

On the Weak Convergence of Stochastic Processes without Discontinuities of the Second Kind

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Summary. Subspaces D_α , $\alpha > 0$, of $D[0, 1]$ are defined and given complete metrics d_α which are stronger than the Prokhorov metric. The spaces (D_α, d_α) are shown to be separable, and their pre-compact subsets are characterized. A condition which is known to guarantee weak pre-compactness of sets of probability measures over $D[0, 1]$ is shown to also guarantee weak pre-compactness of probability measures over D_α for appropriate values of α . Applications are made to the weak convergence of measures induced by stochastic processes, and some examples are included.

1. Introduction

The spaces $C[0, 1]$ and $D[0, 1]$ have been the setting for much of the recent work on the weak convergence of stochastic processes, and several criteria which guarantee the weak precompactness of sets of measures in $C[0, 1]$ and $D[0, 1]$ are now known. LAMPERTI [3] observed that one of these criteria, due essentially to KOLMOGOROV ([4], p. 519), actually guarantees weak precompactness with respect to the topology of $\text{Lip}_\alpha[0, 1]$ for appropriate values of α . The transition from $C[0, 1]$ to $\text{Lip}_\alpha[0, 1]$ is desirable because, as explained in [3], it enlarges the class of continuous functionals.

Here we obtain an improvement of theorems on the weak convergence of stochastic processes in $D[0, 1]$ which is analogous to that given by LAMPERTI for $C[0, 1]$. To do so we need analogues of (1) KOLMOGOROV's criterion and (2) the spaces $\text{Lip}_\alpha[0, 1]$. The former has been given by CHENTSOV [1]. The latter, which we have denoted by D_α , are discussed in section two and are to the best of our knowledge new. In section three we show that (a slightly modified version of) the CHENTSOV criterion does, indeed, guarantee weak precompactness with respect to the topology of D_α . And in section four we give examples of stochastic processes and functionals to which our results apply.

2. The Spaces D_α

By (D, d) we will understand the complete, separable metric space which is constructed in Chapter VII of [5] and is a variation on the space $D[0, 1]$ of [6]. Thus the elements of D are right continuous, real valued functions which are defined on $[0, 1]$, have no discontinuities of the second kind, and are continuous at zero and one. The metric d is defined by

$$d(f, g) = p(\Gamma_f, \Gamma_g) + L(m_f, m_g),$$

where $p(\Gamma_f, \Gamma_g)$ denotes the divergence between the graphs of f and g ([5], [6]),

L denotes the Lévy metric for monotone functions on the line ([2], pp. 38, 39), and m_f is defined as follows: for $0 < \delta \leq 1$ let $\bar{\omega}_f(\delta)$ be the maximum of

$$\sup_{0 \leq t \leq \delta} |f(t) - f(0)|, \quad \sup_{1-\delta \leq t \leq 1} |f(t) - f(1)|,$$

and

$$\hat{\omega}_f(\delta) = \sup \min \{|f(t_1) - f(t_0)|, |f(t_0) - f(t_2)|\},$$

where the third supremum is taken over $t_1, t_0, t_2 \in [0, 1]$ for which

$$t_0 - \delta \leq t_1 \leq t_0 \leq t_2 \leq t_0 + \delta;$$

then $m_f(z) = \bar{\omega}_f(e^z + 0)$ for $z \leq 0$ and $m_f(z) = m_f(0)$ for $z \geq 0$. For $\alpha > 0$ we define D_α to be the set of $f \in D$ for which

$$\int_{-\infty}^{\infty} m_f(z) e^{-\alpha z} dz < \infty,$$

and we topologize D_α with the metric d_α defined by

$$d_\alpha(f, g) = p(\Gamma_f, \Gamma_g) + L_\alpha(m_f, m_g),$$

$$L_\alpha(m_f, m_g) = \int_{-\infty}^{\infty} |m_f(z) - m_g(z)| e^{-\alpha z} dz.$$

d_α is a metric because both p and L_α are and because $f = g$ if and only if $\Gamma_f = \Gamma_g$. Moreover, d_α is stronger than d because L_α is stronger than L .

Our immediate goal is to ascertain some of the topological properties of the spaces (D_α, d_α) , which we will denote simply by D_α in the following. They resemble and to some extent follow from those of (D, d) , which we will denote by D in the following.

