Z. Wahrscheinlichkeitstheorie verw. Gebiete
46, 317 – 324 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

The Ergodic Behaviour of Piecewise Monotonic Transformations

Gerold Wagner
Mathematisches Institut der Universität, Pfaffenwaldring 57 D-7000 Stuttgart 80

1. Main Result

Let T be a piecewise monotonic transformation of the unit interval $I=[0, 1)$ onto itself satisfying some additional conditions to be specified later. Then T splits into a finite number of ergodic components in the following sense: There is a decomposition of I into finitely many disjoint measurable sets $A_1, \ldots, A_s$ and B such that

(a) the A_ρ's are invariant ($TA_\rho = A_\rho \bmod 0$) and the restrictions $T|_{A_\rho} = T_\rho$ are ergodic with respect to Lebesgue measure m,

(b) on each A_ρ there exists an invariant measure μ_ρ which is equivalent to the restriction of m to A_ρ,

(c) the set B satisfies the relations $T^{-1}B \subset B$ and $\lim_{n\to\infty} m(T^{-n}B)=0$.

(d) to each component A_ρ there corresponds a power $T_\rho^* = T_\rho^{m(\rho)}$ of the transformation T_ρ and a disjoint decomposition $A_\rho = A_{\rho 1} \cup \ldots \cup A_{\rho m(\rho)}$ such that $T_\rho^*(A_{\rho\sigma}) = A_{\rho\sigma} \bmod 0 (\sigma = 1, \ldots, m(\rho))$ and T_ρ^* is an exact endomorphism on each $A_{\rho\sigma}$. The transformation T_ρ permutes the $A_{\rho\sigma}$ cyclically.

(The set B may have measure 0.)

2. Preliminaries

Let $\mathscr{P} = \{P_1, \ldots, P_L\}$ be a partition of I into intervals. With every P_j we associate a strictly monotone, absolutely continuous function V_j which maps P_j onto an interval $Q_j \subset I$. Without loss of generality we may assume that for $i \neq j$ we have either $Q_i \cap Q_j = \emptyset$ or $Q_i = Q_j$. This can be achieved by choosing an appropriate refinement $\mathscr{P}'$ of the partition $\mathscr{P}$, if necessary. The transformation T is defined by $Tx = V_j(x)$ for $x \in P_j$. We further assume $TI = I \bmod 0$. Set-theoretic relations are always meant to be modulo a set of Lebesgue measure 0 unless indicated otherwise. We point out the different use of the notion of an "invariant set". A set $M \subset I$ is called invariant with

0044-3719/79/0046/0317/$01.60

respect to a transformation T, if $TM=M \bmod 0$, and strictly invariant, if $T^{-1}M=M \bmod 0$. Let the functions V_j be such that the inverses $V_j^{-1}=f_j$ satisfy the following conditions:

(C1) $0<c_1\leqq|f_j'(x)|\leqq c_2<1$ for all $x\in Q_j$ and some positive constants c_1 and c_2.

(C2) $|f_j''(x)|\leqq K$ for all $x\in Q_j$ and some $K>0$.

Note. As I learnt from Li, the results (a) to (c) are contained in a paper by Li and Yorke, Trans. Amer. Math. Soc. **235** (1978), 183–192. These authors use a quite different method. However, the proofs of Lemma 2.5. and 2.7. are not satisfactory, using relations (2.6.) (a) and (c) which are false.

Using an ingenious method, Lasota and Yorke [1] proved for such transformations the existence of a T-invariant measure $\mu\ll m$ on I. Similar results were obtained by the author in this thesis [4], using a short of renewal argument, the basic idea of which is taken from Fischer's paper [2]. In the author's investigation a major role is played by cylinders $(a_1, \dots, a_s)$ with the property that the domain of definition of the corresponding function $f_{a_1}\circ\cdots\circ f_{a_s}$ is just the interval Q_{a_s}. A cylinder of this type will be called a Q_{a_s}-cylinder. In the following we make the additional assumption $\sup|f_j'(x)|\leqq\frac{\delta}{2}$ for some $\delta\in(0,1)$. We will return to the general case later on. As proved in [4], the transformation T and its iterates T^k can be described essentially in terms of Q_j-cylinders. Denote by $E_{m,n}$ $(0<m\leqq n)$ the union of all Q_j-cylinders $(j=1, \dots, L)$ of rank s satisfying $m\leqq s\leqq n$. The following lemma is an immediate consequence of Lemma 5.1. in [4]. For sake of completeness, however, a separate proof will given below.

