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Non-Linear Equivalence Transformations of Brownian Motion

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1. Introduction

Let $\beta(\cdot,\cdot)$ be a continuous real function of two real variables. Let $(x_t(\omega), t \geqq 0, \omega \in \Omega)$ be a standard Brownian motion process with $x_0 \equiv 0$. The main result of this paper (Theorem 5) is that for any $T > 0$ the process $(\beta(x_t(\omega), t), 0 \leqq t \leqq T, \omega \in \Omega)$ has a probability law in function space absolutely continuous with respect to that of $x_t(\omega)$, $0 \leqq t \leqq T$ (in symbols: $\mathscr{L}(\beta(x_t, t)_0^T) \prec \mathscr{L}((x_t)_0^T)$) if and only if either

$$\beta(x,t) \equiv x + \varphi(t) \quad \text{or} \quad \beta(x,t) \equiv -x + \varphi(t)$$

for $0 \leqq t \leqq T$ and all $x \in R$, where $\varphi(t) = \int_0^t \psi(s)\,ds$ for some $\psi \in L^2([0,T], \lambda)$. (In this paper, λ will always denote Lebesgue measure.)

It is well known [10, pp. 22–23] that if β has the particular forms just stated, then $\mathscr{L}(\beta(x_t, t)_0^T) \prec \mathscr{L}((x_t)_0^T)$. What is new is that to allow functions $\beta(x,t)$ which may not be affine in x, for fixed t, does not actually lead to any new cases of absolute continuity.

If β does not depend on t, then $\beta(x) \equiv x$ or $\beta(x) \equiv -x$. This was conjectured by Segal, who also proposed the problem answered by Theorem 5. This special case is relatively easy to treat, yet it illustrates the methods used in the general case. Therefore we prove it separately as Theorem 1.

The other results in the paper are all used in the proof of Theorem 5. Although they may well be of independent interest, I am not so sure that they are all new. We use the known fact (Denjoy-Young-Saks) that for any $f: R \to R$,

$$\lambda\{x: \lim_{h \downarrow 0} (f(x+h) - f(x))/h = +\infty\} = 0.$$

Theorems 2 and 3 are laws of the (iterated) logarithm for Brownian motion along suitable sequences converging to 0. Theorem 4 states that Brownian motion hits with positive probability the graph of any finite measurable function (even if there are absorbing barriers above and below the graph). This result indicates the possibility of developing the "local time" spent in a graph, although we shall not do so.

A priori, the process $\beta(x_t, t)$ need not be Gaussian nor Markov. Thus we are forced to rely on general properties of continuous or measurable functions and on properties of Brownian motion x_t (e.g. Theorems 2–5). The proof of Theorem 5 has a more probabilistic flavor than the known proofs of the Segal-Feldman-

Hajek theorem on affine equivalence transformations of Gaussian processes [10, Theorem 3, pp. 22–23]; [3, 4]. The latter theorem reduces the problem in question to determining what operators between given Hilbert spaces are Hilbert-Schmidt operators. Such determinations, of course, belong more to functional analysis than to probability.

2. The Case $\beta(x)$

The proof of this first theorem is not given in the fullest detail, since it also is a corollary of Theorem 5.

Theorem 1. *Let β be a continuous real function of a real variable, and $T>0$. Then $\mathscr{L}(\beta(x_t)_0^T) \prec \mathscr{L}((x_t)_0^T)$ iff either $\beta(x) \equiv x$ or $\beta(x) \equiv -x$.*