Theorem 2.1. *A subset F of D_α is precompact in D_α if and only if*

$$(2.1) \quad \sup_{f \in F} \sup_{0 \leq t \leq 1} |f(t)| < \infty,$$

$$(2.2) \quad \lim_{a \rightarrow -\infty} \sup_{f \in F} \int_{-\infty}^a m_f(z) e^{-\alpha z} dz = 0.$$

Proof. (Sufficiency.) Let F satisfy (2.1) and (2.2); then the inequality,

$$(2.3) \quad \bar{\omega}_f(\delta) \leq e \delta^\alpha \int_{-\infty}^{(\log \delta)+1} m_f(z) e^{-\alpha z} dz, \quad 0 < \delta < e^{-1},$$

shows that $\bar{\omega}_f(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly in $f \in F$, so that F is precompact in D ([5], Ch. VII, Thm. 6.2). Therefore, if $f_n \in F$, $n \geq 1$, there exists a subsequence n_k , $k \geq 1$, and an $f_0 \in D$ for which $d(f_{n_k}, f_0) \rightarrow 0$ as $k \rightarrow \infty$. Moreover, $f_0 \in D_\alpha$ by FАТОВ's Lemma, (2.1) and (2.2). Now the convergence of f_{n_k} to f_0 in D implies the convergence of $m_{f_{n_k}}(z)$ to $m_{f_0}(z)$ for all z at which m_{f_0} is continuous and, in particular, almost everywhere with respect to Lebesgue measure. (2.2) now implies $d_\alpha(f_{n_k}, f_0) \rightarrow 0$ as $k \rightarrow \infty$. The sufficiency follows.

(Necessity.) If F is precompact in D_α , then, since d_α is stronger than d , F is precompact in D so that (2.1) follows from the theorem cited above. If (2.2) were false, then there would exist sequences $a_n \in R^1$ and $f_n \in F$ for which $a_n \rightarrow -\infty$ as $n \rightarrow \infty$ and

$$(2.4) \quad \liminf_{n \rightarrow \infty} \int_{-\infty}^{a_n} m_{f_n}(z) e^{-\alpha z} dz > 0.$$

There would also be a subsequence n_k , $k \geq 1$, and an $f_0 \in D_\alpha$ for which $d_\alpha(f_{n_k}, f_0) \rightarrow 0$ as $k \rightarrow \infty$ and, in particular,

$$(2.5) \quad \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} |m_{f_{n_k}}(z) - m_{f_0}(z)| e^{-\alpha z} dz = 0.$$

The necessity of (2.2) follows from the incompatibility of (2.4) and (2.5). Q.E.D.

Corollary 2.1. D_α is complete.

Corollary 2.2. Let $f_n \in D_\alpha$, $n \geq 0$; sufficient conditions for $d_\alpha(f_n, f_0) \rightarrow 0$ as $n \rightarrow \infty$ are that (1) $f_n(t) \rightarrow f_0(t)$ for all t belonging to a dense subset of $(0, 1)$; and (2) that $m_{f_n}(\cdot) e^{-\alpha(\cdot)}$ be dominated by an integrable function on $(-\infty, 0]$.

Proof. Corollary 2.1 follows from the easily verified fact that a Cauchy sequence satisfies (2.1) and (2.2). To establish Corollary 2.2 we need only observe that f_n is precompact and that if f is any limit point of f_n , then $p(\Gamma_f, \Gamma_{f_0}) = 0$ so that $f = f_0$. Q.E.D.

Theorem 2.2. D_α is separable.

Proof. Let D^n be the set of those $f \in D_\alpha$ which take a constant, rational value on each of the intervals $[(i-1)/n, i/n)$, $i = 1, \dots, n \geq 1$ and let $D^0 = \bigcup_{n=1}^{\infty} D^n$.

If $f \in D_\alpha$ and t_1^n, \dots, t_n^n are continuity points of f for which $i-1 < n t_i^n < i$, $i = 1, \dots, n$, then there is an $f_n \in D^n$ which differs from f by less than $n^{-2\alpha}$ at each of the points t_i^n . It follows easily that $f_n(t) \rightarrow f(t)$ as $n \rightarrow \infty$ at all continuity points t of f , the totality of which are dense in $(0, 1)$. Moreover, we have $\overline{\omega}_{f_n}(\delta) \leq \overline{\omega}_f(\delta + (2/n)) + 2n^{-2\alpha}$, $0 < \delta \leq 1 - (2/n)$.