Lemma 2.1. *Let* $\varepsilon>0$ *and* $n_1\in\mathbb{N}$. *Then there is an integer* $n_2\in\mathbb{N}$ *such that* $m(I\setminus E_{n_1,n_2})<\varepsilon$.

Proof. Let $\mathscr{R}(n_1)$ be the class of all cylinders of rank n_1 which are not Q_j-cylinders for any $j\in\{1, \dots, L\}$. Let $R(n_1)$ be their union, and denote their number by $\tau(n_1)$.

Consider an arbitrary cylinder $(a_1, \dots, a_{n_1})\in\mathscr{R}(n_1)$, and let $(i)=P_i$ $(i\in\{1, \dots, L\})$ be a cylinder of rank 1.

Three cases are possible:

(1) If (i) is contained in the interval $T^{n_1}(a_1, \dots, a_{n_1})$, then $(a_1, \dots, a_{n_1}, i)$ is a Q_i-cylinder of rank (n_1+1).

(2) If we have $m((i)\cap T^{n_1}(a_1, \dots, a_{n_1}))=\emptyset$, the cylinder $(a_1, \dots, a_{n_1}, i)$ is not defined at all.

(3) If we have $0<m((i)\cap T^{n_1}(a_1, \dots, a_{n_1}))<m(i)$, the cylinder $(a_1, \dots, a_{n_1}, i)$ is defined but is not a Q_i-cylinder.

The latter case can occur for at most two different values of i.

Now let $\mathscr{R}(n_1+1)$ be the class of all cylinders of rank (n_1+1) which are contained in $R(n_1)$, and which are not Q_j-cylinders. Let $R(n_1+1)$ be again their union, and let $\tau(n_1+1)$ be their number. By the argument above we have $\tau(n_1+1)\leqq 2\tau(n_1)$.

Defining $\mathscr{R}(n_1+s)$, $R(n_1+s)$ and $\tau(n_1+s)$ $(s=1, 2, \dots)$ inductively, and repeating the argument, we get $\tau(n_1+s)\leqq 2^s\tau(n_1)$.

We shall estimate $m(R(n_1+s))$. By our additional assumption, the measure of a cylinder of rank (n_1+s) is at most $(\frac{1}{2}\delta)^{n_1+s}$.

This implies $m(R_1(n+s)) \leqq 2^s \tau(n_1)(\frac{1}{2}\delta)^{n_1+s} \leqq \delta^{n_1+s} \cdot \tau(n_1)$.

We thus have $m(I \setminus E_{n_1, n_1+s}) \leqq m(R(n_1+s)) \leqq \delta^{n_1+s} \cdot \tau(n_1)$.

Now choose s in accordance with $\varepsilon > 0$, and set $n_2 = n_1 + s$. This completes the proof.

It turns out to be convenient in many cases to restrict oneself to points in I which are not endpoints of any cylinder interval. Throughout this article they will be called "interior points". Note that if x is interior, the same is true for Tx. The set of points which are not interior is, of course, countable..

By the next definition we introduce an essential concept.

Definition 2.1. A point $x \in I$ is said to be a Q_j-point if x is interior and is contained in infinitely many Q_j-cylinders. A set M of positive measure is said to be a Q_j-set if almost all points $x \in M$ are Q_j-points.
(Note that a Q_j-cylinder, if considered as a subset of I, need not be a Q_j-set.)

Using Lemma 2.1. we can prove the following assertion:

Lemma 2.2. *For almost all $x \in I$ there exists at least one interval Q_j (depending on x) such that x is a Q_j-point.*

Proof. Choose an arbitrary sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive real numbers satisfying $\sum_{\nu=1}^{\infty} \varepsilon_\nu < \infty$. Lemma 2.1, we can find a sequence $0 < n_0 < n_1 < n_2 < \ldots$ of integers such that $m(I \setminus E_{n_{\nu-1}, n_\nu}) < \varepsilon_\nu$ $(\nu = 1, 2, \ldots)$.