Proof. "If" is clear; we prove "only if". Clearly $\beta(0)=0$. We first claim that β must be monotone. If not, then we can assume there exist b, c with $0<b<c$ and $\beta(b)>\max(0, \beta(c))$. (Note that $\mathscr{L}(-\beta(x_t)) \prec \mathscr{L}(-x_t) = \mathscr{L}(x_t)$ and $\mathscr{L}(\beta(-x_t)) = \mathscr{L}(\beta(x_t))$.) We can also assume

$$\beta(b) = \max(\beta(x):\ 0 \leqq x \leqq c).$$

The first passage time τ for x_t through $\beta(b)$ is a Markov time [5, pp. 22–25]. Then by the strong Markov property of x_t at τ and the properties of x_t at 0 we know that

$$Pr\big(\text{for some } t>0,\ x_t = \beta(b) \text{ and for some } \delta>0,\ x_s \leqq \beta(b) \\ \text{whenever } t \leqq s \leqq t+\delta\big) = 0.$$

Yet if we replace x_t by $\beta(x_t)$, the probability becomes positive. This contradiction proves that β is monotone.

Now β must be strictly monotone, since otherwise $\beta(x_t)$ spends positive time at some point with positive probability. (Only now can we assert that $\beta(x_t)$ is a Markov process.)

Replacing β by $-\beta$ if necessary, we can assume β is strictly increasing. β is differentiable almost everywhere. If it is not absolutely continuous, i.e. not the indefinite integral of β', then $\beta'(x) = +\infty$ for some x.

[*Proof.* For some compact K,

$$\lambda(K) = 0 < \int_K d\beta(x) = \lambda(\beta(K)).$$

Now if γ is the inverse of β, then $\gamma'(y)=0$ for almost all $y \in \beta(K)$, and then $\beta'(\gamma(y)) = +\infty$.]

Thus it suffices to show that whenever $\beta'(x)$ is defined (finite or infinite), then $\beta'(x)=1$. If $\beta'(x) \neq 1$, let τ be the first passage time through x. By the strong Markov property at τ, we have the iterated logarithm law [7, p. 73]: with probability 1,

$$\overline{\lim_{h \downarrow 0}}\, (x_{\tau+h} - x_\tau)/(2h \log|\log h|)^{\frac{1}{2}} = 1.$$

This implies that with probability 1,

$$\overline{\lim_{h \downarrow 0}}\, (\beta(x_{\tau+h}) - \beta(x_\tau))/(2h \log|\log h|)^{\frac{1}{2}} = \beta'(x) \neq 1.$$

Thus if σ is the first passage time of x_t through $\beta(x)$, $\mathcal{L}(\beta(x_t)) \prec \mathcal{L}(x_t)$ implies that with positive probability,

$$\lim_{h\downarrow 0} (x_{\sigma+h} - x_\sigma)/(2h\log|\log h|)^{\frac{1}{2}} \neq 1.$$

This contradiction completes the proof.

3. Logarithm Laws for Brownian Motion

The two theorems in this section are laws of the (iterated) logarithm along suitable sequences converging to 0, for standard Brownian motion x_t with $x_0=0$. The simplest sequences in question are the geometric ones $t_n \equiv \alpha^n$, $0<\alpha<1$. For this case, the results are implicit in Khinchin's original proof of the local law of the iterated logarithm [7, pp. 73–75].

We abbreviate "$\log|\log t|$" by "$\lg_2 t$". Note that $\lg_2 \alpha^n$ is asymptotic to $\log n$.

Theorem 2. *Let $t_n \downarrow 0$ where for some α, $0<\alpha<1$, $t_{n+1} < \alpha t_n$ for all n. Then the following are equivalent:*

(a) *For all γ with $0<\gamma<1$,*

$$\sum (\lg_2 t_n)^{-\frac{1}{2}} \exp(-\gamma \lg_2 t_n) = +\infty.$$

(b) $\overline{\lim}_{n\to\infty} x_{t_n}(2t_n \lg_2 t_n)^{-\frac{1}{2}} = 1$ *almost surely, and* $\liminf_{n\to\infty} x_{t_n}(2t_n \lg_2 t_n)^{-\frac{1}{2}} = -1$ *almost surely.*

Proof. In any case the $\overline{\lim}$ in (b) is $\leqq 1$ and the lim inf $\geqq -1$ by the usual iterated logarithm law. $Pr(x_t \geqq c)$ is asymptotic to $(t/2\pi c^2)^{\frac{1}{2}} \exp(-c^2/2t)$ as $c t^{-\frac{1}{2}} \to \infty$. Thus if the sum in (a) is finite for some γ, $0<\gamma<1$, then

$$\sum Pr\big(x_{t_n} \geqq \gamma (2t_n \lg_2 t_n)^{\frac{1}{2}}\big) < \infty,$$

so the $\overline{\lim}$ in (b) is $\leqq \gamma$ a.s. Hence (b) implies (a).