For example,

$$\begin{aligned} \sup_{0 \leq t \leq \delta} |f_n(t) - f_n(0)| &= \max_{1 \leq i \leq k} |f_n(t_i^n) - f_n(0)| \\ &\leq \max_{1 \leq i \leq k} |f(t_i^n) - f(0)| + 2n^{-2\alpha} \\ &\leq \sup_{0 \leq t \leq \delta + (1/n)} |f(t) - f(0)| + 2n^{-2\alpha} \end{aligned}$$

where k is one plus the greatest integer in $n\delta$. Since $\overline{\omega}_{f_n}(\delta) = 0$ for $0 < \delta < (1/n)$, Corollary 2.2 applies to show that $d_\alpha(f_n, f) \rightarrow 0$ as $n \rightarrow \infty$; and since D^0 is countable, the theorem follows. Q.E.D.

We conclude this section with a discussion of the subspace $C_\alpha = D_\alpha \cap C[0, 1]$. The inequality ([5], p. 235)

$$(2.6) \quad \omega_f(\delta) \leq 2\overline{\omega}_f(\delta) + \eta(f)$$

where ω_f denotes the modulus of continuity and $\eta(f)$ the maximal discontinuity of f shows (1) that $C_\alpha \subset \text{Lip}_\alpha[0, 1]$; (2) that C_α is closed in D_α (Ascoli's Theorem);

and (3) that the identity mapping of C_α into $\text{Lip}_\alpha[0, 1]$ is continuous. In the other direction we may only conclude that if $f \in \text{Lip}_\alpha[0, 1]$, then $f \in C_\beta$ for all $\beta < \alpha$ and that the identity mapping from $\text{Lip}_\alpha[0, 1]$ into C_β is continuous.

3. Measures in D_α

By a measure in D_α we will mean a probability measure defined on the Borel sets B_α of D_α , and we will denote the space of all measures in D_α by M_α . The topology of M_α is the weak topology it inherits as a subset of $C(D_\alpha)^*$, the dual space to the space of continuous functionals on D_α . The following lemma with D_α replaced by D is proved in [5], p. 251; the proof given there extends to D_α in a straightforward manner.

Lemma 3.1. *B_α is the smallest σ -algebra with respect to which the functionals, $\pi_t: f \rightarrow f(t)$, $0 \leq t \leq 1$, are measurable.*

If $X(t)$, $0 \leq t \leq 1$, is a stochastic process, and $\mu \in M_\alpha$, we will say that μ has been induced by $X(t)$, $0 \leq t \leq 1$, if and only if the following equality holds for every finite subset T of $[0, 1]$ and every choice of $x_t \in R^1$, $t \in T$:

$$(3.1) \quad \mu(\{f \in D_\alpha: f(t) \leq x_t, t \in T\}) = \Pr(X(t) \leq x_t, t \in T).$$

By Lemma 3.1 a stochastic process can induce at most one measure.

Theorem 3.1. *Let $X(t)$, $0 \leq t \leq 1$, be a stochastically right continuous stochastic process. If there exist monotone functions $q_i: (0, 1] \rightarrow [0, \infty)$ for which $q_i(h) \rightarrow 0$ as $h \rightarrow 0$, $i = 1, 2, 3$, $\sum_{k=1}^{\infty} q_i(2^{-k}) < \infty$ $i = 1, 2$, and*

$$(3.2a) \quad \Pr(|X(t_{i+1}) - X(t_i)| \geq h^\alpha q_1(h), i = 1, 2) \leq h q_2(h),$$

$$(3.2b) \quad \Pr(|X(i) - X(i + h(-1)^i)| \geq h^\alpha q_1(h)) \leq q_3(h), \quad i = 0, 1,$$

whenever $0 \leq t_1 \leq t_2 \leq t_3 \leq 1$, $t_3 - t_1 \leq h$, and $0 < h \leq 1$, then $X(t)$, $0 \leq t \leq 1$, induces a measure in D_α .