Since there is only a finite number of different intervals Q_j, the Borel-Cantelli lemma implies for almost all $x \in I$ the existence of a sequence of cylinders $c_1 \supset c_2 \supset \ldots$ which contain the point $x \in I$ and which are Q_j-cylinders for a certain index j. This completes the proof.

We now prove three more lemmata which will be of use later on.

Lemma 2.3. *If $x \in I$ is a Q_j-point then the same is true for Tx.*

Proof. Let $c_1 \supset c_2 \supset \ldots$ be a sequence of Q_j-cylinders with ranks $1 \leqq r_1 < r_2 < \ldots$ such that $x \in \bigcap_{m=1}^{\infty} c_m$.

It is true that the image of a cylinder need not be a cylinder again, but the image of Q_j-cylinder of rank $r \geqq 2$ is always a Q_j-cylinder. Therefore, $Tc_2 \supset Tc_3 \supset \ldots$ is a sequence of Q_j-cylinders with $Tx \in \bigcap_{m=2}^{\infty} Tc_m$. Moreover, the point Tx is interior. This completes the proof.

Lemma 2.4. *If some set $A \subset I$ of positive measure is a Q_j-set then* $\limsup T^n A \supset Q_j$.

Proof. It will be sufficient to show $\bigcup_{p=n}^{\infty} T^p A \supset Q_j$ for every $n = 1, 2, \ldots$. By Lebesgue's density theorem and our assumption, there exists a point $z \in A$ with the following two properties:

(1) $z \in A$ is a point of density 1 and

(2) $z \in A$ is a Q_j-point.

Let $c_1 \supset c_2 \supset \dots$ be a sequence of Q_j-cylinders with ranks $1 \leqq r_1 < r_2 < \dots$ containing the point $z \in A$. Fix $n \in \mathbb{N}$. For every $\varepsilon > 0$ there is a cylinder $c_k = (a_1, \dots, a_{r_k})$ satisfying $r_k > n$ and $m(c_k \cap A) \geqq (1-\varepsilon) \cdot m(c_k)$. For an arbitrary measurable set $B \subset c_k$ we shall estimate $m(T^{k(r)} B)$. We note that conditions (C1) and (C2) imply Renyi's condition. That is, for some constant $K > 0$ depending only on the transformation T, we have

$$\sup \left| \frac{d}{dx} f_{a_1} \circ \dots \circ f_{a_k} \right| \leqq K \cdot \inf \left| \frac{d}{dx} f_{a_1} \circ \dots \circ f_{a_k} \right|$$

for every cylinder $(a_1, \dots, a_k)$.

This implies $m(T^{r_k} B) \leqq K \cdot m(Q_j) \cdot m(B) \cdot (m(c_k))^{-1}$ for every $B \subset c_k$.

For $B = c_k \setminus A$ we get $m(T^{r_k}(c_k \setminus A)) \leqq K \varepsilon \cdot m(Q_j)$, hence $m(T^{r_k}(c_k \cap A)) \geqq (1 - K\varepsilon) \cdot m(Q_j)$, the sets $T^{r_k}(c_k \setminus A)$ and $T^{r_k}(c_k \cap A)$ being disjoint.

By choice of k we have $T^{r_k}(c_k \cap A) \supset \bigcup_{\rho=n}^{\infty} T^{\rho} A$.

This implies $m\left(Q_j \setminus \bigcup_{\rho=n}^{\infty} T^{\rho} A\right) \leqq K\varepsilon \cdot m(Q_j)$, which proves the assertion.

Lemma 2.5. *If some set $A \subset I$ of positive measure is a Q_j-set then the interval Q_j itself is a Q_j-set*

Proof. By Lemma 2.3., the set $\bigcup_{n=0}^{\infty} T^n A$ is a Q_j-set which contains the interval Q_j by Lemma 2.4.

