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Infinitely Divisible Representations of Clifford Algebras

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0. Introduction

In a previous paper [1], a non-commutative generalization of the concept of an infinitely divisible random variable was introduced. The objects under discussion there were the enveloping algebra of a Lie algebra $\mathfrak{G}$, and linear functionals on it. The fact that the polynomial algebra of $\mathfrak{G}$ is *graded* allows a consistent definition of generalized moments and cumulants. It is clear that the theory can be generalized to a number of other associative algebras. In this paper, we study the algebra of anti-commutation relations, and define the notion of infinite divisibility. We show that a representation is infinitely divisible if and only if it is quasi-free, that is, the generalized cumulants vanish beyond the second.

1. Basic Concepts

Let $\mathfrak{T}$ be a complex Hilbert space, the "test function space" with scalar product (f, g), $f, g \in \mathfrak{T}$. (As is well known, in the case of canonical anti-commutation relations (CAR's), the test function space can be assumed to be complete without loss of generality [4].) We shall discuss a certain class of representations of the canonical anti-commutation relations over $\mathfrak{T}$, [2]:

$$\begin{aligned} &\{a(f), a(g)\}_+ \equiv a(f)\,a(g) + a(g)\,a(f) = 0 \\ &\{a^*(f), a(g)\}_+ = (f, g)\,\mathbb{1} \end{aligned} \tag{1}$$

where $\mathbb{1}$ is the unit, and $f, g \in \mathfrak{T}$.

We shall construct the CAR algebra $\overline{\mathfrak{A}(\mathfrak{T})}$, i.e. the unique C^*-algebra generated by (1) whose representations are in 1-1 correspondence with representations of (1):

Let us denote by $\mathfrak{T}^*$ the dual space to $\mathfrak{T}$. By the Riesz theorem, there is a canonical bijection $C: \mathfrak{T} \to \mathfrak{T}^*$ which is defined by $\langle Cf, g\rangle = (f, g)$ $(g \in \mathfrak{T})$. Let $\mathfrak{T}_0$ be the complex linear space $\mathfrak{T} \oplus \mathfrak{T}^*$. We can define on $\mathfrak{T}_0$ a conjugation, denoted by *, as follows:

$$\begin{aligned} f^* &= Cf \\ (Cf)^* &= f \end{aligned} \qquad (f \in \mathfrak{T}).$$

Let now $\mathfrak{P}(\mathfrak{T})$ denote the polynomial algebra over $\mathfrak{T}_0$ with the property that the embedding $\mathfrak{T}_0 \to \mathfrak{P}(\mathfrak{T})$ is linear. The conjugation * on $\mathfrak{T}_0$ extends in a unique way to an involution on $\mathfrak{P}(\mathfrak{T})$. Let $\mathfrak{C}_1$ be the 2-sided *-ideal in $\mathfrak{P}$ generated by

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$\{\{f,g\}_+, \{f^*,g\}_+ - \langle f^*,g\rangle \mathbb{1};\ f,g\in\mathfrak{X}\}$. The algebra $\mathfrak{A}(\mathfrak{X})=\mathfrak{P}(\mathfrak{X})/\mathfrak{C}_1$ inherits the *-structure from $\mathfrak{P}(\mathfrak{X})$ and it is a well-known fact that there is a unique norm in $\mathfrak{A}(\mathfrak{X})$ defining a C^*-algebra $\overline{\mathfrak{A}(\mathfrak{X})}$. We shall denote the canonical map $\mathfrak{P}(\mathfrak{X})\to\overline{\mathfrak{A}(\mathfrak{X})}$ by a.

By the Gelfand-Segal theorem, to every state ω on $\overline{\mathfrak{A}(\mathfrak{X})}$ there corresponds a unique cyclic unitary representation $(\pi_\omega, \mathscr{K}_\omega, \Omega_\omega)$ of $\overline{\mathfrak{A}(\mathfrak{X})}$ (where $\Omega_\omega\in\mathscr{K}_\omega$ is the cyclic vector, and $\|\Omega_\omega\|=1$) such that $\omega(x)=(\Omega_\omega, \pi_\omega(x)\,\Omega_\omega)$ for all $x\in\overline{\mathfrak{A}(\mathfrak{X})}$.