Proof. Define stochastic processes $X_k(t)$, $0 \leq t \leq 1$, $k \geq 1$, by

$$X_k(t) = \sum_{i=1}^{2^k} X(i 2^{-k}) I(2^k, i; t),$$

where $I(n, i; \cdot)$ is the indicator function of $[i - 1)/n, i/n)$, $i = 1, \dots, n - 1$, and $I(n, n; \cdot)$ is the indicator function of $[(n - 1)/n, 1]$; then each $X_k(t)$, $0 \leq t \leq 1$, induces a measure μ_k in D_α . We will show first that the sequence μ_k is precompact. Let $B_{k,i}^c$ be the set of $f \in D_\alpha$ for which

$$|f((j - 1) 2^{-k}) - f(j 2^{-k})| \geq q_1(2^{-k}) 2^{-\alpha k}$$

for $j = i, i + 1$ if $1 \leq i < 2^k$, for $j = 1$ if $i = 0$, and for $j = 2^k$ if $i = 2^k$, and

let $E_n = \bigcap_{k=n}^{\infty} \bigcap_{i=0}^{2^k} B_{k,i}$. Then

$$(3.3a) \quad E_n \subset \{f \in D_\alpha: \bar{\omega}_f(2^{-p}) \leq q(2^{-p}) 2^{-\alpha p}, p \geq n\} = F_n \quad \text{say,}$$

$$(3.3b) \quad \mu_k(E_n^c) \leq 2 \left(\sum_{p=n}^{\infty} q_2(2^{-p}) + q_3(2^{-n}) \right), \quad k \geq 1,$$

for $n \geq 1$ where $q(h) = 16(1 - 2^{-\alpha})^{-1}q_1(2h)$, $0 < h \leq (1/2)$. That $\hat{\omega}_f(2^{-p}) \leq q(2^{-p})2^{-\alpha p-1}$, $p \geq n$, for $f \in E_n$ is essentially shown in [1], Lemmas 1 and 2; moreover, $|f(\delta) - f(0)| \leq w$ and $\hat{\omega}_f(\delta) \leq w$ imply $\sup_{0 \leq t \leq \delta} |f(t) - f(0)| \leq 2w$. Next let G_b be the set of $f \in D_\alpha$ which are bounded by $2b$; then for $b > q((1/2))$ and $n \geq 1$ we have

$$(3.4) \quad \mu_k(G_b^c) \leq \Pr \left(\max_{0 \leq i \leq 2^n} |X(i2^{-n})| \geq b \right) + \mu_k(E_n^c), \quad k \geq 1,$$

which may be made arbitrarily small by taking n and then b sufficiently large. Since $F_n \cap G_b$ is compact for every n and b , it follows from (3.3) and (3.4) that μ_k , $k \geq 1$, is precompact.

Now we claim that any limit point of μ_k , $k \geq 1$, must satisfy (3.1) — i.e. that μ_k converges to a measure induced by $X(t)$, $0 \leq t \leq 1$. We may assume that $X(t)$, $0 \leq t \leq 1$, is separable and that $T \subset S$, the set of dyadic rationals of $(0, 1)$. Let μ be a limit point of μ_k , $k \geq 1$, let T be a finite subset of S , and for each $t \in T$ let $h_{n,t}$ be a continuous function for which $0 \leq h_{n,t} \leq 1$, $h_{n,t}(s) = 0$ if either $s \leq t$ or $s \geq t + 2^{-n+1}$, and $h_{n,t}(t + 2^{-n}) = 1$. Then, since the functionals

$$f \rightarrow \sup_{0 \leq s \leq 1} h_{n,t}(s)f(s)$$

are continuous on D_α for every $n \geq 1$ and $t \in T$ ([5], p. 249), we must have

$$\mu \left(\left\{ f: \sup_{0 \leq s \leq 1} h_{n,t}(s)f(s) \leq x_t, t \in T \right\} \right) = \Pr \left(\sup_{s \in S} h_{n,t}(s)X(s) \leq x_t, t \in T \right)$$

for every choice of $x_t \in R^1$, $t \in T$. When $n \rightarrow \infty$, (3.1) follows from (3.3 b) and the stochastic right continuity of $X(t)$, $0 \leq t \leq 1$. Q.E.D.

The next theorem is a straightforward generalization of Theorem 3.1, the key observation being that $\mu(F_n^c) \leq \liminf_{k \rightarrow \infty} \mu_k(F_n^c)$ where μ and μ_k are as in Theorem 3.1.