For the converse, given a positive integer k, let $Z_n \equiv x_{t_{nk}} - x_{t_{(n+1)k}}$. Then the Z_n are independent Gaussian, $EZ_n \equiv 0$, and $\sigma^2 Z_n = t_{nk} - t_{(n+1)k} \geqq t_{nk}(1-\alpha^k)$. For any $C>0$, $Pr\big(Z_n \geqq C(t_{nk} \lg_2 t_{nk})^{\frac{1}{2}}\big)$ is asymptotic to

$$\begin{aligned}(t_{nk}-t_{nk+k})^{\frac{1}{2}}\, &C^{-1}(2\pi t_{nk} \lg_2 t_{nk})^{-\frac{1}{2}} \exp[-C^2 t_{nk}(\lg_2 t_{nk})/2(t_{nk}-t_{nk+k})] \\ &\geqq (1-\alpha^k)\, C^{-1}(2\pi \lg_2 t_{nk})^{-\frac{1}{2}} \exp[-C^2(\lg_2 t_{nk})/2(1-\alpha^k)].\end{aligned}$$

Now $\sum_n (\lg_2 t_{nk})^{-\frac{1}{2}} \exp(-\gamma \lg_2 t_{nk}) = +\infty$ for all $\gamma<1$ by (a) since the t_{nk} are decreasing. Thus

$$\sum_n Pr\big(Z_n \geqq C(t_{nk} \lg_2 t_{nk})^{\frac{1}{2}}\big) = \infty$$

whenever $0<C<[2(1-\alpha^k)]^{\frac{1}{2}}$. Now $x_{t_{nk+k}}$ is centered Gaussian and independent of Z_n, so

$$Pr\big(x_{t_{nk}} \geqq C(t_{nk} \lg_2 t_{nk})^{\frac{1}{2}}\big) \geqq \tfrac{1}{2} Pr\big(Z_n \geqq C(t_{nk} \lg_2 t_{nk})^{\frac{1}{2}}\big).$$

Hence with positive probability, $x_{t_{nk}} \geqq C(t_{nk} \lg_2 t_{nk})^{\frac{1}{2}}$ for infinitely many values of n [1, Lemma 1]. This positive probability must be 1 by the zero-one law. Now letting $k \to \infty$ we obtain this for every $C < 2^{\frac{1}{2}}$, and hence (b) holds, Q.E.D.

Theorem 3. *Let $t_n \downarrow 0$ and for some α, $0 < \alpha < 1$, $t_{n+1} < \alpha t_n$ for all n. Then with probability* 1,

$$\overline{\lim_{n\to\infty}}\, x_{t_n}(2 t_n \log n)^{-\frac{1}{2}} = 1$$

and the lim inf *of the same sequence is* -1.

Proof. For any $C > 0$, $Pr(x_{t_n} > C(t_n \log n)^{\frac{1}{2}})$ is asymptotic to $C^{-1}(2\pi \log n)^{-\frac{1}{2}} \cdot n^{-C^2/2}$. The sum of these probabilities converges if $C > 2^{\frac{1}{2}}$, so the $\overline{\lim}$ is $\leqq 1$ and the lim inf is $\geqq -1$.

To prove the $\overline{\lim}$ is $\geqq 1$, again take a positive integer k and let $Z_n = x_{t_{nk}} - x_{t_{nk+k}}$. By the previous proof with $\log(n\,k)$ in place of $\lg_2 t_{nk}$, we obtain the desired result. The proof for the lim inf is of course symmetrical. Q.E.D.