In what follows, we shall find it more convenient to consider, instead of a state ω on $\overline{\mathfrak{A}(\mathfrak{X})}$, the unique linear functional W on $\mathfrak{P}(\mathfrak{X})$ with the property that

$$\begin{array}{ccc} \mathfrak{P}(\mathfrak{X}) & \xrightarrow{W} & \mathbb{C} \\ \big\downarrow a & & \big\downarrow 1 \\ \overline{\mathfrak{A}(\mathfrak{X})} & \xrightarrow{\omega} & \mathbb{C} \end{array}$$

is a commutative diagram. The linear functional W has the properties:

(i) $W(x^*x)\geqq 0$ for all $x\in\mathfrak{P}(\mathfrak{X})$

(ii) $W(1)=1$

(iii) $W(\mathfrak{C}_1)=0$.

Conversely, every linear functional W on $\mathfrak{P}(\mathfrak{X})$ satisfying (i), (ii) and (iii) defines a unique state ω on $\overline{\mathfrak{A}(\mathfrak{X})}$ such that $W=\omega\circ a$. We shall call W the *expectation functional of* π_ω.

It will be useful to have a slightly more general concept than representation.

Definition 1.1. A *-homomorphism π' of $\mathfrak{P}(\mathfrak{X})$ into $\mathfrak{B}(\mathscr{K})$ will be called a *k-extended representation* of the CAR's ($k>0$), if it satisfies the relations

$$\{\pi'(f), \pi'(g)\}_+ = 0$$
$$\{\pi'(f^*), \pi'(g)\}_+ = k\langle f^*, g\rangle$$

for all $f,g\in\mathfrak{X}\subset\mathfrak{P}(\mathfrak{X})$.

Let $\mathfrak{C}_k$ now denote the two-sided *-ideal in $\mathfrak{P}(\mathfrak{X})$ generated by $\{f,g\}_+$ and $\{f^*,g\}_+ - k\langle f^*,g\rangle\mathbb{1}$ $(f,g\in\mathfrak{X}\subset\mathfrak{P}(\mathfrak{X}))$. As above, for each linear functional W' on $\mathfrak{P}(\mathfrak{X})$ satisfying (i), (ii) and

(iii′) $W'(\mathfrak{C}_k)=0$,

there exists a unique k-extended representation of the CAR's over $\mathfrak{X}$.

Let γ be the unique *-automorphism of $\overline{\mathfrak{A}(\mathfrak{X})}$ with the property

$$\gamma(a(F)) = -a(F), \qquad F\in\mathfrak{X}_0\subset\mathfrak{P}(\mathfrak{X}).$$

Definition 1.2. A state ω of $\overline{\mathfrak{A}(\mathfrak{X})}$ is said to be *even* if it is invariant under γ, i.e.

$$\omega(x)=\omega(\gamma x) \qquad \text{for all } x\in\overline{\mathfrak{A}(\mathfrak{X})}.$$

Let ω be an even state of $\overline{\mathfrak{A}(\mathfrak{X})}$ and let $(\pi, \mathscr{K}, \Omega)$ be the corresponding cyclic representation. Then, there is a unitary operator U_γ on $\mathscr{K}$ such that for all $x\in\overline{\mathfrak{A}(\mathfrak{X})}$, $U_\gamma\,\pi(x)\,U_\gamma^{-1}=\pi(\gamma(x))$ and $U_\gamma\,\Omega=\Omega$. Further, U_γ is hermitian [3].

Powers [3] has also shown that if $\mathfrak{T}=L^2(\mathbb{R}^3)$, then a translation invariant state is even.

We shall restrict ourselves to even states, and shall call the corresponding expectational functional W even. An even functional has the extra property

(iv) W vanishes on the homogenous polynomials of odd degree.

