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An Iterated Logarithm Law for the Maximum in a Stationary Gaussian Sequence*

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Summary. Let $\{X_n, n=1, 2, \ldots\}$ be the successive terms of a discrete coordinate stationary Gaussian stochastic process. Assume, without loss of generality, that $EX_n=0$ and $r_0=EX_n^2=1$ for all n. Let $r_n \equiv EX_k X_{k+n}$ be the covariance function. If either there exists an $\alpha>0$ such that

$$\lim_{n\to\infty} n^\alpha r_n=0, \quad \text{or} \quad \sum_{n=-\infty}^{\infty} r_n^2<\infty,$$

then

$$P\{\liminf_{n\to\infty}(2\log n)^{\frac{1}{2}}(Z_n-(2\log n)^{\frac{1}{2}})/\log\log n=-\tfrac{1}{2},\ \limsup_{n\to\infty}(2\log n)^{\frac{1}{2}}(Z_n-(2\log n)^{\frac{1}{2}})/\log\log n=\tfrac{1}{2}\}=1,$$

where

$$Z_n \equiv \operatorname*{Sup}_{1\leq k\leq n} X_k.$$

It is not sufficient that

$$\lim_{n\to\infty} r_n=0.$$

Section 1

Let $\{X_n, n=1, 2, \ldots\}$ be the successive terms of a stationary Gaussian sequence. It is assumed, without loss of generality, that $EX_n=0$, $EX_n^2=1$, for all n. We define the covariance function

$$r_n \equiv EX_k X_{k+n}.$$

By stationarity, of course, the covariance does not depend on k. The purpose of this paper is to establish Theorem 1.1.

Theorem 1.1. *If either*

$$\exists\, \alpha>0\colon \lim_{n\to\infty} n^\alpha r_n=0, \tag{1.1}$$

or

$$\sum_{n=-\infty}^{\infty} r_n^2<\infty,$$

then

$$P\{\liminf_{n\to\infty}(2\log n)^{\frac{1}{2}}(Z_n-(2\log n)^{\frac{1}{2}})/\log\log n \\ =-\tfrac{1}{2},\ \limsup_{n\to\infty}(2\log n)^{\frac{1}{2}}(Z_n-(2\log n)^{\frac{1}{2}})/\log\log n=\tfrac{1}{2}\}=1, \tag{1.2}$$

where

$$Z_n=\operatorname*{Sup}_{1\leq k\leq n} X_k.$$

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The theorem follows as a consequence of the results of Sections 2 and 3. If $r_n = 0$, for all $n \neq 0$, the variables X_i are mutually independent, and identically distributed. In this case, (1.2) follows as a special case of the general result proved in [5].

Section 2

In this section, we consider the asymptotic behavior of the limit supremum. The main result for this is Theorem 2.3.

Theorem 2.1. *Let $\{X_n, n=0,1,2,\ldots\}$ be a sequence of normalized Gaussian variates. Then*

$$P\{\limsup_{n\to\infty} (2\log n)^{\frac{1}{2}}(X_n - (2\log n)^{\frac{1}{2}})/\log\log n \leq \tfrac{1}{2}\} = 1.$$

Remark. No conditions on the joint distributions of the variates are required.

Proof. Let

$$c_n \equiv (2\log n)^{\frac{1}{2}} + \theta(2\log n)^{-\frac{1}{2}} \log\log n. \tag{2.1}$$

It is known that

$$P\{X_n > x\} \sim \phi(x) \equiv (2\pi)^{-\frac{1}{2}} x^{-1} e^{-x^2/2} \tag{2.2}$$

as $x \to \infty$ (Cramér [2] p. 374). Clearly $-\log\phi(c_n) = \frac{1}{2}\log 2\pi + \log c_n + c_n^2/2$. But

$$\begin{aligned}\log c_n &= \tfrac{1}{2}(\log 2 + \log\log n) + \log\left(1 + \theta(2\log n)^{-1}\log\log n\right)\\ &= \tfrac{1}{2}(\log 2 + \log\log n) + o(1) \quad \text{as} \quad n\to\infty,\end{aligned}$$

and

$$c_n^2/2 = \log n + \theta\log\log n + o(1) \quad \text{as} \quad n \to\infty.$$