Theorem 3.2. *Let $X_n(t)$, $0 \leq t \leq 1$, $n \geq 0$, be a sequence of stochastically right continuous stochastic processes, each of which satisfies (3.2) with q_1 , q_2 , and q_3 independent of n . Also let μ_n be the measure induced by $X_n(t)$, $0 \leq t \leq 1$, $n \geq 0$. A sufficient condition for μ_n , $n \geq 1$, to be precompact (converge to μ_0) is that for every finite subset $T = \{t_1, \dots, t_k\} \subset [0, 1]$ the family of joint distributions of $(X_n(t_1), \dots, X_n(t_k))$, $n \geq 1$, be precompact (converge to the joint distribution of $(X_0(t_1), \dots, X_0(t_k))$).*

We conclude this section with a lemma which gives sufficient conditions for (3.2) to hold. It is an easy consequence of Markov's Inequality.

Lemma 3.2. *If there are positive constants K , b , b_1 , b_2 , β , and γ for which*

$$(3.5a) \quad E(|X_n(t_1) - X_n(t_2)|^{b_1} |X_n(t_2) - X_n(t_3)|^{b_2}) \leq Kh^{1+\beta},$$

$$(3.5b) \quad E(|X_n(i) - X_n(i+h(-1)^i)|^b) \leq Kh^\gamma, \quad i = 0, 1,$$

whenever $0 \leq t_1 \leq t_2 \leq t_3 \leq 1$, $t_3 - t_1 \leq h$, and $n \geq 0$, then for any

$$\alpha < \min\{\beta/(b_1 + b_2), \gamma/b\}$$

the processes $X_n(t)$, $0 \leq t \leq 1$, $n \geq 0$, satisfy (3.2) with $q_1(h) = h^\delta$ and $q_2(h) = q_3(h) = Kh^\varepsilon$ for appropriate $\varepsilon, \delta > 0$.

4. Examples

Example 1. (The sample distribution function.) Let X_1, X_2, \dots be independent, identically distributed random variables whose common distribution function F has a bounded density which vanishes off $[0, 1]$. Define

$$X_n(t) = \sqrt{n}(F_n(t) - F(t)), \quad 0 \leq t \leq 1, \quad n \geq 1,$$

where F_n denotes the sample distribution function of X_1, \dots, X_n . Then the following is true (cf. [1] and [5], pp. 259–267): (1) the finite dimensional distributions of $X_n(t)$, $0 \leq t \leq 1$, converge to those of a normal process $X_0(t)$, $0 \leq t \leq 1$, with covariance function $K(s, t) = F(s)(1 - F(t))$, $0 \leq s \leq t \leq 1$; and (2) (3.5) is satisfied with $b = b_1 = b_2 = 2$ and $\beta = \gamma = 1$. It follows from Theorem 3.2 that the measures induced in D_α by $X_n(t)$, $0 \leq t \leq 1$, converge to that induced by $X_0(t)$, $0 \leq t \leq 1$, for $0 < \alpha < (1/4)$.

Example 2. (Convergence to an infinitely divisible law: bounded variances case.) Let $X_{n,1}, \dots, X_{n,k_n}$, $n \geq 1$, be a triangular array of random variables which are independent and identically distributed in each row. Suppose that the distribution function and variance of $S_{n,k_n} = \sum_{k=1}^{k_n} X_k$ converge to those of an infinitely divisible law (with finite, positive variance). It follows from the theory presented in [2] that the finite dimensional distributions of

$$(4.1) \quad X_n(t) = \sum_{k=1}^{k_n} S_{n,k} I(k_n, k; t), \quad 0 \leq t \leq 1, \quad n \geq 1,$$

converge as $n \rightarrow \infty$ to those of an infinitely divisible process, $X_0(t)$, $0 \leq t \leq 1$. Moreover, if $0 \leq t_1 \leq t_2 \leq t_3 \leq 1$, $|t_3 - t_1| \leq h$, and $n \geq 0$ then

$$E\left(\prod_{i=1}^2 (X_n(t_i) - X(t_{i+1}))\right)^2 \leq K(h\sigma^2 + 4(h\Theta)^2)$$

where Θ and σ^2 are respectively the mean and variance of the limit law and K is independent of n . Thus (3.5a) is satisfied with $b_1 = b_2 = 2$ and $\beta = 1$, and, similarly, we may verify that (3.5b) is satisfied with $b = 2$ and $\gamma = 1$. Therefore, by Theorem 3.2 the measures induced by $X_n(t)$, $0 \leq t \leq 1$, converge in D_α to that induced by $X_0(t)$, $0 \leq t \leq 1$, for $0 < \alpha < (1/4)$.