For functionals satisfying (i), (ii), (iii′) and (iv), we can define the truncated functionals, also called the cumulants, W_T, on $\mathfrak{P}_1(\mathfrak{T})$ — the subalgebra of $\mathfrak{P}(\mathfrak{T})$ consisting of all polynomials with no constant term, by induction:

$$\begin{aligned} &W_T(F)=0 \\ &W(F_1\dots F_n)=\sum_I(-1)^{\varepsilon(I)}\,W_T(F_{i_1}\dots F_{i_{l_1}})\dots W_T(F_{i_{l_{k-1}+1}}\dots F_{i_{l_k}}) \\ &(F_i\in\mathfrak{T}_0) \end{aligned} \tag{2}$$

where the sum is taken over all partitions of $(1\dots n)$ into disjoint subsets $(i_1,\dots,i_{l_1})\dots(i_{l_{k-1}+1},\dots,i_{l_k})$ and the trivial partition $(1\dots n)$ is included. The indices in each cluster appear in their natural order and $\varepsilon(I)$ is the parity of the permutation $(1,\dots,n)\to(i_1,\dots,i_{l_1},\dots,i_{l_k})$.

W_T is well-defined since W_T vanishes on the elements $x\in\mathfrak{P}_1$ of odd degree and hence the permutation of clusters does not alter $\varepsilon(I)$. Conversely, W is uniquely determined by W_T on $\mathfrak{P}_1$, and $W(1)=1$. Moreover, the following lemma holds.

Lemma 1.3. *Let W be an even expectation functional of a k-extended representation and let W_T denote the corresponding truncated functional. Then W_T vanishes on the *-ideal $\mathfrak{J}\subset\mathfrak{P}_1$ generated by $(\{F,G\}_+\,H)$ with $F,G,H\in\mathfrak{T}_0$, and*

$$\left.\begin{aligned} &W_T(\{f,g\}_+)=0 \\ &W_T(\{f^*,g\}_+)=k\langle f^*,g\rangle \end{aligned}\right.\quad (f,g\in\mathfrak{T}).$$

Proof. The second part of the statement is obvious; hence it remains to prove that W_T vanishes on $\mathfrak{J}$. Clearly, it is sufficient to prove it for the elements of the form

$$H_1\dots H_m\{F,G\}_+\,H_{m+1}\dots H_n\qquad(n\geqq 1). \tag{3}$$

We shall proceed by induction on n. The case $n=1$ is obvious, since W_T vanishes on terms of degree 3. In (2), we split the sum into several parts, as follows:

$$\begin{aligned} &-W_T(H_1\dots H_m\{F,G\}_+\,H_{m+1}\dots H_n)=-W(H_1\dots H_m\{F,G\}_+\,H_{m+1}\dots H_n) \\ &+\sum_{I'}(-1)^{\varepsilon(I')}\,W_T(H_{i_1}\dots\{F,G\}_+\dots H_{i_{l_1}})\dots W_T(H_{i_{l_{k-1}+1}}\dots H_{i_n}) \\ &+\sum_{I'}(-1)^{\varepsilon(I')}\,W_T(\{F,G\}_+)\,W_T(H_{i_1}\dots H_{i_{l_1}})\dots W_T(H_{i_{l_{k-1}+1}}\dots H_{i_n}) \\ &+\sum_{I'}(\pm 1)\,W_T(H_{i_1}\dots F\dots H_{i_{l_1}})\,W_T(H_{i_{l_1+1}}\dots G\dots H_{i_{l_2}})\dots W_T(H_{i_{l_{k-1}+1}}\dots H_{i_n}) \\ &+\sum_{I'}(\pm 1)\,W_T(H_{i_1}\dots G\dots H_{i_{l_1}})\,W_T(H_{i_{l_1+1}}\dots F\dots H_{i_{l_2}})\dots W_T(H_{i_{l_{k-1}+1}}\dots H_{i_n}) \end{aligned}$$

where I' are all partitions of $(1,2,\dots,n)$ and the sign $(-1)^{\varepsilon(I')}$ corresponds to the permutation $(1,\dots,n)\to(i_1\dots i_n)$ and the sign $\pm$ in the last two sums corresponds to the permutation

$$(1,2\ldots m\cdot F, G, \ldots n)\to(i_1,\ldots,F,\ldots i_{l_1}, i_{l_1+1}\ldots G\ldots i_{l_2}\ldots i_n).$$

