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l^1-Convolution Algebras: Representation and Factorization

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Summary. Let $*$ denote a convolution with respect to which l^1 becomes a Banach algebra. Necessary and sufficient conditions are given for $(l^1, *)$ to be represented by pointwise products of series of orthogonal polynomials. Properties of the polynomials are related to properties of the convolution; and, in the case of positive convolutions, an analogue of Hinčin's factorization theorem is obtained through the use of Delphic semigroups.

I. Introduction

Let $\mathscr{P}$ denote the probability measures on $N=\{0,1,2,\dots\}$. That is, $u=(u^0,u^1,u^2,\dots)$ belongs to, $\mathscr{P}$ if and only if $u^n\geqq 0$ for all n and $\sum_{n=0}^{\infty} u^n=1$. Let $*$ denote a commutative, associative binary operation on $\mathscr{P}$ so that $(\mathscr{P}, *)$ becomes a semigroup; such an operation is called a *convolution* after the classic example

$$u*v=w \Leftrightarrow w^k=\sum u^{k-n} v^n$$

(the unmarked summation sign $(\sum)$ indicates that the sum is to be performed as all repeated indices range over N). Observe that the relation above is equivalent to

$$\sum w^n x^n=(\sum u^n x^n)(\sum v^n x^n).$$

This can be generalized: suppose that for each $n\in N$, R_n is a polynomial of degree exactly n; then let

$$u*v=w \Leftrightarrow (\sum u^n R_n)(\sum v^m R_m)=\sum w^k R_k. \tag{1}$$

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If R_n are the normalized Gegenbauer polynomials $P_n^\lambda(x)/P_n^\lambda(1)$ then $(\mathscr{P}, *)$ is a semigroup (see Hirschman [7, 8]). This is also the case for the Jacobi polynomials; see [1] and [4] for a discussion of these and for additional references.

Given a semigroup $(\mathscr{P}, *)$ one is naturally led to questions of structure such as infinite divisibility, factorization, etc. In the classical case such a study may be conducted by means of generating functions ([5, 6]) or characteristic functions ([2, 12, 14, 17, 18]).

Any convolution on $\mathscr{P}$ can be extended to give a convolution on all of l^1 (the absolutely convergent series), so that $(l^1, *)$ becomes a Banach algebra. (Note that if l^1 is a Banach algebra with respect to an operation $*$, $\mathscr{P}$ need not be a semigroup with respect to that operation.) Theorem 1 gives necessary and sufficient conditions that a convolution on $\mathscr{P}$, or even on l^1, the space of absolutely convergent series, is given by (1) for some sequence of orthogonal polynomials. The fact that (1) holds leads to information about the polynomials which can be used, in turn, to obtain results about the structure of $(l^1, *)$; this is done in the proof of Theorem 1 and in Section III. Sections IV, V and VI contain a study of positive convolution; that is, it is assumed that $u * v$ is a non-negative sequence whenever u and v are. In this case it is possible to obtain sharp bounds for the polynomials on the support of the measure with respect to which they are orthogonal (Theorem 5). This is done by introducing a canonical semigroup $(\mathfrak{P}, *)$ which has a characteristic function similar to the object of that name studied elsewhere ([12, 14]). Some of the properties of the characteristic function are listed in Theorem 7; a discussion of infinitely divisible characteristic functions including an analogue of the Levy-Hinčine formula is contained in Theorem 8. Finally Section VI contains analogs of Hinčin's factorization theorems ([11, 12]) for the semigroup $(\mathfrak{P}, *)$ (Theorem 9) and many semigroups of non-negative sequences for which $(\mathscr{P}, *)$ is not a semigroup (Theorem 10). Theorem 9 is proved with the help of the delphic semigroups of Kendall [9], an approach suggested to the author by Richard Askey of the University of Wisconsin.

The operation defined by (1) can be given a more explicit form. Let $C_{n,m}^k$ be the *linearization constants* determined by

$$R_n R_m = \sum C_{n,m}^k R_k \tag{2}$$

then

$$u * v = w \Leftrightarrow w^k = \sum C_{n,m}^k u^n v^m. \tag{3}$$

It is often useful to know, and difficult to determine, whether the $C_{n,m}^k$ are non-negative (see [1] and [4]); for instance, if $R_n(1) = 1$ for all n an $C_{n,m}^k \geqq 0$ then $\mathscr{P}$ is a semigroup with respect to $*$. Necessary conditions for the non-negativity of the linearization constants can be obtained from Theorem 5, these are expressed as the negative results of Theorem 6.

Notations and Definitions. Let $N = \{0, 1, 2, \ldots\}$ and $N' = N - \{0\}$. An unmarked summation sign, $\sum$, indicates summation as all repeated indices range over N; $\sum'$ indicates the same for N'. *If* $u = (u^0, u^1, u^2, \ldots)$ let $\|u\| = \sum |u^n|$ and write

$u \in l^1$ if $\|u\| < \infty$,

$u \in c_c$ if $u^k = 0$ except for k in a finite set,

$u \geqq 0$ if $u^k \geqq 0$ for all $k \in N$,

$u \in l^1_+$ if $u \geqq 0$ and $u \in l^1$,

and $|u|$ for the sequence $(|u^0|, |u^1|, |u^2|, \dots)$.

A *convolution* is any operation, $*$, on l^1 with respect to which l^1 becomes a commutative Banach algebra denoted $(l^1, *)$.

Let π^n be the projection operator defined by $\pi^n(u) = u^n$, and define the basis elements E_n by $\pi^n(E_n) = 1$ and $\pi^k(E_n) = 0$ if $n \neq k$. If $*$ is a convolution on l^1 and $u \in l^1$, define $u^{*0} = E_0$ and $u^{*n} = u * u^{*(n-1)}$ for $n \geqq 1$. If $Q(x) = \sum a_n x^n$ is a polynomial let $Q(u) = \sum a_n u^{*n}$.