4. Hitting Graphs

Let b_t be a Brownian motion process (not necessarily starting at 0), with $c < b_0 < d$ for some real c and d. Let $T = T_{cd}(b_t)$ be the least time such that $b_t = c$ or d (or $T = +\infty$ if there is no such t). Let $y_t = b_t$ for $0 \leqq t \leqq T$ and let $y_t = b_T$ for $t \geqq T$. Then one calls y_t a Brownian motion with *absorbing barriers* at c and d.

Theorem 4. *Let $\{y_t, t \geqq 0\}$ be a Brownian motion with absorbing barriers at c and d, $c < d$. Let K be a compact set of positive numbers with $\lambda(K) > 0$. Let f be continuous: $K \to R$ with $c < \min_K f \leqq \sup_K f < d$. Then $Pr(y_t = f(t)$ for some $t \in K) > 0$.*

Proof. Let $\inf K = m > 0$, $\sup K = M < \infty$. We can assume $a \leqq y_0 \leqq b$ where

$$c < a \leqq \min_K f \leqq \sup_K f \leqq b < d.$$

y_t is Markov with stationary transition probabilities. For each x with $c \leqq x \leqq d$ let $y_t^{(x)}$ be an absorbing barrier process with $y_0^{(x)} \equiv x$. The distribution of $y_t^{(x)}$ is a probability measure P_x on continuous paths with range in $[c, d]$. If $x = c$ or d then P_x is concentrated in the constant function c or d. Let

$$P_x(s, B) \equiv P_x\{g\colon g(s) \in B\}.$$

For $0 < s < t$, and any measurable set $A \subset [c, d]$, we have the Markov composition law (Chapman-Kolmogorov equation)

$$P_x(t, A) = \int_c^d P_y(t-s, A)\, P_x(s, dy).$$

Now if $A \subset (c, d)$ we have $P_x(t, A) \leqq Pr(x + x_t \in A)$ where x_t is standard Brownian motion. Hence the measure $P_x(t, \cdot)$ restricted to (c, d) has a density $p_x(t, u)$,

$$P_x(t, A) = \int_A p_x(t, u)\, du.$$

We can take

$$0 \leqq p_x(t, y) \leqq g_x(t, y) \equiv (2\pi t)^{-\frac{1}{2}} \exp[-(x-y)^2/2t]$$

everywhere (the inequality must hold for almost all y in any case). We shall need the following:

Lemma 4A. *p_x can be chosen so that*

$$\inf\{p_x(t, y) \colon 0 \leqq t \leqq M, a \leqq x \leqq b, a \leqq y \leqq b\} > 0.$$

Proof. Given $t, 0 \leqq t \leqq M$, let $(z_s, 0 \leqq s \leqq t)$ be a Brownian motion tied down at 0 and t. z_s is Gaussian with $z_0 \equiv z_t \equiv E z_s \equiv 0$, $E z_r z_s = r(t-s)/t$, $0 \leqq r \leqq s \leqq t$. If $(x_s, 0 \leqq s \leqq t)$ is standard Brownian and we set $z_s = x_s - s x_t/t$, then z_s will be Brownian tied down at 0 and t, as one sees from the covariances. Thus for any $\varepsilon > 0$, we have

$$Pr(|z_s| \leqq \varepsilon, 0 \leqq s \leqq t) \geqq Pr(|x_s| \leqq \varepsilon/2, 0 \leqq s \leqq t) > 0,$$

a proof of the latter known fact being omitted.

Now let $\varepsilon = \min(a-c, d-b)$.

Let x be a random variable with the distribution of y_0 and let G be a Gaussian random variable with $EG=0$, $EG^2=t$. Let x, G, and $(z_s, 0 \leqq s \leqq t)$ be independent. Then we have a version of b_s:

$$b_s = x + sG/t + z_s.$$

Then if $|z_s| < \varepsilon/2$ for $0 \leqq s \leqq t$, and $|G| < \varepsilon/2$, we have $c < b_s < d$ for $0 \leqq s \leqq t$, and thus y_s coincides with b_s. Then we have

$$p_x(t, y) \geqq Pr(|z_s| \leqq \varepsilon/2, 0 \leqq s \leqq M) > 0$$

under the given conditions, proving the Lemma.