The last two sums cancel out because the parity of any permutation is altered by exchanging F and G. The first sum vanishes by the induction hypothesis. Hence

$$\begin{aligned}-W_T(H_1\ldots H_m\{F,G\}_+ H_{m+1}\ldots H_n) &= -W(H_1\ldots H_m\ldots H_n)\,W(\{F,G\}_+)\\ &+\sum_{I'}(-1)^{\varepsilon(I')}W_T(\{F,G\}_+)\,W_T(H_{i_1}\ldots H_{i_{l_1}})\ldots W_T(H_{i_{l_{k-1}+1}}\ldots H_{i_n})\end{aligned}$$

where we used (iii′) and (ii) for W. Using the relation $W_T(\{F,G\}_+)=W(\{F,G\}_+)$ and (2) we obtain $W_T(H_1\ldots H_m\{F,G\}_+ H_{m+1}\ldots H_n)=0$.

2. Infinitely Divisible Representations of the CAR's

Definition 2.1. Let $\mathfrak{T}=\bigoplus_n \mathfrak{T}_n$. A state ω of $\overline{\mathfrak{A}(\mathfrak{T})}$ is said to be a *product state with respect to this decomposition* of $\mathfrak{T}$ if for each n, the following condition is satisfied: if $x\in\overline{\mathfrak{A}(\mathfrak{T}_n)}$, $y\in\overline{\mathfrak{A}(\mathfrak{T}_n^\perp)}$, then

$$\omega(x\,y)=\omega(x)\,\omega(y).$$

It can be shown [3] that if $\mathfrak{T}=\bigoplus_{n=1}^{N}\mathfrak{T}_n$, and ω_n is an even state of $\overline{\mathfrak{A}(\mathfrak{T}_n)}$, for each n, then there is a unique product state

$$\omega=\omega|_{\overline{\mathfrak{A}(\mathfrak{T}_1)}}\otimes\omega|_{\overline{\mathfrak{A}(\mathfrak{T}_2)}}\otimes\cdots\otimes\omega|_{\overline{\mathfrak{A}(\mathfrak{T}_N)}}$$

with $\omega|_{\overline{\mathfrak{A}(\mathfrak{T}_n)}}=\omega_n$ for each n. In fact, if $(\pi_n,\mathcal{K}_n,\Omega_n)$ is the cyclic *-representation corresponding to ω_n for each n, then ω corresponds to $\left(\bigwedge_n \pi_n, \otimes\mathcal{K}_n, \otimes\Omega_n\right)$ where $\wedge$ is defined as follows:

Let $f=f_1\oplus f_2\in\mathfrak{T}_1\oplus\mathfrak{T}_2$. We define

$$(\pi_1\wedge\pi_2)\big(a(f)\big)=\pi_1\big(a(f_1)\big)\otimes U_\gamma+1\otimes\pi_2\big(a(f_2)\big). \tag{4}$$

It is easy to see that $\pi_1\wedge\pi_2$ is a cyclic representation of $\overline{\mathfrak{A}(\mathfrak{T}_1\oplus\mathfrak{T}_2)}$ of which the state is $\omega=\omega_1\otimes\omega_2$. Clearly, if π_1 and π_2 are k-extended representations of the CAR's over $\mathfrak{T}_1$ and $\mathfrak{T}_2$ respectively, and if we define $\pi_1\wedge\pi_2$ by (4), then it is again a k-extended representation of the CAR's over $\mathfrak{T}_1\oplus\mathfrak{T}_2$. It is easily checked that this product is associative; indeed, $\pi_1\wedge\pi_2$ is the unique cyclic representation corresponding to $\omega_1\otimes\omega_2$, so the associativity follows from that of the corresponding product state $\omega=\omega_1\otimes\omega_2\otimes\cdots\otimes\omega_N$.