The convolution is evidently determined by the numbers

$$C^k_{n,m} = \pi^k(E_n * E_m)$$

so that $E_n * E_m = \sum C^k_{n,m} E_k$ and (3) holds. The Banach algebra inequality $\|u * v\| \leqq \|u\| \, \|v\|$ with $u = E_m$ and $v = E_n$ implies

$$\sum_{k=0}^{\infty} |C^k_{n,m}| \leqq 1 \qquad (n, m \in N).$$

II. Representation of l^1-Convolution Algebras

Let α be a measure on $(-\infty, \infty)$; the *support* of α consists of those x such that $\alpha([x-\varepsilon, x+\varepsilon]) > 0$ for every ε. A measure is *degenerate* if it is supported on a finite set of points.

Theorem 1. *The following two sets of conditions on an l^1-convolution algebra are equivalent*

I. $E_0 * u = u \qquad (u \in l^1),$ (1)

$\pi^0(E_n * E_n) > 0 \qquad (n \in N),$ (2)

$\pi^k(E_n * E_1) = 0 \qquad$ if $|n-k| > 1$ $(n, k \in N),$ (3)

$\pi^{n+1}(E_n * E_1) \neq 0 \qquad (n \in N).$ (4)

II. *There is a positive non-degenerate measure α supported on a subset S of $[-1, 1]$ and polynomials $\{P_n\}_{n \in N}$ orthogonal with respect to α and bounded by 1 on S such that*

$$P_0(x) = 1 \quad \textit{and} \quad P_1(x) = x,$$

and if u and v are in l^1 then

$$u * v = w \Leftrightarrow \left(\sum u^n P_n\right)\left(\sum v^m P_m\right) = \sum w^k P_k. \tag{5}$$

The convolution and the polynomials are related by

$$E_n = P_n(E_1). \tag{6}$$

Remark 1. Hypotheses (1)–(4) are related to properties of orthogonal polynomials; i.e., (2) corresponds to the fact that $\int P_n^2\, d\alpha > 0$, (3) to the recursion relation, and (4) to the observation that $P_1 P_n$ has degree $n+1$.

Remark 2. The relation $P_1(x) = x$ is not typical of orthogonal polynomials, but the restriction is easily removed. Indeed, if $\{R_n\}_{n\in N}$ is a sequence of polynomials orthogonal with respect to a measure β and if the convolution satisfies Equation (1), Section I then $R_n(x) = P_n(R_1(x))$ and if R_1 is chosen to be an increasing function there is $\lambda > 0$ such that $\beta(E) = \lambda\alpha(\{R_1(x): x \in E\})$ for measurable sets E.

Remark 3. The set S need not be all of $[-1, 1]$, for if $R_n(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ are the normalized Jacobi polynomials (see [16]) then

$$\int_{-1}^{1} R_n(x) R_m(x)(1-x)^\alpha (1+x)^\beta\, dx = \int_{-1}^{1} P_n(R_1(x)) P_m(R_1(x))(1-x)^\alpha (1+x)^\beta\, dx$$

and the change of variables $y = R_1(x)$ in the last integral shows that

$$S = \left[-\frac{1+\beta}{1+\alpha}, 1\right] \subsetneqq [-1, 1].$$

Proof of Theorem 1. Assume (1)–(4) and let

$$C_{n,m}^k = \pi^k(E_n * E_m) \tag{7}$$

then

$$C_{0,n}^k = E_n^k, \tag{1'}$$

$$C_{n,n}^0 > 0, \tag{2'}$$

$$C_{n,1}^k = 0 \quad \text{if } |n-k| > 1, \tag{3'}$$

$$C_{n,1}^{n+1} \neq 0. \tag{4'}$$

Application of π^k to $E_n * E_m = E_m * E_n$ and to $(E_p * E_q) * E_r = E_p * (E_q * E_r)$ yields

$$C_{n,m}^k = C_{m,n}^k$$

and

$$\sum C_{p,q}^j C_{j,r}^k = \sum C_{p,j}^k C_{q,r}^j. \tag{8}$$

An inductive argument based on (3′) and (8) yields.

$$C_{n,m}^k = 0 \qquad \text{unless } |n-m| \leqq k \leqq n+m. \tag{9}$$

If now $k=0$ in (8), (9) implies there is only one non-zero term on each side so that $C_{p,q}^r C_{r,r}^0 = C_{p,p}^0\, C_{q,r}^p$ which combined with (2′) and (4′) yields

$$C_{n,1}^{n+1} \cdot C_{1,n+1}^n > 0. \tag{10}$$

Finally $\|E_n * E_m\| \leqq 1$ implies

$$|C^k_{n,m}| \leqq \sum_{k=0}^{\infty} |C^k_{n,m}| \leqq 1. \tag{11}$$

The relations (3′) and (4′) together with

$$\begin{aligned} &P_0(x)=1, \quad P_1(x)=x, \\ &P_1 P_{n-1} = \sum C^k_{n-1,1} P_k \quad (n>1) \end{aligned} \tag{12}$$

define a unique sequence of polynomials $\{P_n\}_{n\in N}$ with degree of P_n being exactly n. It can be shown inductively that $E_1^{*n} = \sum_{k=0}^{n} A^j_k E_k$ with $A^n_n \neq 0$ so that $\{E_1^{*n}\}_{n\in N}$ is a linearly independent subset of l^1, thus if Q and R are polynomials then $Q=R$ if and only if $Q(E_1)=R(E_1)$. Hence (12) is equivalent to

$$\begin{aligned} &P_0(E_1)=E_0, \quad P_1(E_1)=E_1, \\ &P_1(E_1) * P_{n-1}(E_1) = \sum C^k_{n-1,1} P_k(E_1) \quad (n>1) \end{aligned}$$

so (6) follows inductively.