So

$$\phi(c_n) \sim (4\pi)^{-\frac{1}{2}} n^{-1} (\log n)^{-(\theta+\frac{1}{2})} \tag{2.3}$$

as $n\to\infty$, and

$$\sum_{n=1}^{\infty} \phi(c_n) < \infty, \tag{2.4}$$

iff $\theta > \frac{1}{2}$. By the Borel-Cantelli Theorem (Loeve [3] p. 228) $X_n > c_n$ only a finite number of times with probability one. Equivalently

$$(2\log n)^{\frac{1}{2}}(X_n - (2\log n)^{\frac{1}{2}})/\log\log n > \theta$$

only a finite number of times with probability one, provided $\theta > \frac{1}{2}$. The theorem is proved.

Theorem 2.2. *If*

$$\exists\, \gamma < 1 : \sum_{k=1}^{\infty} k^{-\gamma} |r_k| < \infty, \tag{2.5}$$

and

$$\lim_{n\to\infty} r_n = 0, \tag{2.6}$$

then

$$P\{\limsup_{n\to\infty} (2\log n)^{\frac{1}{2}}(X_n - (2\log n)^{\frac{1}{2}})/\log\log n = \tfrac{1}{2}\} = 1. \tag{2.7}$$

Proof. Let A_n be the event $\{a_n < X_n \leqq b_n\}$, where $a_n \equiv (2\log n)^{\frac{1}{2}} + \theta(2\log n)^{-\frac{1}{2}} \cdot \log\log n$, $b_n \equiv (2\log n)^{\frac{1}{2}} + \frac{1}{2}(1+\varepsilon)(2\log n)^{-\frac{1}{2}} \log\log n$, $\varepsilon > 0$ is arbitrarily chosen and $\theta < \frac{1}{2}$. By (2.1), and Theorem 2.1, $X_n > b_n$ only a finite number of times, with probability one. Therefore, it is sufficient to prove that with probability one, infinitely many of the events A_k occur. Let n_0 be so chosen that if $n \geqq n_0$,

$$(2\log n)^{\frac{1}{2}} \leqq a_n, \qquad b_n \leqq (2\log n)^{\frac{1}{2}} + \varepsilon. \tag{2.8}$$

Let

$$J_n = \sum_{k=n_0}^{n} I_k,$$

where I_k is the indicator function for A_k. By (2.3), it follows that

$$EI_k \sim (4\pi)^{-\frac{1}{2}} k^{-1}\big((\log k)^{-(\frac{1}{2}+\theta)} - (\log k)^{-(1+\varepsilon/2)}\big) \sim (4\pi)^{-\frac{1}{2}} k^{-1} (\log k)^{-(\frac{1}{2}+\theta)}, \tag{2.9}$$

as $k \to \infty$. Since the term on the right side of (2.9) is not summable, then for any $n_1 \geqq n_0$,

$$\begin{aligned} EJ_n &\sim \sum_{k=n_1}^{n} EI_k \sim (4\pi)^{-\frac{1}{2}} \int_{n_1}^{n} t^{-1} (\log t)^{-(\frac{1}{2}+\theta)}\, dt \\ &= (\tfrac{1}{2}-\theta)^{-1} (4\pi)^{-\frac{1}{2}} (\log t)^{(\frac{1}{2}-\theta)} \big|_{n_1}^{n} \sim (\tfrac{1}{2}-\theta)^{-1} (4\pi)^{-\frac{1}{2}} (\log n)^{(\frac{1}{2}-\theta)}, \end{aligned} \tag{2.10}$$

as $n \to \infty$. It is sufficient to prove that for any integer n_2,

$$\lim_{n\to\infty} P\{J_n \leqq n_2\} = 0.$$

But by the Chebycheff inequality,

$$P\{J_n \leqq n_2\} = P\{J_n - EJ_n \leqq n_2 - EJ_n\} \leqq \operatorname{Var} J_n/(n_2 - EJ_n)^2 \sim \operatorname{Var} J_n/(EJ_n)^2.$$