Let $m>0$ be an integer, and let $\Delta_m\subset\underbrace{\mathfrak{T}\oplus\mathfrak{T}\oplus\cdots\oplus\mathfrak{T}}_{m}$ be the diagonal, i.e. $\Delta_m=\{f\oplus f\oplus\cdots\oplus f;\, f\in\mathfrak{T}\}$; Δ_m is isomorphic to $\mathfrak{T}$ as a linear space. Let us denote this isomorphism by ξ_m, $\xi_m(f)=f\oplus\cdots\oplus f\in\Delta_m$. Then $\langle\xi_m(f),\xi_m(g)\rangle=m\langle f,g\rangle$ $(f,g\in\mathfrak{T})$. Hence, if π' is a k-extended representation of the CAR's over Δ_m, then π defined on $\mathfrak{T}$ by

$$\pi(f)=\pi'(f\oplus\cdots\oplus f)$$

extends to a km-extended representation of CAR's over $\mathfrak{T}$. Let us denote this representation by $\xi(\pi')$.

Definition 2.2. A cyclic representation $(\pi, \mathscr{K}, \Omega)$ of $\overline{\mathfrak{A}(\mathfrak{T})}$ is said to be *infinitely divisible* if, for each $m \in \mathbb{N}$, there exists an extended representation $(\pi^{1/m}, \mathscr{K}^{1/m}, \Omega^{1/m})$ of the CAR's over $\mathfrak{T}$ such that $(\pi, \mathscr{K}, \Omega)$ is equivalent to

$$\xi(\underbrace{\pi^{1/m} \wedge \pi^{1/m} \wedge \cdots \wedge \pi^{1/m}}_{m}|_{\mathfrak{P}(\Delta_m)}).$$

In what follows we shall find all the infinitely divisible representations $(\pi, \mathscr{K}, \Omega)$ of $\overline{\mathfrak{A}(\mathfrak{T})}$. We shall need the following lemma. For the proof, see [3].

Lemma 2.3. *Let* $\mathfrak{T} = \bigoplus_{n=1}^{N} \mathfrak{T}_n$ *and let* W_n *be a functional on* $\mathfrak{P}(\mathfrak{T}_n)$ *satisfying* (i), (ii), (iii′) *and* (iv) *with the same value of k for each n. Then, if* W_T *is the truncated functional of* $W = W_1 \otimes \cdots \otimes W_N$, *the following relation holds:*

$$W_T(F_1 \dots F_m) = \sum_{n=1}^{N} (W_n)_T((E_n F_1)(E_n F_2) \dots (E_n F_m)),$$

where E_n *is the orthogonal projection* $\mathfrak{T}_0 \to \mathfrak{T}_{n,0}$ $(n = 1, 2 \dots N)$.

Corollary 2.4. *If* W_T *is the truncated functional corresponding to the k-extended representation* π *of the CAR's over* $\mathfrak{T}$, *and* W_T' *that of the k m-extended representation* $\xi(\pi \wedge \cdots \wedge \pi|_{\mathfrak{P}(\Delta_m)})$, *then*

$$W_T' = m\, W_T.$$

Corollary 2.5. *A representation* $(\pi, \mathscr{K}, \Omega)$ *of* $\overline{\mathfrak{A}(\mathfrak{T})}$ *is infinitely divisible if and only if for every* $m \in \mathbb{N}$, $W_T' = \frac{1}{m} W_T$ *is the truncated functional corresponding to some extended cyclic representation* π' *of the CAR's over* $\mathfrak{T}$.

Proof. The necessity follows from Corollary 2.4. To show sufficiency, let $m \in \mathbb{N}$ and define an extended representation $\pi'' = \xi(\underbrace{\pi' \wedge \cdots \wedge \pi'}_{m}|_{\mathfrak{P}(\Delta_m)})$. By Corollary 2.4, π'' has truncated functional $W_T'' = m\, W_T' = W_T$. Hence, the corresponding expectation functionals W'' and W agree on the whole of $\mathfrak{P}(\mathfrak{T})$. This implies that $\pi \cong \pi''$, in particular π'' is a representation. Hence π is ∞-divisible.