Now (7) can be written

$$E_m * E_n = \sum C^k_{m,n} E_k \tag{13}$$

so that (6) yields

$$P_m P_n = \sum C^k_{m,n} P_k. \tag{14}$$

Equation (13) implies

$$u * v = w \Leftrightarrow w^k = \sum C^k_{n,m} u^n v^m$$

thus (5) holds.

Now let k_n be the leading coefficient in P_n, so $k_n \neq 0$, and from (12) and (3′) $k_{n-1} = C^n_{n-1,1} \cdot k_n$; thus if $p_n = P_n/k_n$, (12) can be transformed into

$$\begin{aligned} &p_0(x)=1, \quad p_1(x)=x, \\ &p_n(x)=(x-c_n)\, p_{n-1}(x) - \lambda_n p_{n-2}(x) \quad (n>1) \end{aligned}$$

where $c_n = C^{n-1}_{n-1,1}$ and $\lambda_n = C^{n-2}_{n-1,1} C^{n-1}_{n-2,1}$. Equation (10) implies $\lambda_n > 0$ and (11) implies $0 \leqq |c_n|$, $\lambda_n \leqq 1$. It then follows from [3] or [15] that there is a positive measure α of total mass 1 supported in a compact set S with respect to which $\{p_n\}_{n\in N}$ and hence $\{P_n\}_{n\in N}$ are orthogonal polynomials. Integration of (14) with respect to α yields

$$\int P_m P_n = C^0_{m,n}, \tag{15}$$

so if $u \in c_c$

$$\begin{aligned}\int (\sum u^n P_n)^2\, d\alpha &\leqq \sum C^0_{m,n} u^m \cdot u^n \\ &= \pi^0(u * u) \\ &\leqq \|u * u\| \leqq \|u\|^2. \end{aligned} \tag{16}$$

Now let $u = E_m^{*k}$, then $\sum u^n P_n(E_1) = \sum u^n E_n = E_m^{*k} = [P_m(E_1)]^{*k} = P_m^k\,(E_1)$, so $\sum u^n P_n = P_m^k$ and (16) implies

$$\int |P_m^k|^2\, d\alpha \leqq \|E_m^{*k}\|^2 \leqq 1$$

or

$$\{\int |P_n|^{2k}\, d\alpha\}^{1/2k} \leqq 1 \quad (k = 0, 1, 2, \ldots).$$

These inequalities and the continuity of P_n imply

$$|P_n(x)| \leqq 1 \qquad (x \in S). \tag{17}$$

Let $n = 1$ in (17); the fact that $P_1(x) = x$ implies that $S \subset [-1, 1]$. Equations (15) and (2′) imply that $\int P_n^2\, d\alpha > 0$ for all n, so the orthogonality of $\{P_n\}_{n \in N}$ implies that α is non-degenerate.

Now suppose that the hypotheses II hold. Let $h_k = \{\int P_k^2\, d\alpha\}^{-1}$ and define constants $C^k_{m,n}$ by Equation (14) so that

$$C^k_{m,n} = h_k \int P_k P_m P_n\, d\alpha$$

then

$$C^k_{0,n} = h_k \int P_k P_n\, d\alpha = E_n^k, \tag{1″}$$

$$C^0_{n,n} = h_0 \int P_n^2\, d\alpha > 0, \tag{2″}$$

$$C^k_{n,1} = h_k \int P_k P_n P_1\, d\alpha = 0 \quad \text{if } |n-k| > 1, \tag{3″}$$

$$C^{n+1}_{n,1} = h_{n+1} \int P_{n+1} P_n P_1\, d\alpha \neq 0. \tag{4″}$$

If the right-hand member of (5) is multiplied by $h_l P_l$ and integrated it will yield

$$w^l = \sum C^l_{m,n} u^m v^n$$

so that $\pi^l(E_m * E_n) = C^l_{m,n}$ and hence (1″)–(4″) imply (1)–(4).

III. Harmonic Analysis of $(l^1, *)$ and the Non-Existence of Generating Functions

Let $(l^1, *)$ and $\{P_n\}_{n \in N}$ be as in Theorem 1. The representation of $(l^1, *)$ as a space of continuous functions on S may or may not be the Gelfand representation, but is closely related to it. Let Δ be the set of complex numbers defined by

$$\Delta = \{z : |P_n(z)| \leqq 1 \text{ for all } n \in N\}$$

and, when defined,

$$h_z(u) = \sum u^n P_n(z).$$

Theorem 2. *Each complex homomorphism h of $(l^1, *)$ has the form h_z for a unique $z \in \Delta$.*

Proof. The ideas of the proof is essentially that of Hirschman [7], Lemma 2d, but since that paper is not generally available an argument is given here.

If $z \in \Delta$, then $h_z(u)$ is defined for each u, and h_z is a homomorphism because of Theorem 1. The uniqueness follows from the observation that $h_z(E_1) = P_1(z) = z$.

Now assume that h is a homomorphism of $(l^1, *)$, let $z = h(E_1)$ then $z h(E_{n-1}) = \sum C_{n-1,1}^k h(E_n)$ for $n > 1$ so that $h(E_n) = P_n(z)$ for all n. But h is a bounded linear functional on l^1 with norm no greater than one [13, p. 206] so $|P_n(z)| = |h(E_n)| \leq \|E_n\| = 1$, thus $z \in \Delta$.