Now by Fubini's theorem,

$$\liminf_{h \downarrow 0} E\lambda\{t \in K \colon f(t) < y_t < f(t) + h\}/h \equiv \kappa > 0,$$

and since $p_x \leqq g_x$,

$$\limsup_{h \downarrow 0} E\lambda^2\{t \in K \colon f(t) < y_t < f(t) + h\}/h^2 \equiv L < \infty.$$

By the Schwartz inequality for any random variable X,

$$E(X)^2 < E(X^2)\, Pr(X \neq 0),$$

$$\liminf_{h \downarrow 0} P_x(f(t) \leqq y_t \leqq f(t) + h \text{ for some } t \in K) \geqq \kappa^2/L > 0,$$

and

$$Pr(\text{for all } n, f(t_n) < y_{t_n} < f(t_n) + 1/n \text{ for some } t_n \in K) \geqq \kappa^2/L.$$

On this event, $y_t = f(t)$ for some $t \in K$, namely any accumulation point of the t_n, Q.E.D.

5. The Main Theorem

We are now ready to state

Theorem 5. *Let* $T>0$ *and let* β *be continuous:* $R\times[0,T]\to R$. *Then* $\mathscr{L}(\beta(x_t,t)_0^T)\prec\mathscr{L}(\mathscr{L}((x_t)_0^T)$ *iff either* $\beta(x,t)\equiv x+\varphi(t)$ *or* $\beta(x,t)\equiv -x+\varphi(t)$ *where* $\varphi(t)=\int_0^t\psi(s)\,ds$, $\psi\in L^2([0,T],\lambda)$.

Proof. It is well known that "if" holds [10, p. 22]. For the converse we first prove:

Lemma 5A. *If* β *is continuous and* $\mathscr{L}(\beta(x_t,t)_0^T)\prec\mathscr{L}((x_t)_0^T)$ *then for* $0<t\leqq T$, $\beta(\cdot,t)$ *is a monotone function.*

Proof. If not, then by symmetry we may assume there is a $t\in(0,T]$ and $c,y,d\in R$ such that $c<y<d$, $\beta(y,t)>\max(\beta(c,t),\beta(d,t))$. Then for some u and v, $0<u<t<v$, and whenever $u\leqq s\leqq v$, there are $f(s)$ and $g(s)$ such that $c<f(s)<d$ and

$$\max_{c\leqq x\leqq d}\beta(x,s)=g(s)=\beta(f(s),s))>\max(\beta(c,s),\beta(d,s)).$$

g is continuous and we can take f to be measurable. Then by Lusin's theorem there is a compact $K\subset[u,v]$ with $\lambda(K)>0$ such that f restricted to K is continuous. From now on, we shall use only that restriction of f.

By the Denjoy-Young-Saks theorem [8, p. 18], there is a compact $M\subset K$ with $\lambda(M)>0$ such that for all $t\in M$,

$$L(t)\equiv\liminf_{h\downarrow 0}(g(t+h)-g(t))/h<\infty.$$

L is measurable, and the lim inf can be taken through positive rational values of h only. We may assume $\sup\{x\colon x\in M\}<v$.

There is positive probability that $c<x_u<d$. Then, we apply Theorem 4 to the process starting at time u and obtain that the following event has positive probability: $c<x_t<d$ and hence $\beta(x_t,t)\leqq g(t)$ whenever $u\leqq t\leqq v$ and for some $s\in M$, $x_s=f(s)$ and hence $\beta(x_s,s)=g(s)$.

Let s be the first time t in M such that $x_t=g(t)$, or $+\infty$ if there is no such t. Then s is a Markov time, and $Pr(M_s)>0$, where $M_s=\{\omega\colon s(\omega)<\infty\}$, by Theorem 4. Probabilities of events A depending on s below signify normalized probabilities $Pr(A\cap M_s)/Pr(M_s)$.