Consequently, it is sufficient to prove that

$$\lim_{n\to\infty} \operatorname{Var} J_n/(EJ_n)^2 = 0, \tag{2.11}$$

since n_2 is arbitrary. But

$$\operatorname{Var} J_n = \sum_{k=n_0}^{n} \operatorname{Var} I_k + \sum_{k \neq l = n_0}^{n} \operatorname{cov}(I_k, I_l). \tag{2.12}$$

Clearly

$$\sum_{k=n_0}^{n} \operatorname{Var} I_k \leqq \sum_{k=n_0}^{n} EI_k = o\Big(\sum_{k=n_0}^{n} EI_k\Big)^2,$$

as $n \to \infty$. So the first term on the right side of (2.12) can be neglected in establishing (2.11). Clearly, if $k < l$,

$$\operatorname{cov}(I_k, I_l) = (2\pi)^{-1} \int_{a_k}^{b_k} \int_{a_l}^{b_l} D(r_{k-l}, s, t)\, ds\, dt,$$

where

$$D(r_{k-l}, s, t) = (1-r_{k-l}^2)^{-\frac{1}{2}} e^{-(s^2+t^2-2r_{k-l}st)/2(1-r_{k-l}^2)} - e^{-(s^2+t^2)/2} = \sum_{i=1}^{3} D_i(r_{k-l}, s, t),$$

$$D_1(r_{k-l}, s, t) = (1-r_{k-l}^2)^{-\frac{1}{2}} e^{-s^2/2} \left(e^{-(t-r_{k-l}s)^2/2(1-r_{k-l}^2)} - e^{-(t^2/2(1-r_{k-l}^2))}\right),$$

$$D_2(r_{k-l}, s, t) = (1-r_{k-l}^2)^{-\frac{1}{2}} e^{-s^2/2} \left(e^{-t^2/2(1-r_{k-l}^2)} - e^{-t^2/2}\right),$$

$$D_3(r_{k-l}, s, t) = \left((1-r_{k-l}^2)^{-\frac{1}{2}} - 1\right) e^{-(s^2+t^2)/2}.$$

Clearly,

$$e^{-(t-r_{k-l}s)^2/2(1-r_{k-l}^2)} - e^{-t^2/2(1-r_{k-l}^2)} \leqq (r_{k-l})_+ (1-r_{k-l}^2)^{-\frac{1}{2}} s\, e^{-(t-(r_{k-l})_+ s)^2/2(1-r_{k-l}^2)},$$

where $(r_{k-l})_+ = \max(r_{k-l}, 0)$. Consequently,

$$D_1(r_{k-l}, s, t) \leqq (r_{k-l})_+ (1-r_{k-l}^2)^{-\frac{1}{2}} s\, e^{-(s^2+t^2-2(r_{k-l})_+ st)/2(1-r_{k-l}^2)} \leqq (r_{k-l})_+ (1-r_{k-l}^2)^{-\frac{1}{2}} s\, e^{-(s^2+t^2-2(r_{k-l})_+ st)/2}.$$