Corollary. *Let z be any complex number, then the sequence $\{P_n(z)\}_{n \in N}$ is either unbounded or bounded by unity.*

Proof. If $\{P_n(z)\}_{n \in N}$ is bounded h_z is a homomorphism of l^1.

Remark. Theorem 1 implies that $S \subset \Delta$ and (cf. Remark 3 in Section II) the inclusion may be proper. It would be desirable to have a condition on the convolution which would guarantee equality.

The results of Szegö [16] will be used to show equality holds for many systems; but unfortunately the hypothesis must be given in terms of the measure $d\alpha$.

Theorem 3. *Assume that $\{P_n\}_{n \in N}$ are as in Theorem 1 but that $d\alpha(x) = w(x)\,dx$, and suppose that $\int_0^\pi |\log w(\cos\theta)|\,d\theta$ exists, then $\Delta = [-1, 1]$.*

The proof here is also inspired by [7].

Proof. The integral condition implies $w(x) > 0$ for almost all x, so that the support of α is $[-1, 1]$. Thus Theorem 1 implies $[-1, 1] \subset \Delta$.

The hypotheses of the Theorem ensure the validity of the asymptotic formula

$$P_n(z) \cong D\, C_{n,n}^{0\ \frac{1}{2}} \zeta^n \tag{1}$$

where z is a complex number not in $[-1, 1]$, $z = \frac{1}{2}(\zeta + \zeta^{-1})$ with $|\zeta| > 1$, and D depends only on z and is not zero ([16], Theorem 12.1.2). Now assume by way of contradiction that $z \in \Delta - [-1, 1]$. Let $|\zeta| > 1$ be such that $z = \frac{1}{2}(\zeta + \zeta^{-1})$. Define $\zeta_1 = |\zeta|$ and $z_1 = \frac{1}{2}(\zeta_1 + \zeta_1^{-1})$. Since $z \in \Delta$, $P_n(z)$ is bounded, so that (1) implies $C_{n,n}^{0\ \frac{1}{2}} \zeta^n$ is also. Thus $C_{n,n}^{0\ \frac{1}{2}} \zeta_1^n$ is bounded and (1) shows that $P_n(z_1)$ must be as well. The Corollary to Theorem 2 implies that $z_1 = P_n(z_1) \leq 1$ which contradicts $z_1 = \frac{1}{2}(|\zeta| + |\zeta|^{-1}) > 1$.

Many of the results proved below can be obtained when $(l^1, *)$ admits a generating function (see [6]). However it will be shown here that a large class of convolutions which satisfy (1)–(4) including those arising from the Gegenbauer or more generally the Jacobi polynomials do not admit of generating functions.

A *generating function* ([6]) is an injective mapping from $B=\{u\in l^1_+ : \|u\|\leqq 1\}$ into the holomorphic functions on the unit disk such that if $u, v\in B$, $a, b\geqq 0$, and $a+b\leqq 1$ then $\Phi(au+bv)=a\Phi(u)+b\Phi(v)$, $\Phi(u*v)=\Phi(u)\Phi(v)$, and if $\{u_n\}_{n\in N}\subset B$ satisfies $\pi^k u_n\to\pi^k u$ for some $u\in B$ and each $k\in N$, then $\Phi(u_n)$ converges uniformly on compact subsets of the disk to $\Phi(u)$, Φ can be extended to give a representation of $(l^1,*)$, which will also be called a *generating function*, and (cf. [13], p. 206)

$$|\Phi(u)(z)|\leqq\|u\| \qquad (u\in l^1, |z|<1). \tag{2}$$

Theorem 4. *If the hypotheses of Theorem 3 are satisfied, then* $(l^1,*)$ *does not admit a generating function.*

Proof. Assume, by way of contradiction, that Φ is a generating function, let $|z|<1$ and $w=\Phi(E_1)(z)$. Observe that $P_n(w)=P_n(\Phi(E_1)(z))=\Phi(P_n(E_1))(z)=\Phi(E_n)(z)$, so (2) implies $|P_n(w)|\leqq 1$ for all n. This implies, by Theorem 3 that $-1\leqq w\leqq 1$, so the Open Mapping Theorem allows us to conclude that $z\to\Phi(E_1)(z)$ is a constant function with value, say, t_1. Now choose k, m, n such that $P_k(t_1)\leqq P_m(t_1)\leqq P_n(t_1)$, so there is a $\lambda\in[0,1]$ such that $P_m(t_1)=\lambda P_k(t_1)+(1-\lambda)P_n(t_1)$. It then follows that $\Phi(E_m)=\Phi(\lambda E_k+(1-\lambda)E_n)$ which violates the injectivity of Φ.

IV. Positive Convolution Algebras

To the assumptions that $(l^1,*)$ and $\{P_n\}_{n\in N}$ are as in Theorem 1 add now the requirement that $C^k_{m,n}\geqq 0$ $(k,m,n\in N)$ so that $(l^1_+,*)$ becomes a semigroup. The aim of the present section is to obtain a sharp bound on S for the polynomials $P_n(x)$.

It will be useful to study a larger algebra than $(l^1,*)$: let $b=\operatorname{lub} S$, and let

$$q_n=P_n(b).$$

The definition of the larger algebra requires a

Lemma. $0<q_n\leqq 1$.

Proof. If k_n is the leading coefficient of P_n then $k_n=k_{n-1}/C^n_{n-1,1}$ and $k_0=1$ so that $k_n>0$. Since all the zeros of P_n lie in $(-1,b)$ the first inequality follows. The second is a consequence of Theorem 1.

