$$

In particular

$$\int_0^\infty G_n(x)\, d\kappa(x) < \infty$$

so the asymptotic behaviour of $G_n(x)$, cf. 4.2, implies that κ satisfies (17).

That conversely any function of the form (16), with κ satisfying (17), belongs to $\mathscr{T}_n$ is an easy consequence of Lemma 4.2. □

4.4. *Remark.* Let $f = -\log \mathscr{L}\mu \in \mathscr{T}_n$ with $\mu \in \Gamma_n(\mathbb{R}_+)$. Since $S^k f \in \mathscr{T}_0$ for $k = 0, 1, \ldots, n$ there exist numbers $b_k \geqq 0$ and positive measures κ_k on $]0, \infty[$ such that

$$S^k f(s) = b_k s + \int_0^\infty \log(1+sx)\, d\kappa_k(x). \tag{19}$$

We claim that $b_0 = b_1 = \ldots = b_n$ and that $S^* \kappa_k = \kappa_{k+1}$ for $k = 0, 1, \ldots, n-1$ in the Schwartz distribution sense, hence $S^{*k} \kappa_0 = \kappa_k$ for $k \leqq n$. In fact, from (19) with $k = 0$ we get

$$S f(s) = b_0 s + \int_0^\infty \frac{sx}{1+sx}\, d\kappa_0(x),$$

and from (19) with $k=1$ we get

$$\begin{aligned} S f(s) &= b_1 s + \int_0^\infty \left(\int_0^x \frac{s}{1+su}\, du\right) d\kappa_1(x) \\ &= b_1 s + \int_0^\infty \frac{su}{1+su} \frac{\kappa_1(]u, \infty[)}{u}\, du \end{aligned}$$

which show that $b_0 = b_1$ and that κ_0 has the density $\kappa_1(]u, \infty[)/u$ with respect to Lebesgue measure, hence $S^* \kappa_0 = \kappa_1$.

Conversely, let $f = -\log \mathscr{L}\mu \in \mathscr{T}_0$ with $\mu \in \Gamma_0(\mathbb{R}_+)$ and

$$f(s) = b_0 s + \int_0^\infty \log(1+sx)\, d\kappa_0(x).$$

If $S^*\kappa_0 \geqq 0$ on $]0,\infty[$ in the distribution sense then $\mu\in\Gamma_1(\mathbb{R}_+)$. In fact, the condition $S^*\kappa_0\geqq 0$ implies that κ_0 has a density of the form $h(u)/u$ for some decreasing function h: $]0,\infty[\to\mathbb{R}_+$ which may be assumed continuous from the right. Since

$$\kappa_0(]e,\infty[)\leqq\int_e^\infty \log x\, d\kappa_0(x)<\infty$$

there exists a non-negative (Radon) measure κ_1 on $]0,\infty[$ such that $\kappa_1(]u,\infty[) = h(u)$, $u>0$, and the above calculation shows that

$$S f(s) = b_0 s + \int_0^\infty \log(1+sx)\, d\kappa_1(x)$$

hence $Sf\in\mathscr{T}$ and $\mu\in\Gamma_1(\mathbb{R}_+)$. Iteration of this argument leads to

4.5. **Theorem.** *Let* $f=-\log\mathscr{L}\mu$ for $\mu\in\Gamma_0(\mathbb{R}_+)$ *have the representation*

$$f(s)=bs+\int_0^\infty \log(1+sx)\,d\kappa(x).$$

Then we have for $n\geqq 1$

$$\mu\in\Gamma_n(\mathbb{R}_+)\Leftrightarrow S^{*k}\kappa\geqq 0 \quad \text{on }]0,\infty[\quad \textit{for } k=1,\ldots,n.$$

Let us finish by the discrete analogues of the classes Γ_n.

Consider the set $\Gamma(\mathbb{Z}_+)$ of so-called *generalized negative binomial convolutions*, introduced by Bondesson [5] to be the set of weak limits of finite convolutions of negative binomial distributions on $\mathbb{Z}_+$.

The set $\Gamma(\mathbb{Z}_+)$ is the discrete analogue of $\Gamma(\mathbb{R}_+)$. In fact, it is easy to see that for $\mu\in P(\mathbb{R}_+)$ we have

$$\mu\in\Gamma(\mathbb{R}_+)\Leftrightarrow\forall\,\theta>0:\ \pi_\mu(\theta)\in\Gamma(\mathbb{Z}_+),$$

(see also Bondesson [5], Theorem 2.2). Moreover, for each fixed $\theta>0$, the mapping $\mu\mapsto\pi_\mu(\theta)$ is a bijection of $\Gamma(\mathbb{R}_+)$ onto $\Gamma(\mathbb{Z}_+)$.

Introducing discrete self-decomposability classes $\Gamma_n(\mathbb{Z}_+)$ for $n\in\mathbb{N}\cup\{\infty\}$ we find that for each fixed $\theta>0$ and $n\in\mathbb{N}\cup\{\infty\}$ the mapping $\mu\mapsto\pi_\mu(\theta)$ is a bijection of $\Gamma_n(\mathbb{R}_+)$ onto $\Gamma_n(\mathbb{Z}_+)$. In particular

$$\Gamma_\infty(\mathbb{Z}_+)=L_\infty(\mathbb{Z}_+).$$

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