Recalling (2.8), if $k<l$,

$$\begin{aligned}(2\pi)^{-1} \int_{a_k}^{b_k} \int_{a_l}^{b_l} D_1(r_{k-l}, s, t) &\leqq (2\pi)^{-1}(b_k-a_k)(b_l-a_l)\, b_l (r_{k-l})_+ (1-r_{k-l}^2)^{-\frac{1}{2}} k^{-1} l^{-(1-2(r_{k-l})_+(1+\varepsilon))} \\ &\leqq (8\pi)^{-1}(1+\varepsilon-2\theta)^2 (2\log k)^{-\frac{1}{2}} (2\log l)^{-\frac{1}{2}} \left((2\log l)^{\frac{1}{2}}+\varepsilon\right) \\ &\quad\cdot (\log\log k)(\log\log l)(r_{k-l})_+ (1-r_{k-l}^2)^{-\frac{1}{2}} k^{-1} l^{-(1-2(r_{k-l})_+(1+\varepsilon))}.\end{aligned} \tag{2.13}$$

$$(2\pi)^{-1} \int_{a_k}^{b_k} \int_{a_l}^{b_l} D_2(r_{k-l}, s, t)\, ds\, dt \leqq 0. \tag{2.14}$$

$$\begin{aligned}(2\pi)^{-1} \int_{a_k}^{b_k} \int_{a_l}^{b_l} D_3(r_{k-l}, s, t)\, ds\, dt &\leqq (2\pi)^{-1}\left((1-r_{k-l}^2)^{-\frac{1}{2}}-1\right)(b_k-a_k)(b_l-a_l)(k\,l)^{-1} \\ &\leqq (8\pi)^{-1}(1+\varepsilon-2\theta)^2 (\log k)^{-\frac{1}{2}} (\log l)^{-\frac{1}{2}} (\log\log k) \\ &\quad\cdot (\log\log l)\left((1-r_{k-l}^2)^{-\frac{1}{2}}-1\right)(k\,l)^{-1}.\end{aligned} \tag{2.15}$$

By (2.6),

$$\sup_{n\geqq 1} r_n < 1. \tag{2.16}$$

Therefore, the expression $(1-r_{k-l}^2)^{-\frac{1}{2}}$ is dominated by a constant for all $k \neq l$. In (2.13) the expressions $(2\log k)^{-\frac{1}{2}} \log\log k$, and $(2\log l)^{-\frac{1}{2}}\left((2\log l)^{\frac{1}{2}}+\varepsilon\right)$ are dominated by constants for all k and l, and in summing the terms (2.13), $\log\log l$ will be dominated by $\log\log n$. In (2.15), $(\log k)^{-\frac{1}{2}}\log\log k$, and $(\log l)^{-\frac{1}{2}}\log\log l$ are dominated by constants for all k and l. Consequently, there exists a constant C_0, such that, on regrouping terms

$$\sum_{k\neq l=n_0}^{n} \operatorname{cov}(I_k, I_l)/EJ_n^2 \leqq C_0(\log n)^{-1+2\theta}(\log\log n)$$

$$\cdot\left(\sum_{k=1}^{n}(r_k)_+\sum_{p=n_0}^{\infty} p^{-1}(p+k)^{-(1-2(r_k)_+(1+\varepsilon))}\right. \tag{2.17}$$

$$\left.+\sum_{k=1}^{n}\left((1-r_k^2)^{-\frac{1}{2}}-1\right)\sum_{p=n_0}^{\infty} p^{-1}(p+k)^{-1}\right).$$

By (2.6), since $\varepsilon>0$ is arbitrarily chosen, it is possible to chose it so that

$$1-2(r_k)_+(1+\varepsilon)>1-\gamma>0,$$

for all sufficiently large k. Note that for any β, $0<\beta<1-\gamma$,

$$\sum_{p=n_0}^{\infty} p^{-1}(p+k)^{-(1-\beta)}\leqq \int_{n_0-1}^{\infty} t^{-1}(t+k)^{-(1-\beta)}\,dt=\int_{n_0-1}^{k} t^{-1}(t+k)^{-(1-\beta)}\,dt$$

$$+\int_{k}^{\infty} t^{-1}(t+k)^{-(1-\beta)}\,dt\leqq k^{-(1-\beta)}\int_{n_0-1}^{k} t^{-1}\,dt+\int_{k}^{\infty} t^{-2+\beta}\,dt$$

$$=k^{-(1-\beta)}(\log k-\log(n_0-1))+(1-\beta)^{-1}k^{-(1-\beta)}\leqq k^{-\gamma}$$