Example 3. (Convergence to a stable law.) Let X_1, X_2, \dots be independent, identically distributed random variables whose common distribution function F belongs to the domain of normal attraction of a stable law of characteristic exponent γ , $0 < \gamma < 2$. For the sake of convenience, we suppose also that F is symmetric if $\gamma \geq 1$. Let $X_n(t)$, $0 \leq t \leq 1$, be defined by (4.1) with $X_{n,k} = X_k n^{-(1/\gamma)}$, $k = 1, \dots, n$; then, as above, the finite dimensional distributions of $X_n(t)$, $0 \leq t \leq 1$, will converge to those of a stable process, $X_0(t)$, $0 \leq t \leq 1$. We show that (3.2) is satisfied for any α , $0 < \alpha < (1/2\gamma)$. Indeed, since $x^\gamma(1 - F(x) + F(-x))$

is convergent as $x \rightarrow \infty$ ([2], p. 182) and therefore bounded, we have by a standard argument that

$$(4.2) \quad \sup_{t \geq 0} t^{-\nu} |\varphi(t) - 1| \leq B$$

where φ is the characteristic function of X_1 . The inequality, $|z^k - 1| \leq k|z - 1|$, $|z| \leq 1$, now shows that (4.2) also holds when φ is replaced by φ_k , the characteristic function of $k^{-(1/\nu)} \sum_{i=1}^k X_i$; and from the truncation inequality ([4], p. 196) we may now infer that

$$x^\nu \Pr \left(\left| k^{-(1/\nu)} \sum_{i=1}^k X_i \right| \geq x \right) \leq B_1$$

is uniformly bounded in $x \geq 0$ and $k \geq 1$. (3.2) is an easy consequence: for example

$$\Pr(|X_n(t) - X_n(0)| \geq t^\alpha) \leq B_1 t^{1-\alpha\nu}, \quad 0 \leq t \leq 1, \quad n \geq 1.$$

A similar analysis will show that the limit process also satisfies (3.2) for any $\alpha < 1/2\nu$ so that Theorem 3.2 applies.

Examples of continuous functionals on D_α which are not continuous on all of D are easy to construct. In particular, any functional of the form

$$\Phi(f) = \int_{-\infty}^0 \varphi(z) m_f(z) e^{-\alpha z} dz, \quad f \in D_\alpha,$$

where φ is a bounded measurable function on $(-\infty, 0]$ are continuous on D_α and not even defined on all of D . Such functionals might be of interest if, for example, one thinks of $\bar{\omega}_f$ as measuring the smoothness of f . Another class of functionals to which our results apply is illustrated by

Example 4. (The sample density.) Let F_n and $X_n(t)$, $0 \leq t \leq 1$, $n \geq 1$, be as in Example 1 and define

$$f_n(t) = (F_n(t + h_n) - F_n(t - h_n))/2h_n, \quad 0 \leq t \leq 1, \quad n \geq 1,$$

where $0 < h_n \rightarrow 0$ with $nh_n^{2(1-\alpha)} \rightarrow \infty$, $\alpha < \frac{1}{4}$. f_n provides an estimate of the density f of X_1, X_2, \dots (See [7] and the references given there.) Now

$$(4.3) \quad \sup_{0 \leq t \leq 1} |f_n(t) - E(f_n(t))| = \sup_{0 \leq t \leq 1} |X_n(t + h_n) - X_n(t - h_n)|/2h_n\sqrt{n}$$

where $X_n(t) = 0$ for $t < 0$ and $t > 1$. It follows easily from (2.6) and Example 1 that the right side of (4.3) converges to zero in probability as $n \rightarrow \infty$. If f is continuous, then we also have $E(f_n(t)) \rightarrow f(t)$ uniformly in t ([7]) so that f_n will be uniformly consistent. While stronger results on the uniform consistency of sample densities are available ([7] and the references given there), this is apparently the first which makes use of the theory of weak convergence of stochastic processes.

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