There exist rational $h_n=h_n(s)\downarrow 0$ such that $g(s+h_n)-g(s)<[L(s)+1]\,h_n$. (For more details on the choice of $h_n(s)$, see below.) We can assume $h_{n+1}<h_n/2$ for all n. Then by Theorem 3, almost surely

$$\overline{\lim}\,(x_{s+h_n}-x_s)/(2h_n\log n)^{\frac{1}{2}}=1,$$

which implies $x_{s+h_n}>g(s+h_n)$ for some n (recall $x_s=g(s)$). Thus $Pr(x_t\leqq g(t)$ for $u\leqq t\leqq v$ and $x_s=g(s)$ for some $s\in[u,v])=0$. But we saw earlier that the same event has positive probability for $\beta(x_t,t)$, contradicting $\mathscr{L}(\beta(x_t,t)_0^T)\prec\mathscr{L}((x_t)_0^T)$.

To apply the strong Markov property at the random time s, we must make h_n depend on s in a suitably measurable way. The function L is measurable and could be taken to be continuous, if we like, on M. Let

$$\varepsilon = d - \sup\{x:\ x \in M\} > 0.$$

Let $\{r_m\}_{m=1}^{\infty}$ be an enumeration of the positive rational numbers $<\varepsilon$. Given $n \geqq 0$ and $h_1(s), \ldots, h_n(s)$, let $h_{n+1}(s) = r_m$ for the least m such that

$$g(s+r_m) - g(s) < \big(L(s)+1\big)\, r_m$$

and (if $n \geqq 1$) $r_m < h_n(s)/2$. Then the h_n are measurable functions of s and we can apply the strong Markov property ([2, Satz 5.1, p. 94, Folgerung p. 96]; set $\eta_n \equiv s + h_n$). Thus the proof of Lemma 5A is complete.

Now that we know $\beta(\cdot, t)$ is monotone for each $t \in (0, T]$, we can say that it is strictly monotone, else $\beta(x_t, t)$ takes some value with positive probability. Now we know that $\beta(x_t, t)$ is a (strong) Markov process.

It is easily seen that $\beta(\cdot, t)$ is always monotone in the same sense. So we can assume $\beta(x, t)$ increases as x increases, for each $t \in (0, T]$. Thus the derivative $\partial\beta(x, t)/\partial x$ exists and is finite for almost all x and t. $\partial\beta/\partial x$ is measurable.

Lemma 5B. $\partial\beta(x, t)/\partial x = 1$ *for almost all* $t \in (0, T]$ *and* $x \in R$.

Proof. If the Lemma were false, then by Fubini's theorem there exists an x such that $\partial\beta(x, t)/\partial x$ exists but is $\neq 1$ for t in a set of positive measure. For some $\delta > 0$ and compact set K, $\lambda(K) > 0$, we have either

(I) $\partial\beta(x, t)/\partial x \leqq 1 - \delta$ for all $t \in K$, or

(II) $\partial\beta(x, t)/\partial x \geqq 1 + \delta$ for all $t \in K$.

Taking a smaller δ and K if necessary, we can assume that $[\beta(y, t) - \beta(x, t)]/(y - x) \leqq 1 - \delta$ in case (I), or $\geqq 1 + \delta$ in case (II), whenever $0 < |y - x| \leqq \delta$ and $t \in K$. There is a compact set $M \subset K$, with $\lambda(M) > 0$, all of whose points are density points of K, i.e. for all $t \in M$,

$$\lim_{h \downarrow 0} \lambda(K \cap [t-h, t+h])/2h = 1.$$

Given $s \in M$, there exist $h_n(s)$ such that $2^{-2n-1} \leqq h_n \leqq 2^{-2n}$ and $s + h_n \in K$ for all sufficiently large n. For definiteness, we take h_n as small as possible. Then the iterated logarithm law of Theorem 2 holds for the sequence $h_n \downarrow 0$.