Now define for sequences u

$$\|u\|_b=\sum|u^n|q_n,$$
$$l^1_b=\{u: \|u\|_b<\infty\},$$
$$Q_n(x)=P_n(x)/q_n$$

and, if $u\in c_c$ let

$$u\tilde{}(x)=\sum u^n q_n Q_n(x)$$

so that if $v \in c_c$

$$(u * v)^{\sim} = u^{\sim} v^{\sim}. \tag{1}$$

The convolution is readily extended to an operation of l_b^1 which satisfies

$$\|u * v\|_b \leqq \|u\|_b \|v\|_b,$$

while

$$\|u * v\|_b = \|u\|_b \|v\|_b \quad \text{if } u \geqq 0 \text{ and } v \geqq 0. \tag{2}$$

Note also that the Lemma implies $l^1 \subset l_b^1$. The utility of $(l_b^1, *)$ rests largely on the following

Theorem 5. *If* $C_{n,m}^k \geqq 0$ *then* $|P_n(x)| < P_n(b)$ *almost everywhere with respect to* α.

Proof. Recall that $\int P_n^2 \, d\alpha = C_{n,n}^0$ so that imitating an argument in the proof of Theorem 1

$$\int |u^{\sim}|^2 \, d\alpha = \sum (u^n)^2 \, C_{n,n}^0 = \pi^0(u * u) \leqq \|u * u\|_b.$$

Thus

$$\int |u^{\sim}|^2 \, d\alpha \leqq \|u\|_b^2. \tag{3}$$

Now $E_n^{\sim} = q_n Q_n$ so that $(E_n^{*k})^{\sim} = (q_n Q_n)^k$ whence from (3) and (2)

$$\int [q_n Q_n)^k]^2 \, d\alpha \leqq \|E_n^{*k}\|_b^2 = \|E_n\|_b^{2k} = q_n^{2k}$$

so that

$$\{\int |Q_n|^{2k} \, d\alpha\} \leqq 1$$

which implies that

$$|Q_n(x)| \leqq 1 \qquad \text{a.e. } (d\alpha). \tag{4}$$

The polynomials $\{Q_n\}_{n \in N}$ form a complete orthogonal system in $L^2(d\alpha)$ with

$$\int Q_n^2 \, d\alpha = C_{n,n}^0 q_n^{-2}, \tag{5}$$

so that if $\varphi \in L^2(d\alpha)$ and $\varphi^{\hat{}}(n) = \int \varphi Q_n \, d\alpha$

$$\sum [\varphi^{\hat{}}(n)]^2 \, q_n^2 / C_{n,n}^0 = \int \varphi^2 \, d\alpha. \tag{6}$$

But (4) and (5) imply $q_n^2 / C_{n,n}^0 \geqq 1$ so that (6) leads to

$$\lim_{n \to \infty} \varphi^{\hat{}}(n) = 0. \tag{7}$$

Now let m be fixed and let

$$A = \{x : Q_m(x) = 1\}.$$

Then if $x \in A \cap S$, the relation (14) in Section 2 implies

$$1 = Q_m^2(x) = \sum_{k=0}^{\infty} q_k q_m^{-2} C_{m,m}^k Q_k(x)$$
$$\leqq \sum_{k=0}^{\infty} q_k q_m^{-2} C_{m,m}^k = 1$$

because of (4). Thus $C_{m,m}^k Q_k(x) = C_{m,m}^k$, but $C_{m,m}^{2m} \neq 0$ so that $Q_{2m}(x) = 1$. The argument can be iterated to show

$$Q_{2^k m}(x) = 1 \qquad (k \in N', x \in A \cap S).$$

Thus if χ is the characteristic function of A,

$$\hat{\chi}(2^k m) = \int \chi Q_{2^k m}(x)\, d\alpha = \alpha(A) \qquad (k \in N')$$

so that $\alpha(A) = 0$ by (7), whence $|Q_m(x)| < 1$ almost everywhere $(d\alpha)$ and the Theorem follows.

Corollary. *If $C_{n,m}^k \geqq 0$ then $\alpha(\{b\}) = 0$ and b is a limit point of S.*

Proof. $Q_n(b) = 1$.

Remark 1. The methods of Theorem 5 can be used to obtain a result in more general cases; for example, if $C_{n,n}^{2n} \geqq 0$ $(n \in N)$ then $|P_n(x)| < 1$ a.e. $(d\alpha)$.

Remark 2. Because P_n is a polynomial the set of x for which $|P_n(x)| = P_n(b)$ or $|P_n(x)| = 1$ is finite so the strong inequality is a restriction on the location of point masses of α.

Theorem 5 can also be used to obtain some results on the positivity of linearization constants of orthogonal polynomials:

Theorem 6. *Let $\{R_n\}_{n \in N}$ be a sequence of polynomials orthogonal with respect to a measure β, define $b_{n,m}^k$* by

$$R_n R_m = \sum_{k=0}^{\infty} b_{n,m}^k R_k. \tag{8}$$

(i) *If β has unbounded support then $\sup_{n,m} \sum |b_{m,n}^k| = \infty$.*

(ii) *If β has unbounded support and $R_n(1) = 1$ then $b_{m,n}^k$ is sometimes negative.*

(iii) *Suppose β is supported in $[-1, 1]$ and $R_n(1) > 0$; let c be the least upper bound of the support of β. Then $\beta(\{c\}) > 0$ implies that $b_{m,n}^k$ is sometimes negative.*

Proof. (i) Assume, by way of contradiction, that $\sup_{n,m} \sum_{k=1}^{\infty} |b_{n,m}^k| = M < \infty$. Let $r_n = M^{-1} R_n$ and $C_{n,m}^k = M^{-1} b_{n,m}^k$ so that $r_n r_m = \sum C_{n,m}^k r_k$ and $C_{n,m}^k$ defines an l^1-convolution which satisfies the conditions of Theorem 1. Then if $\{P_n\}_{n \in N}$ and α are as in Theorem 1, it follows from Remark 2 after Theorem 1 that $r_n = P_n(r_1)$ so that $\{r_n\}_{n \in N}$ is a family of polynomials orthogonal with respect to a compactly supported measure γ. This implies (cf. [16], Theorems 3.1.1 and 6.1.1) that β has compact support.