Proposition 2.6. *If* $(\pi, \mathscr{K}, \Omega)$ *is* ∞*-divisible, then the corresponding truncated functional* W_T *is positive semi-definite on* $\mathfrak{P}_1(\mathfrak{T})$.

Proof. Let $x \in \mathfrak{P}_1(\mathfrak{T})$ be an arbitrary fixed element

$$x = \sum_{i=1}^{N} F_{i1} F_{i2} \dots F_{ii} \qquad (F_{ij} \in \mathfrak{T}_0)$$

and choose $m \in \mathbb{N}$. By Corollary 2.5, the functional $W_{m,T}' = \frac{1}{m} W_T$ is the truncated functional of some extended representation, say π_m' of the CAR's over $\mathfrak{T}$. The corresponding expectation functional, W_m' say, satisfies (i), (ii), (iii′) and (iv); hence

$$W_m'(x^* x) \geqq 0.$$

Thus

$$0 \leqq W'_m(x^* x) = \sum_{i,j=1}^{N} W'_m(F^*_{jj} \dots F^*_{j1} F_{i1} \dots F_{ii})$$
$$= \sum_{i,j=1}^{N} \sum_{I} (-1)^{\varepsilon(I)} W'_{m,T}(I_1) \dots W'_{m,T}(I_q)$$

where $I=(I_1, \dots, I_q)$ is a partition of $((jj),(j,j-1),\dots(j,1)(i,1)\dots(ii))$ into $q \geqq 1$ parts, $I_1, \dots, I_q$, and $W'_{m,T}(I_\alpha)$ denotes the value of $W'_{m,T}$ on the product of elements with indices in I_α. Thus

$$0 \leqq \sum_{i,j=1}^{N} \frac{1}{m} W_T(F^*_{jj} \dots F^*_{j1} F_{i1} \dots F_{ii}) + \sum_{i,j=1}^{N} \sum_{I,\, q \geqq 2} (-1)^{\varepsilon(I)} \frac{1}{m^q} W_T(I_1) \dots W_T(I_q).$$

This inequality is valid for all $m \in \mathbb{N}$. Thus, letting $m \to \infty$, we obtain

$$W_T\left(\sum_{i,j=1}^{N} F^*_{jj} \dots F^*_{j1} F_{i1} \dots F_{ii}\right) = W_T(x^* x) \geqq 0.$$

Theorem 2.7. *An even cyclic representation of* $\overline{\mathfrak{A}(\mathfrak{T})}$ *is* ∞*-divisible if and only if it is quasi-free.*

Proof. Let $(\pi, \mathscr{K}, \Omega)$ be infinitely divisible. By Proposition 2.6 and Lemma 1.3, the corresponding truncated functional is a positive semi-definite functional on $\mathfrak{P}_1(\mathfrak{T})$ vanishing on $\mathfrak{J}$. We shall prove that such a functional necessarily vanishes on $(\mathfrak{P}_1)^3$, i.e. it vanishes on all homogeneous polynomials whose degree is $\geqq 3$. Let y be such a polynomial, i.e.

$$y = xFG, \qquad \text{where } x \in \mathfrak{P}_1, \text{ and } F, G \in \mathfrak{T}_0.$$

By the generalized Schwarz inequality

$$|W_T(xFG)|^2 \leqq W_T(xx^*)\, W_T(G^* F^* FG). \tag{5}$$

Now, $W_T(G^* F^* FG) \geqq 0$ and the same holds for $W_T(G^* F F^* G)$. But

$$W_T(G^* F^* F G) + W_T(G^* F F^* G) = W_T(G^* \{F^* F\}_+ G) = 0.$$

Hence $W_T(G^* F^* FG) = 0$ and the left-hand side of (5) is zero.