for all sufficiently large k. Clearly for all sufficiently large k, $1-2(r_k)_+(1+\varepsilon)>1-\beta>\gamma>0$, so that the first sum on p in (2.17) is dominated by $k^{-\gamma}$. The second sum on p is dominated by the first one, and for sufficiently large k, $\left((1-r_k^2)^{-\frac{1}{2}}-1\right)\leqq|r_k|$. Therefore, there exists a positive real constant C_1 such that

$$\sum_{k\neq l=n_0}^{n} \operatorname{cov}(I_k, I_l)/EJ_n^2\leqq C_1(\log n)^{1-2\theta}\log\log n$$

$$\cdot\sum_{k=1}^{n} k^{-\gamma}|r_k|+o(1),$$

as $n\to\infty$. The theorem is proved.

Theorem 2.3. *The result* (2.7) *of Theorem* 2.2 *holds, provided either*

$$\exists\,\alpha>0:\ \lim_{n\to\infty} n^{\alpha} r_n=0, \tag{2.18}$$

or

$$\sum_{n=-\infty}^{\infty} r_n^2<\infty. \tag{2.19}$$

Proof. Clearly both conditions imply (2.6). Suppose (2.19) holds

$$\sum_{n=1}^{\infty} n^{-\frac{3}{4}}|r_n|\leqq\sum_{n=1}^{\infty} n^{-\frac{3}{2}}+\sum_{n=1}^{\infty} r_n^2<\infty.$$

So (2.5) holds. Now assume (2.18). For any positive finite constant $C, r_n < Cn^{-\alpha}$ for all sufficiently large n, and $n^{-1+\alpha/2} r_n < Cn^{-1-\alpha/2}$ which is summable. So (2.5) holds.

With probability one, $Z_n > c_n$ infinitely many times with probability one, if and only if $X_n > c_n$ infinitely many times with probability one. This follows from the fact that c_n is a non-decreasing sequence. So the results of this section hold if X_n is replaced by Z_n.

Section 3

In this section we consider the behavior of the limit infimum as n approaches infinity.

Lemma 3.1. *If the X_i are independent, and normal with means zero and variances* 1, *and* $\theta = -\frac{1}{2}$, *then*

$$\lim_{n\to\infty} p\{Z_n > c_n\} = \exp - (4\pi)^{-\frac{1}{2}}.$$

Proof. For any d.f. $F(x)$, $(1-F(x)) \sim -\log F(x)$, as $F(x) \to 1$. This can be seen by expanding $-\log F(x)$ in a power series about $F(x)=1$. Thus,

$$-\log F^n(c_n) = -n \log F(c_n) \sim n(1-F(c_n)) \sim n\,\phi(c_n) = (4\pi)^{-\frac{1}{2}} + o(1), \qquad \text{as } n \to \infty,$$

by (2.3). The lemma is proved.

Berman [1] has shown that for any non-negative integer n, and real c,

$$|P\{Z_n \leqq c\} - \bar{P}\{Z_n \leqq c\}| \leqq D_n(c), \tag{3.1}$$

where

$$D_n(c) \equiv \sum_{j=1}^{n-1} (n-j)\,|r_j|\, Q(c, |r_j|), \tag{3.2}$$

$$Q(c, |r_j|) \equiv (1-|r_j|^2)^{-\frac{1}{2}} \exp\{-c^2/(1+|r_j|)\},$$

and the measures $P(\cdot)$ and $\bar{P}(\cdot)$ refer respectively to that of the given process with covariance function r_n, and that of a sequence of i.i.d. normalized Gaussian variates.