(ii) If $R_n(1)=1$, evaluation of (8) at $x=1$ leads to $\sum_{k=1}^{\infty} b_{n,m}^k=1$ which contradicts (i) unless some $b_{n,m}^k$ are negative.

(iii) Assume, by way of contradiction, that $b_{n,m}^k$ is always non-negative. Let $r_n(x)=R_n(x)/R_n(1)$, $C_{n,m}^k=b_{n,m}^k R_k(1)/R_n(1)\,R_m(1)$ and observe as in part (i) that $C_{n,m}^k$ defines an l^1-convolution which, in this case, is positive. If now P_n, α and γ have the same meaning as in part (i), then γ has compact support and $\int r_n\,d\gamma = \int r_n\,d\beta = 0$ for all $n \geqq 1$. Thus the Stone-Weierstrass Theorem and the Riesz Representation Theorem imply that β is a multiple of γ. It now follows from the Corollary to Theorem 5 and Remark 2 following Theorem 1 that $R_1(b)=c$ and $\beta(\{c\})=\alpha(\{b\})=0$.

V. Characteristic Functions

Assume $*$ is a positive convolution (that is $u*v \geqq 0$ if $u \geqq 0$ and $v \geqq 0$) which satisfies the hypotheses of Theorem 1. Let

$$\mathfrak{P}=\{u: u\in l_b^1,\ u\geqq 0, \text{ and } \|u\|_b=1\}.$$

In analogy with [12] and [14] the members of $\mathfrak{P}$ will be called *distribution sequences* or simply *distributions*, and the function $u^{\sim}(x)=\sum u^n q_n Q_n(x)$ (which converges uniformly on S) is called the *characteristic function* of u. Write $\mathfrak{P}^{\sim}$ for the class of characteristic functions.

The structure of distribution sequences and characteristic functions presented here has most of the features of that contained in [14] with the exception that the interval of orthogonality, I, is replaced by the support set S. One has (cf. [14])

Theorem 7. *Let $\{u_j\}_{j\in N}$ be a sequence of distributions.*

(a) *Suppose u is a distribution and $\pi^k u_j \to \pi^k u$ for each k, then $u_j^{\sim}$ converges uniformly on S to $u^{\sim}$.*

(b) *Suppose $u_n^{\sim}$ converges in S to a function f which is continuous at b, then u_j converges pointwise to a distribution u, $u^{\sim}=f$, and thus the convergence of $u_j^{\sim}$ to f is uniform on S.*

(c) *If $u\in\mathfrak{P}$ and $\alpha(\{x: u^{\sim}(x)=1\})>0$, then $u=E_0$; equivalently, if $f\in\mathfrak{P}^{\sim}$ and $\alpha(\{x: f(x)=1\})>0$ then $f(x)=1$ for all $x\in S$.*

Proof. Parts (a) and (b) can be established by modifying the corresponding proofs in [14]. Part (c) follows from Theorem 5 because if $u\in\mathfrak{P}$ and $u^{\sim}(x)=1$, then $\sum u^n q_n Q_n(x)=1=\sum u^n q_n$ so $\sum' u^n q_n(1-Q_n(x))=0$ for x belonging to a set of positive α-measure. But $1-Q_n(x)>0$ almost everywhere with respect to α by Theorem 5 so $u^n q_n=0$ if $n\geqq 1$, whence $u=E_0$.

A characteristic function f is said to be *infinitely divisible* if corresponding to each $n\in N'$ is $f_n\in\mathfrak{P}^{\sim}$ such that $f=(f_n)^n$.

Theorem 8. (a) *Infinitely divisible characteristic functions have no zeros in S.*

(b) *Products of infinitely divisible characteristic functions are infinitely divisible.*

(c) *If* $\{f_j\}_{j\in N}$ *is a sequence of infinitely divisible characteristic functions which converges pointwise to a characteristic function f, then f is infinitely divisible.*

(d) *Suppose* $u^n \geqq 0$ *and* $\sum' u^n q_n < \infty$ *then*

$$f = \exp[\textstyle\sum' u^n q_n(Q_n - 1)] \tag{1}$$

is an infinitely divisible characteristic function; conversely, every infinitely divisible characteristic function can be expressed in the form (1).

Proof. Let f be an infinitely divisible characteristic function, and choose characteristic functions f_n so that $(f_n)^n = f$. Then $f = (f_2)^2$ so $f \geqq 0$, and $f = [(f_{2n})^2]^n$ so it may be assumed that $f_n \geqq 0$ and $f_n = f^{1/n}$. It then follows that $g(x) = \lim_{n\to\infty} f^{1/n}(x)$ exists in S. The Corollary to Theorem 5 states that b is a limit point of S; since $f(b) = 1$ and since f is continuous on S, there is $c < b$ such that $f(x) > 0$ if $x \in S \cap [c, b]$, whence

$$g(x) = 1 \quad \text{if } x \in S \cap [c, b]. \tag{2}$$

Thus g is continuous at b so that Theorem 7(b) implies g is a characteristic function. Finally (2) and Theorem 7(c) imply $g(x) \equiv 1$ which yields (a).

The proofs of (b)–(d) can be obtained by modifying arguments in [12] and [14].