Let $A = \{n:\ \beta(x, s+h_n) \geqq \beta(x, s)\}$. Then either

$$\sum_{n \in A} (\lg_2 h_n)^{-\frac{1}{2}} \exp(-\gamma^2\, 2^{-1} \lg_2 h_n) = +\infty$$

whenever $0 < \gamma < 1$, or else the sum diverges for all such γ with "$n \in A$" replaced by "$n \notin A$".

The functions $h_n(s)$ are measurable, so we can assume either that the sum for $n \in A$ diverges for all $s \in M$, or else that the sum over $n \notin A$ converges for all $s \in M$.

We have now four essentially symmetrical cases to consider. Suppose first that we are in case (I) and that the sum for $n \in A$ is divergent for all $s \in M$. Let $t_r \equiv h_{n(r)}$ where $n(r)\uparrow$ as $r\uparrow$ and $\{n(r):\ r = 1, 2, \ldots\} = A$. Let s be the least t in M such that $x_t = x$, or $+\infty$ if there is no such t. s is finite with positive probability. We have for $x-\delta<y<x$,

$$\beta(x, s+t_n)-\beta(y, s+t_n) \leqq (1-\delta)(x-y),$$
$$\beta(x, s+t_n) \geqq \beta(x, s),$$
$$\beta(y, s+t_n) \geqq \beta(x, s)-(1-\delta)(x-y).$$

By Theorem 2 we have, almost surely where $s<\infty$,

$$\liminf_{n\to\infty}(x_{s+t_n}-x_s)(2t_n \lg_2 t_n)^{-\frac{1}{2}} = -1,$$
$$\liminf_{n\to\infty}\big(\beta(x_{s+t_n}, s+t_n)-\beta(x_s, s)\big)(2t_n \lg_2 t_n)^{-\frac{1}{2}} \geqq -1+\delta.$$

Now for any function f from $[0, T]$ into R, let $\sigma(f)$ be the least t such that $f(t)=\beta(x, t)$, or $+\infty$ if there is no such t. Then σ defines a Markov time for the Brownian motion $(x_t, 0\leqq t\leqq T)$. We thus have almost surely for σ finite,

$$\liminf_{n\to\infty}(x_{\sigma+t_n(\sigma)}-x_\sigma)\big(2t_n(\sigma)\lg_2 t_n(\sigma)\big)^{-\frac{1}{2}} = -1.$$

But $\sigma\big(t\to\beta(x_t, t)\big)=s(t\to x_t)$, since $\beta(x_t, t)=\beta(x, t)$ iff $x_t=x$. Thus we have contradicted the absolute continuity assumption. The other three possible cases can be treated similarly, so Lemma 5B is proved.

Thus by Fubini's theorem, for almost all $t\in(0, T]$ we have $\partial\beta(x, t)/\partial x=1$ for almost all x, and hence

$$|\beta(x, t)-\beta(y, t)| \geqq |x-y|$$

for all x and y. Then by continuity, the last inequality holds for all $t\in[0, T]$, and $\beta(\cdot, t)$ maps R *onto* R.

Lemma 5C. *For each* $t\in(0, T]$, $\beta(x, t)\equiv x+\varphi(t)$ *for some function* φ.

Proof. If not, the only remaining possibility to eliminate is that $\beta(x, t)-\beta(y, t)>x-y>0$ for some x and y. This also holds for all t in a suitable open interval I about the original t. For almost all such t, since $\partial\beta(x, t)/\partial x=1$ a.e., there must be some x with $\partial\beta(x, t)/\partial x\equiv+\infty$, as in the proof of Theorem 1.