Lemma 3.2. *If*

$$\lim_{n\to\infty} D_n(C(n,0)) = 0, \tag{3.3}$$

where

$$C(n,\varepsilon) = c_n, \tag{3.4}$$

and $\theta = -(\frac{1}{2}+\varepsilon)$, *then*

$$P\{\liminf_{n\to\infty} (2\log n)^{\frac{1}{2}} (Z_n - (2\log n)^{\frac{1}{2}})/\log\log n \leqq -\tfrac{1}{2}\} = 1.$$

Proof. Clearly, by (3.2), the conclusion of Lemma 3.1 holds. But this is sufficient to establish the result of the present lemma.

Lemma 3.3. *Suppose that for any $\varepsilon>0$, $n(\varepsilon, m)$ is a sequence of integers, such that*

$$C(n(\varepsilon, m), \varepsilon) \geqq C(n(\varepsilon, m+1), 2\varepsilon) \tag{3.5}$$

for all sufficiently large m. If, for all $\varepsilon>0$

$$Z_{n(\varepsilon, m)} \leqq C(n(\varepsilon, m), \varepsilon), \tag{3.6}$$

only a finite number of times with probability one, then

$$P\{\liminf_{n\to\infty} (2\log n)^{\frac{1}{2}}(Z_n-(2\log n)^{\frac{1}{2}})/\log\log n \geqq -\tfrac{1}{2}\}=1. \tag{3.7}$$

Proof. By definition (2.1) of c_n, and by (3.4) it is sufficient to prove that for any $\varepsilon>0$,

$$Z_n \leqq C(n, \varepsilon) \tag{3.8}$$

only a finite number of times with probability one. Let us assume a realization of the sequence, which is such that (3.6) holds. By assumption, of course, almost every realization has this property. Let n_0 be sufficiently large so that for any $n \geqq n_0$, there exists an m, so that, $n(\varepsilon, m) \leqq n \leqq n(\varepsilon+1, m)$, and (3.6) holds. Clearly

$$Z_n \geqq Z_{n(\varepsilon, m)} \geqq C(n(\varepsilon, m), \varepsilon) \geqq C(n(\varepsilon, m+1), 2\varepsilon) \geqq C(n, 2\varepsilon).$$

So (3.8) holds provided ε is replaced by 2ε. But ε was arbitrarily chosen. The theorem is proved.

Lemma 3.4. *The condition* (3.5) *is satisfied if, for all $\varepsilon>0$,*

$$n(\varepsilon, m) \equiv \exp \varepsilon m. \tag{3.9}$$

Proof. By definitions (2.1) and (3.4), clearly

$$C(n, \varepsilon) \equiv (2\log n)^{\frac{1}{2}} - (\tfrac{1}{2}+\varepsilon)(2\log n)^{-\frac{1}{2}} \log\log n. \tag{3.10}$$

For convenience, let

$$C(n(\varepsilon, m), \varepsilon) - C(n(\varepsilon, m+1), 2\varepsilon) \equiv \sum_{i=1}^{3} D_i(\varepsilon, m), \tag{3.11}$$

where

$$\begin{aligned} D_1(\varepsilon, m) &\equiv (2\log n(\varepsilon, m))^{\frac{1}{2}} - (2\log n(\varepsilon, m+1))^{\frac{1}{2}}, \\ D_2(\varepsilon, m) &\equiv (\tfrac{1}{2}+\varepsilon)\big((2\log n(\varepsilon, m+1))^{-\frac{1}{2}} \log\log n(\varepsilon, m+1) \\ &\qquad -(2\log n(\varepsilon, m))^{-\frac{1}{2}} \log\log n(\varepsilon, m)\big), \end{aligned}$$

and

$$D_3(\varepsilon, m) \equiv \varepsilon(2\log n(\varepsilon, m+1))^{-\frac{1}{2}} \log\log n(\varepsilon, m+1). \tag{3.12}$$