3. Construction of the Ergodic Components

For a fixed Q_j consider the set $F(Q_j)$ of all Q_j-points in I. Lemma 2.5. implies that either the set $F(Q_j)$ has measure 0 or that it contains the interval Q_j. We can disregard the Q_j's for which $m(F(Q_j)) = 0$ holds, thus assuming $m(F(Q_j)) > 0$ in the sequel. It is clear from Lemma 2.3. that $T(F(Q_j)) \subset F(Q_j)$ holds. For any invariant subset $H \subset F(Q_j)$ we can easily prove that H is ergodic with respect to m, but there still remains the possibility that there may exist a proper subset $G \subset H$ satisfying $0 < m(G) < m(H)$ and $TG = G$. This would imply that any invariant measure $\mu \ll m$ on H must vanish on $H \setminus G$. So we are looking for a minimal invariant subset of $F(Q_j)$ which we can find as follows:

Let $F' \subset F(Q_j)$ be an arbitrary subset such that $m(F') > 0$ and $TF' = F'$. Lemma 2.4., the set F' contains the interval Q_j and hence the set $M_j = \bigcup_{\nu=0}^{\infty} T^{\nu} Q_j$. We shall prove that the sets M_j possess the required properties.

Theorem 3.1. *The following statements are true:*

(a) $TM_j = M_j$;

(b) *if $i \neq j$ then the sets M_i and M_j are either disjoint or identical.*

Proof. (a) We have $TM_j = \bigcup_{\nu=1}^{\infty} T^\nu Q_j \subset M_j$. Thus the sequence $M_j, TM_j, T^2 M_j, \ldots$ is descending. By Lemma 2.4., we have $\overline{\lim_{n\to\infty}}\, T^n M_j \supset Q_j$ and hence a fortiori $TM_j \supset Q_j$. This proves $TM_j = M_j$. (Note that, in general, we do not have $T^{-1} M_j = M_j$!)

(b) Let $D = M_i \cap M_j$ and assume that $0 < m(D) < m(M_i)$. As proved in (a), the sets M_i and M_j are invariant. Thus the sequence $D, TD, T^2 D, \ldots$ is descending. By Lemma 2.4., the set D contains the interval Q_i and hence the set M_i. This contradicts the assumption $m(D) < m(M_i)$.

In order to complete the construction, it now suffices to denote by $A_1, A_2, \ldots, A_s$ the different ones among the sets M_j and to let T_ρ ($\rho = 1, 2, \ldots, s$) be the restriction of T to A_ρ. It remains to prove ergodicity and existence of an invariant measure.

Theorem 3.2. (a) *Each transformation T_ρ is ergodic on A_ρ with respect to the restriction $m|_{A_\rho} = m_\rho$.*

(b) *There exists a (unique) T_ρ-invariant measure μ_ρ on A_ρ which is equivalent to m_ρ.*

Proof. (a) This follows from the minimality property of the sets A_ρ. The statement may also be proved along the lines of part (b) of Theorem 3.1.

(b) The existence of an invariant measure $\mu_\rho \ll m$ was proved in [1] and, by another method, in [4].

Let $g_\rho(x)$ be the Radon-Nikodym derivative of μ_ρ with respect to m. We shall prove that $g_\rho(x) > 0$ for m-almost all $x \in A_\rho$.

Let $G_\rho = \{x \in A_\rho | g_\rho(x) > 0\}$. Then the relation $T_\rho G_\rho = TG_\rho \subset G_\rho$ holds, which again implies $G_\rho = A_\rho$. This proves part (b).

Since the measures m_ρ and μ_ρ are equivalent, the transformation T is also ergodic with respect to μ_ρ. As is well-known from ergodic theory, this implies that the measure μ_ρ is the only one which is invariant and absolutely continuous with respect to m_ρ. We are going to study the complement of the union of the A_ρ's.

Theorem 3.3. *Let $B = I \setminus \bigcup_{\rho=1}^{s} A_\rho$. Then the set B satisfies the relation* $\lim_{n\to\infty} m(T^{-n} B) = 0$.

Proof. We proved in Theorem 3.1. that $TA_\rho = A_\rho$ and hence $T^{-1} A_\rho \supset A_\rho$ ($\rho = 1, \ldots, s$). This implies that the sequence $B, T^{-1} B, T^{-2} B, \ldots$ is descending. Denote its limit by C and assume $m(C) > 0$. We know by Lemma 2.2. that C has a Q_j-subset of positive measure for some index $j \in \{1, \ldots, s\}$. Since the set C is strictly invariant, it must contain the interval Q_j and, hence, the set A_ρ for some $\rho \in \{1, \ldots, s\}$. This contradicts the definition of B.