For each $m=1, 2, \ldots$, let

$$B_m=\{(x, t)\colon\ \beta(y, t)-\beta(z, t)\geqq 2(y-z)$$
$$\text{whenever } y>z, |y-x|\leqq 1/m, \text{ and } |z-x|\leqq 1/m\}.$$

Then B_m is a closed set. Let $T_m=\{t\in I\colon (x, t)\in B_m \text{ for some } x\}$. Then each T_m is a countable union of compact sets. (Almost) every $t\in I$ belongs to some T_m. Hence for some m, $\lambda(T_m)>0$. We fix such an m. For each $t\in T_m$ let $y(t)$ be the least nonnegative x such that $(x, t)\in T_m$, or if there is no such x, then the greatest negative x. Then y is a measurable function. (In fact, whenever $0\leqq a\leqq b$, or $a\leqq b<0$, $\{t\in T_m\colon a\leqq y(t)\leqq b\}$ is compact.)

Thus there is a compact $C\subset T_m$ with $\lambda(C)>0$ such that $y\restriction C$ is continuous. Let K be a compact set of density points of C with $\lambda(K)>0$.

By Theorem 1, we can assume that for each $t \in K$ there is $L(t) < \infty$ and $h_n \downarrow 0$ with $t+h_n \in C$ such that for all n,

$$|y(t+h_n)-y(t)| \leqq L(t)\, h_n.$$

By Theorem 4, there is positive probability that $x_t = y(t)$ for some $t \in K$. Let s be the least such t, or $+\infty$ if there is none. s is a Markov time for x_t.

As above in the proof of Lemma 5A, we can choose rational numbers $h_n \downarrow 0$ with $h_{n+1} < h_n/2$ for all n, and h_n depending only and measurably on s, such that

$$|y(s+h_n)-y(s)| \leqq L(s)\, h_n$$

for all n, where L is a measurable function. Passing to a subsequence, we can assume by symmetry that $\beta\big(y(s+h_n), s+h_n\big) \geqq \beta\big(y(s), s\big)$ for all n.

By Theorem 3, we have almost surely

$$\overline{\lim_{n\to\infty}}\, |x_{s+h_n} - x_s|\, (2h_n \log n)^{-\frac{1}{2}} = 1.$$

For any $C>0$ and n large enough,

$$\begin{aligned}
&\beta\big(y(s)+C(h_n \log n)^{\frac{1}{2}}, s+h_n\big) - \beta\big(y(s), s\big)\\
&\quad \geqq \beta\big(y(s)+C(h_n \log n)^{\frac{1}{2}}, s+h_n\big) - \beta\big(y(s+h_n), s+h_n\big)\\
&\quad \geqq 2\big(C(h_n \log n)^{\frac{1}{2}} - L(s)\, h_n\big)\\
&\quad \geqq 2^{\frac{1}{2}}\, C(h_n \log n)^{\frac{1}{2}},
\end{aligned}$$

because $L(s)\, h_n = o(C[h_n \log n]^{\frac{1}{2}})$ and $C(h_n \log n)^{\frac{1}{2}} < 1/m$ for n large enough. Thus almost surely where $s<\infty$, letting $C \uparrow 2^{\frac{1}{2}}$, we have

$$\overline{\lim_{n\to\infty}}\big(\beta(x_{s+h_n}, s+h_n) - \beta(x_s, s)\big)\,(2h_n \log n)^{-\frac{1}{2}} \geqq 2^{\frac{1}{2}}.$$

Now, as in the proof of Lemma 5B, for any function $f\colon [0, T] \to R$, let $\sigma(f)$ be the least $t \in K$ such that $f(t) = \beta\big(y(t), t\big)$, or $+\infty$ if there is no such t. Then $\sigma\big(t \to \beta(x_t, t)\big) = s(t \to x_t)$, and $\sigma(t \to x_t)$ is a Markov time for x_t. So Theorem 3 applied to $x_{\sigma+h_n(\sigma)} - x_\sigma$ yields a contradiction to the absolute continuity assumption, and Lemma 5C is proved.

Now it is well known [10, p. 22] that we must have $\varphi(t) = \int_0^t \psi(s)\, ds$ for some $\psi \in L^2([0, T], \lambda)$, and the proof of Theorem 5 is completed.

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