It is sufficient, of course, to prove that the term on the left side of (3.11) is positive for sufficiently large m. Clearly $(m+1)^{\frac{1}{2}} - m^{\frac{1}{2}} = \frac{1}{2} m^{-\frac{1}{2}}(1+o(1))$, as $m\to\infty$. So

$$D_1(\varepsilon, m) = -\tfrac{1}{2}(2\varepsilon/m)^{\frac{1}{2}}(1+o(1)),$$

as $m \to \infty$. But $D_2(\varepsilon, m) = (2\varepsilon)^{-\frac{1}{2}}(\frac{1}{2}+\varepsilon)(g(m+1)-g(m))$, where $g(m) = m^{-\frac{1}{2}} \log \varepsilon m$. So, for sufficiently large n,

$$D_2(\varepsilon, m) \sim -\tfrac{1}{2} m^{-\frac{3}{2}} \log \varepsilon m + m^{-\frac{3}{2}} \tag{3.13}$$

as $m \to \infty$, and

$$D_3(\varepsilon, m) = \varepsilon(2\varepsilon(m+1))^{-\frac{1}{2}} \log \varepsilon(m+1). \tag{3.14}$$

Combining (3.11), (3.12), (3.13) and (3.14), the result clearly follows. The lemma is proved.

Lemma 3.5. *If the X_i are independent and normalized Gaussian variates, then* (3.7) *holds.*

Proof. Reviewing the reasoning of Lemma 3.1, and (2.3),

$$-\log P\{Z_n \leqq C(n, \varepsilon)\} \sim (4\pi)^{-\frac{1}{2}} (\log n)^{\varepsilon} \qquad \text{as } n \to \infty.$$

So

$$-\log P\{Z_{n(\varepsilon, m)} \leqq C(n(\varepsilon, m), \varepsilon)\} \sim (4\pi)^{-\frac{1}{2}} (\varepsilon m)^{\varepsilon},$$

and there exists a real constant A_1 such that for sufficiently large m,

$$P\{Z_{n(\varepsilon, m)} \leqq C(n(\varepsilon, m), \varepsilon)\} \leqq \exp - A_1 m^{\varepsilon}.$$

Consequently

$$\sum_{m=1}^{\infty} P\{Z_n(\varepsilon, m) \leqq C(n(\varepsilon, m), \varepsilon)\} < \infty. \tag{3.15}$$

By the Borel-Cantelli Theorem (Loeve [3] p. 228), the conditions of Lemma 3.3 are satisfied and consequently its conclusion holds. The lemma is proved.

Under what conditions on the covariance function does the result of Lemma 3.5 hold? Reviewing the proof with particular attention to (3.15), and recalling (3.1), it is sufficient that for all sufficiently small ε,

$$\sum_{m=1}^{\infty} D_{n(\varepsilon, m)}(C(n(\varepsilon, m), \varepsilon)) < \infty.$$

It is sufficient that for all sufficiently small ε,

$$\exists \beta > 1: \lim_{m \to \infty} m (\log m)^{\beta} D_{n(\varepsilon, m)}(C(n(\varepsilon, m), \varepsilon)) = 0.$$

By definition (3.9), it is sufficient that for all sufficiently small ε,

$$\exists \beta > 1: \lim_{n \to \infty} (\log n)(\log \log n)^{\beta} D_n(C(n, \varepsilon)) < \infty. \tag{3.16}$$

It follows from the definitions (3.2) and (3.10) that $D_n(C(n, \varepsilon))$ is monotonically increasing in ε, for every fixed n, and so the condition (3.16) implies (3.3). So the condition of Lemma 3.2 is satisfied. Thus, the following theorem has been proved.

Theorem 3.1. *If* (3.16) *holds,*

$$P\{\liminf_{n \to \infty} (2 \log n)^{\frac{1}{2}} (Z_n - (2 \log n)^{\frac{1}{2}}) / \log \log n = -\tfrac{1}{2}\} = 1. \tag{3.17}$$