Conversely, let $(\pi, \mathscr{K}, \Omega)$ be a quasi-free representation with the truncated functional W_T and expectation functional W defined by (2). Let $m \in \mathbb{N}$ and define $W_{mT} = \frac{1}{m} W_T$. The corresponding functional W_m is now defined by

$$\begin{aligned} &W_m(1) = 1, \\ &W_m(F_1 \dots F_{2N}) = \sum_{I_2} (-1)^{\varepsilon(I_2)} W_{mT}(F_{i_1} F_{j_1}) \dots W_{mT}(F_{i_N} F_{j_N}) \end{aligned} \tag{6}$$

where the summation is taken over all partitions of $(1, \dots, 2N)$ into pairs with appropriate signs. W_m obviously satisfies (i), (iv) and vanishes on $\mathfrak{C}_{1/m}$. It remains to prove that W_m is positive semi-definite and therefore induces a $1/m$-extended representation whose truncated functional is $(W_m)_T$. It is easy to see from (6), that $W_m(F_1 \dots F_{2N}) = \frac{1}{m^N} W(F_1 \dots F_{2N})$. Let now x be an arbitrary element in $\mathfrak{P}$:

$$x = \alpha \mathbb{1} + \sum_{i=1}^{p} F_{i1} \dots F_{ii}.$$

Then

$$W_m(x^* x) = W_m \Big(\alpha \sum_{\substack{i=1 \\ i \text{ even}}}^{p} F_{ii}^* \dots F_{i1}^* + \bar{\alpha} \sum_{\substack{i=1 \\ i \text{ even}}}^{p} F_{i1} \dots F_{ii} + \sum_{\substack{i,j=1 \\ i+j \text{ even}}}^{p} F_{jj}^* \dots F_{j1}^* F_{i1} \dots F_{ii} + |\alpha|^2 \Big)$$

$$= W \Big(\alpha \sum_{\substack{i=1 \\ i \text{ even}}}^{p} \frac{1}{m^{i/2}} F_{ii}^* \dots F_{i1}^* + \bar{\alpha} \sum_{\substack{i=1 \\ i \text{ even}}}^{p} \frac{1}{m^{i/2}} F_{i1} \dots F_{ii} + |\alpha|^2$$

$$+ \sum_{\substack{i,j=1 \\ i+j \text{ even}}}^{p} \frac{1}{m^{\frac{i+j}{2}}} F_{jj}^* \dots F_{j1}^* F_{i1} \dots F_{ii} \Big) = W(y^* y),$$

where

$$y = \alpha \mathbb{1} + \sum_{i=1}^{p} G_{i1} \dots G_{ii}$$

with

$$G_{ik} = m^{-1/2} F_{ik} \qquad (i = 1, \dots, p\; k = 1 \dots i).$$

3. Factorizable Representations of Fermion Fields

We now consider a finite or infinite set of Fermi fields, $\{a_r^{\#}(f)\}_{r \in I}$ where $I = \{1, 2, \dots, N\}$ or $I = \mathbb{N}$, and $f \in L^2(\mathbb{R}^\nu)$. Here, ν is the dimension of space, and $a^{\#}$ means a or a^*. These fields satisfy the anti-commutation relations

$$\begin{aligned} &\{a_r(f), a_s(g)\}_+ = 0, \\ &\{a_r^*(f), a_s(g)\}_+ = \delta_{rs} \int \overline{f(x)}\, g(x)\, dx. \end{aligned} \tag{7}$$

The techniques of continuous tensor products lead to representations of current algebras with ultra-local truncated functions [5], expressed by Eq. (4) of [1]. The next lemma is the analogue of Theorem 6 of [1], and is proved in a similar way.

In fact, (7) are the canonical anti-commutation relations over $\mathfrak{T} \otimes L^2(\mathbb{R}^\nu)$, where $\mathfrak{T}$ is a finite or separable complex Hilbert space with an orthonormal basis labeled by I. Product elements of the form (i, f) $(i \in I, f \in L^2(\mathbb{R}^\nu))$ then form a total set in $\mathfrak{T} \otimes L^2(\mathbb{R}^\nu)$. The elements of form $(i, f)^* = (i^*, f^*)$, $i^* \in I$, where I now labels the dual basis in $\mathfrak{T}^*$ and $f^* \in (L^2)^*$, are total in $(\mathfrak{T} \otimes L^2)^*$.