VI. Factorization Theorems

The purpose of this section is to prove analogs of Hinčin's factorization theorems (cf. [11], Theorems 5.4.2 and 5.5.4 or [12], Theorems 6.2.1 and 6.2.2) which give decompositions of $(l^1_+, *)$ much like the prime factorization of positive integers. Thus if G is a Hausdorff topological abelian semigroup with identity e and if $f \in G$, then f is *decomposable* if there are noninvertible elements g and h of G such that $f = gh$, and f is *infinitely divisible* if corresponding to each $n \in N$ there is $f_n \in G$ such that $f = (f_n)^n$. The semigroup G will be said to have *property H* if it satisfies the two conditions.

H1. If $f \in G$ is decomposable with no indecomposable factor, then f is infinitely divisible.

H2. If $f \in G$ then $f = gh$ where g is infinitely divisible and h is a convergent product of a finite or denumerable sequence of indecomposable factors.

Hinčin's results are that H1 and H2 hold when G is the classical semigroup of probability measures on the line.

The aim of this section is to establish property H for two semigroups of positive sequences. Let $*$ be a convolution which satisfies the hypotheses of Theorem 1.

Assume further that $C^k_{n,m} \geqq 0$ for all $k, n, m \in N$. The two factorization theorems promised are:

Theorem 9. $(\mathfrak{P}, *)$ has property H.

Theorem 10. $(l^1_+, *)$ *has property H provided any of the following conditions hold:*

(1) *There is* $\varepsilon > 0$ *such that*

$$\|u * v\| \geqq \varepsilon \|u\| \cdot \|v\| \qquad (u, v \in l^1_+).$$

(2) $\{P_n(1)\}_{n \in N}$ *is bounded.*

(3) $1 \in S$.

Proof of Theorem 9. The method, which draws upon [2] and [10], is to show that $\mathfrak{P}$ is a union of sub-semigroups which have property H by virtue of being delphic (see Kendall [9]). The proof is contained in the following three lemmas and the fact that $\mathfrak{P}$ and $\mathfrak{P}^{\sim}$ are isomorphic.

Let $H_c = \{f \in \mathfrak{P}^{\sim} : f(x) > 0 \text{ if } x \in S \cap [c, b]\}$ have the topology of uniform convergence on S and let

$$\Delta_c(f) = -\int_c^b \log f \, d\alpha.$$

Lemma 1. *To each* $\varepsilon > 0$ *there corresponds* $\delta > 0$ *such that if* $f \in H_c$ *and* $\Delta_c(f) < \delta$, *then* $1 - f(x) < \varepsilon$ *for all* $x \in S$.

Proof. Let $c^n = \int_c^b Q_n \, d\alpha \Big/ \int_c^b d\alpha$, then (cf. Eq. (7) of Section IV) $\lim_{n \to \infty} c^n = 0$, and by Theorem 5 $c^n < 1$ if $n > 0$ so that $c_0 = \min\{1 - c^n : n > 0\} > 0$. Now let $\varepsilon > 0$, choose $\delta < \frac{1}{2} \varepsilon c_0 \int_c^b d\alpha$ and assume $\Delta_c(f) < \delta$ for some $f \in H_c$. Since $1 - u \leqq -\log u$ it follows that $\int_c^b (1 - f) \, d\alpha < \delta$, so if $f = \sum u^n q_n Q_n$, then

$$\sum' u^n q_n \leqq c_0^{-1} \sum' u^n q_n [1 - c^n] = c_0^{-1} \left[\int_c^b (1 - f) \, d\alpha \right] \Big/ \int_c^b d\alpha < \tfrac{1}{2} \varepsilon,$$

so $1 - f = \sum' u^n q_n [1 - Q_n] < \varepsilon$.

Lemma 2. *Assume* f_{ij} *is a characteristic function for* $1 \leqq j \leqq k(i)$, $i = 1, 2, 3, \ldots$ *and suppose that there is a characteristic function f such that*

$$f(x) = \lim_{i \to \infty} \prod_{j=1}^{k(i)} f_{ij}(x)$$

for each $x \in S$. *Then f is infinitely divisible provided that*

$$\lim_{i \to \infty} \max_{1 \leqq j \leqq k(i)} [1 - f_{ij}(x)] = 0$$

for each $x \in S$.

Proof. If

$$x \in S, \quad f(x) = \lim_{i \to \infty} \exp\left\{ \sum_{j=1}^{k(i)} [\log f_{ij}(x)] \right\}$$
$$= \lim_{i \to \infty} \exp\left\{ \sum_{j=1}^{k(i)} [f_{ij}(x) - 1] \right\}.$$

The Lemma now follows from Theorem 8(d) and (c) because f is a limit of infinitely divisible characteristic functions.

Lemma 3. *$\mathfrak{P}^{\sim}$ is a union of the subsemigroups H_c each of which is delphic under Δ_c, and each of which has the property that if $f \in H_c$, then $F(f) \subset H_c$.*

Proof. If $f \in \mathfrak{P}^{\sim}$ then $f(b)=1$ and since f is continuous on S, $f \in H_c$ for some $c<b$ because b is a limit point of S by the Corollary to Theorem 5.

That H_c satisfies most of the axioms for a delphic semigroup is obvious from the definition of Δ_c and the first two lemmas. The proof is completed by showing

$$F(f)=\{g \in H_c : f=gh \text{ for some } h \in H_c\}$$

is compact for each $f \in H_c$.