Theorem 3.2. *The condition* (3.16) *holds, provided*

$$\exists\, \alpha>0: \lim_{n\to\infty} n^{\alpha-1} \sum_{j=1}^{n-1} |r_j| = 0 \tag{3.18}$$

and

$$\lim_{n\to\infty} r_n = 0.$$

Proof. By the second of the conditions (3.18), it follows that the terms r_j, $1 \leqq j < \infty$, are all smaller than 1 in absolute value. Furtherfore it follows that 1 cannot be a limit point for r_j^2, and so $(1-r_j^2)^{-\frac{1}{2}}$ is bounded away from ∞ on $1 \leqq j < \infty$. Thus, these terms can be replaced, uniformly, by a constant C_1. Recalling (3.2), and (3.10), then

$$\begin{aligned}(\log n)(\log\log n)^{\beta} D_n(C(n,\varepsilon)) &\leqq C_1 \sum_{j=1}^{n-1} |r_j|\, n^{1-2/(1+|r_j|)} (\log n)^{(1+2\varepsilon)/(1+|r_j|)} \\ &= C_1 n^{-1} \sum_{j=1}^{n-1} |r_j|\, n^{2|r_j|/(1+|r_j|)} (\log n)^{(1+2\varepsilon)/(1+|r_j|)}\end{aligned} \tag{3.19}$$

for any $\varepsilon > 0$. Clearly, for sufficiently large n, the term on the right side of (3.19) is less than

$$C_1 n^{\alpha-1} \sum_{j=1}^{n-1} |r_j|.$$

Let n_0 be the smallest value of n, for which this is true. Consider the sum in (3.19), when the sum is from 1 to n_0. This sum approaches zero as $n\to\infty$, since each of its terms does. The theorem is proved.

Theorem 3.3. *The result* (3.17) *holds, provided that either*

$$\exists\, \alpha>0: \lim_{n\to\infty} n^{\alpha} r_n = 0, \tag{3.20}$$

or

$$\sum_{j=1}^{\infty} r_j^2 < \infty. \tag{3.21}$$

Proof. Clearly by Theorem 3.1 and 3.2, it is sufficient to show that, if either (3.20) or (3.21) holds, then (3.18) does. First, assume that (3.20) does. Let $\varepsilon>0$ be arbitrarily chosen. Then, there exists an n_0, such that if $n>n_0$, $r_n \leqq \varepsilon n^{-\alpha}$. Then

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n-1} |r_j| &\leqq \frac{1}{n}\sum_{j=1}^{n_0} |r_j| + \frac{1}{n}\sum_{j=n_0+1}^{n-1} |r_j| \leqq \frac{1}{n}\sum_{j=1}^{n_0-1} |r_j| + \frac{\varepsilon}{n}\int_{n_0-1}^{n} x^{-\alpha}\,dx \\ &\leqq \varepsilon(1-\alpha)^{-1} n^{-\alpha} + o(n^{-1}). \text{ So } \quad n^{\alpha-1}\sum_{j=1}^{n-1} |r_j| \leqq \varepsilon(1-\alpha)^{-1} + o(n^{\alpha-1}),\end{aligned}$$

provided $\alpha<1$. But ε was arbitrarily chosen. If $\alpha \geqq 1$, then, without loss of generality, it can be replaced by a value α' which is not.

Now consider the condition (3.21). By the Cauchy-Schwarz inequality,

$$\left(n^{\alpha-1}\sum_{j=1}^{n-1}|r_j|\right)^2 \leqq n^{2\alpha-1}\sum_{j=1}^{n-1} r_j^2,$$

which approaches 0, for any $\alpha<\frac{1}{2}$. That the second of the conditions (3.18) holds, is immediate. The theorem is proved.

Section 4

Theorem 1.1 has now been proved. The conditions (3.20) and (3.21) cannot be weakened substantially. In particular, in order that the conclusion of Theorem 1.1 hold, the condition

$$\lim_{n\to\infty} r_n = 0, \tag{4.1}$$

is not sufficient. Clearly (1.2) implies that

$$Z_n - (2\log n)^{\frac{1}{2}} \to 0, \tag{4.2}$$

with probability one. But in [4] a class of processes is presented, whose members are such that (4.1) holds, and (4.2) does not.

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