Lemma 3.1. *If the functional*

$$W_T\big((i_1, f_1)^{\#_1} \dots (i_n, f_n)^{\#_n}\big) = \int f_1^{\#_1}(x) \dots f_n^{\#_n}(x)\, dx\; T(i_1^{\#_1} \dots i_n^{\#_n}) \tag{8}$$

(where f^* in the integral means $\bar{f}$) is the truncated functional of some representation of (7), *then $T(i_1^{\#_1} \dots i_n^{\#_n})$ is the truncated functional of some infinitely divisible representation of $\overline{\mathfrak{A}(\mathfrak{T})}$.*

Representations of (7) with truncated functional (8) are called *ultralocal.*

Proof. If we choose a fixed $f = \chi_m$ where χ_m is the characteristic function of a set $K_m \subset \mathbb{R}^\nu$ of measure $1/m$, the restriction of π to this subalgebra gives a $1/m$-extended representation of CAR's over $\mathfrak{T}$, whose truncated functional is, by (8), equal to $1/m\, T$. Since m can be chosen to be any positive integer, T is the truncated functional of an infinitely divisible representation of the CAR's over $\mathfrak{T}$.

As a corollary to this lemma

Theorem 3.2. *Every ultralocal representation of* (7) *is quasi-free.*

The next definition is the analogue of Araki's concept of factorisable representations [6].

Definition. A cyclic representation $(\pi, \mathscr{K}, \Omega)$ of (7) is said to be *factorisable* if, given any division of $\mathbb{R}^\nu$ into a finite number of disjoint measurable sets $M_1, \dots, M_r$, there exist representations $(\pi_1, \mathscr{K}_1, \Omega_1), \dots, (\pi_r, \mathscr{K}_r, \Omega_r)$ of $\overline{\mathfrak{A}(\mathfrak{T} \otimes L^2(M_1))}, \dots, \overline{\mathfrak{A}(\mathfrak{T} \otimes L^2(M_r))}$ respectively, such that

$$\pi \cong \pi_1 \wedge \cdots \wedge \pi_r .$$

Powers [3] has shown that every state, that is an infinite product of states on finite dimensional mutually orthogonal subspaces of $\mathfrak{T}$, is quasi-free, if it is translation invariant. (Powers also considers norm limits of these.) The following theorem is rather similar to this (and also to the analysis of Araki [6]).

Theorem 3.3. *Let $(\pi, \mathscr{K}, \Omega)$ be a factorisable representation of* (7), *such that ω is invariant under translations in $\mathbb{R}^\nu$, i.e.*

$$\omega(A) = \omega(\tau_a(A)), \qquad a \in \mathbb{R}^\nu,$$

*for every $A \in \overline{\mathfrak{A}(\mathfrak{T} \otimes L^2)}$, and τ_a is the *-automorphism of $\overline{\mathfrak{A}(\mathfrak{T} \otimes L^2)}$ induced by $a_i(f) \to a_i(f_a)$ $(i \in I)$ with $f_a(x) = f(x+a)$. Then π is quasi-free.*

Proof. Let f_0 be the characteristic function of a cube K in $\mathbb{R}^\nu$, and divide K into m equal disjoint pieces, $\{M_j\}$, related to each other by translation by $k\,a$ along some direction. We note that the representations $\pi_j = \pi | \overline{\mathfrak{A}(L^2(M_j))}$ $1 \leqq j \leqq m$, are all equivalent; indeed, the unitary equivalence is given by the space translation operator, $U(a)$. Since $E_j f_0$ is just the translate of $E_k f_0$, for $1 \leqq j, k \leqq m$, each $(\pi_j, \mathscr{K}, \Omega)$ provides an equivalent cyclic extended representation of the CAR's over $\mathfrak{T}$. Hence π, restricted to the subalgebra generated by $a_i(f_0)$ $(i \in I)$, is infinitely divisible, and so is quasi-free.

Now, a representation of (7) is uniquely determined by its restriction to the subalgebra generated by the characteristic functions of cubes. Hence π is quasifree.

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