Let $\{g_j\}_{j \in N}$ be any sequence from H_c. For each j there is $u_j \in \mathfrak{P}$ such that $g_j = u_j^{\sim}$, so if $\{g_j\}_{j \in N}$ is replaced by an appropriate subsequence $\pi^k u_j$ will converge for each k as $j \to \infty$. Let u be the limiting sequence, then $\|u\|_b \leqq 1$. Define $g = \sum u^n q_n Q_n$, then g_j converges weakly to g in the sense that

$$\int g_j \varphi \, d\alpha \to \int g \varphi \, d\alpha \qquad (\varphi \in L^1(d\alpha)). \tag{4}$$

Now for each j there is $h_j \in H_c$ such that $g_j h_j = f$, but if $x \in S \cap [c,b]$ $0<h_j(x) \leqq 1$ so that

$$f(x) \leqq g_j(x) \leqq 1 \qquad (x \in S \cap [c,b]). \tag{5}$$

It is easily seen from (4) and (5) that $f(x) \leqq g(x) \leqq 1$ for almost all $x \in S \cap [c,b]$. The continuity of g, the fact that $f(b)=1$, and the Corollary to Theorem 5 imply that $g(b)=1$, whence $u \in \mathfrak{P}$, $g \in H_c$, and (from Theorem 7) $g_j \to g$ uniformly.

This shows every sequence from $F(f)$ has a subsequence which converges uniformly in $F(f)$, hence $F(f)$ is compact and H_c is delphic.

The proof of Theorem 10 is easily obtained by using Theorem 9 together with Lemma 4 and Lemma 6 which are proved below.

The sums $\sum C^j_{n,m} u^n v^m$ may converge even if u and v do not belong to l^1 or even l^1_b, when they do the result will be denoted $u * v$.

Lemma 4. *Assume the hypotheses of Theorem 10 to be satisfied and let $u \in l^1_+$.*

(a) *If $u = v * w$ with $v \geqq 0$ and $w \geqq 0$ then v and w belong to l^1_+.*

(b) *u is decomposable in l^1_+ if and only if $u/\|u\|_b$ is decomposable in $\mathfrak{P}$.*

(c) *u is divisible in l^1_+ if and only if $u/\|u\|_b$ is divisible in $\mathfrak{P}$.*

Proof. By Theorem 1, (3) implies (2); we show below that (2) implies (1), and so we may assume that condition (1) of Theorem 9 holds.

If (2) holds it follows from the Corollary to Theorem 2 that $|P_n(1)| \leqq 1$ for all n. In fact, $P_n(1)=1$ for all n; this is true for $n=0$ and $n=1$. Assume $P_k(1)=1$ for $k<n$ then evaluation of $P_1 P_{n-1} = \sum C^k_{n-1,1} P_k$ at 1 yields

$$\begin{aligned} 1 &= C^{n-2}_{n-1,1} + C^{n-1}_{n-1,1} + C^n_{n-1,1} P_n(1) \\ &\leqq C^{n-2}_{n-1,1} + C^{n-1}_{n-1,1} + C^n_{n-1,1} \leqq 1, \end{aligned}$$

so $\leqq$ can be replaced by $=$ and $P_n(1)=1$ because $C^n_{n-1,1}\neq 0$. Now, evaluating $P_nP_m=\sum C^k_{n,m}P_k$, at 1 yields $\sum_{k\in N} C^k_{n,m}=1$ $(n, m\in N)$ so that if $u\geqq 0$ and $v\geqq 0$, $\|u*v\| =\|u\|\,\|v\|$, and (1) holds with $\varepsilon=1$.

Conclusion (a) of the Lemma is a simple consequence of (1). To prove (b) assume first that $u=v*w$ with u, v, and w in l^1_+, then u, v, and w are in l^1_b, $\|u\|_b =\|v\|_b\,\|w\|_b$ (see Eq. (2) of Section IV) so that

$$\frac{u}{\|u\|_b}=\frac{v}{\|v\|_b}*\frac{w}{\|w\|_b}$$

is as required. The other implication in (b) follows from (a). Finally (c) is readily obtained from (a) and (b).

Lemma 5. *Suppose $\{f_k\}_{k\in N}$ is a sequence of characteristic functions, then $\prod_{k=1}^{\infty} f_k$ converges to a characteristic function if and only if there is $n_0\in N$ such that*

$$\left|\prod_{k=m}^{n} f_k(x)-1\right|\leqq\varepsilon \quad \text{if } x\in S \text{ and } n_0<m\leqq n. \tag{6}$$

Proof. Suppose $\prod_{k=1}^{\infty} f_k$ converges to the characteristic function f, then by Lemma 3 $f\in H_c$ for some $c<b$. Let $\varepsilon>0$ be given and let δ be as in Lemma 1. Then it is possible to choose n_0 so large that

$$\left|\Delta_c\left(\prod_{k=1}^{p} f_k\right)-\Delta_c(f)\right|\leqq\delta/2 \quad \text{if } p\geqq n_0. \tag{7}$$

Now if $n_0<m\leqq n$, then (7) implies

$$\Delta_c\left(\prod_{k=m}^{n} f_k\right)=\Delta_c\left(\prod_{k=1}^{n} f_k\right)-\Delta_c\left(\prod_{k=1}^{m-1} f_k\right)\leqq\delta,$$

so that (6) follows by Lemma 1. The converse is merely Cauchy's criterion for uniformly convergent infinite products.

Lemma 6. *Suppose $\{w_r\}_{r\in N}$ is a sequence of distributions and $\prod_{r=1}^{\infty} * w_r$ converges to a distribution w. Assume $w\in l^1$, then $w_r\in l^1$ for each r.*

Proof. Let $j\in N'$, then the product $\prod_{r=1, r\neq j}^{\infty} * w_r$ converges in $\mathfrak{P}$ to some distribution w_j' because of Lemma 5, and since $*$ is commutative $w_j*w_j'=w$ so that Lemma 4 implies that $w_j\in l^1_+$.

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