We can describe the sets A_ρ as follows: Each A_ρ contains those intervals Q_j for which $m(A_\rho \cap F(Q_j)) > 0$ holds. Furthermore, the set A_ρ may contain measurable subsets of intervals Q_j for which $m(F(Q_j)) = 0$ is valid.

The Theorems 3.1. through 3.3. were proved under the additional assumption that $\sup|f_j'(x)|<\frac{1}{2}$. In the general case, choose a suitable power $S=T^g$ such that S will satisfy this assumption. Let $C_1, \ldots, C_t, D$ be the components of S. We let $B=D$ and prove that $\lim_{n\to\infty} m(T^{-n}B)=0$.

Let $n=p\cdot g+q$ $(0\leqq q<g)$. We have the obious inequality $m(T^{-n}B)\leqq L^q\cdot m(S^{-p}D)$, L being the number of cylinders of rank 1. So $\lim_{n\to\infty} m(T^{-n}B)=0$.

We define sets A'_σ by $A'_\sigma=\bigcup_{\nu=0}^{g-1} T^\nu C_\sigma$. Clearly, one has $TA'_\sigma=A'_\sigma$.

By arguments very similar to those used in the proofs of Theorems 3.1. and 3.2., we can establish the following properties:

(a) For $\sigma\neq\tau$ the sets A'_σ and A'_τ are either identical or disjoint. (In fact, each set A'_σ is the union of sets C_λ but we do not use this.)

(b) If U_σ is a subset of A'_σ of positive measure satisfying $TU_\sigma\subset U_\sigma$ then $U_\sigma=A'_\sigma$.

Denoting the different ones among the sets A'_σ by $A_1, \ldots, A_s$, we obtain the desired decomposition $I=A_1\cup\cdots\cup A_s\cup B$ for which the Theorems 3.1. through 3.3. remain valid. This settles the general case.

A special case. Let T be an F-expansion (that is, let $F\colon [0,1)\to\mathbb{R}$ be a strictly monotone, absolutely continuous function; define T by $Tx=F(x) \bmod 1$). The corresponding f_i's are assume to satisfy conditions (C1) and (C2).

Theorem 3.4. *If T is an F-expansion then there is only a single component A_1.*

Proof. For an arbitrary interval $J\subset I$ consider the sequence $J, TJ, T^2J, \ldots$. Since we have assumed $\sup|f_j'(x)|<1$, there is an integer $k\in\mathbb{N}$ such that T^kJ contains an interval which intersects two cylinders of rank 1. In case of an F-expansion this means that $T^{k+1}J$ contains, for some $\varepsilon>0$, a set of the form $(0,\varepsilon)\cup(1-\varepsilon,1)$. By the definition of the A_ρ's, every component A_ρ would contain a set of this form for some $\varepsilon>0$. Since different components A_ρ are disjoint, there can only be a single one. This concludes the proof. One might think that, at least in the case of F-expansions, the endomorphism (T_1, A_1, μ_1) would be exact. This is not so, however. For a counterexample, consider the transformation T:

$Tx=\frac{6}{5}x+\frac{2}{5} \bmod 1$. For every sufficiently small $\varepsilon>0$ ($\varepsilon<\frac{1}{10}$ will do) we have $T^i(0,\varepsilon)\subset(\frac{2}{5},\frac{3}{5})$ if i is odd, and $T^i(0,\varepsilon)\subset(0,\frac{1}{5})\cup(\frac{4}{5},1)$ if i is even. Since $(0,\varepsilon)\subset A_1$ for some $\varepsilon>0$, T_1 cannot be exact.

But we shall prove that there is always a power T^g of T such that T^g splits into exact components.

In the author's thesis [4] several sufficient conditions were established which imply the existence of an invariant measure $\mu\equiv m$ on I such that T is exact. This is true, for example, if T is an F-expansion satisfying $\sup|f_i'(x)|<\frac{\delta}{2}$ for some $\delta\in(0,1)$.

4. Decomposition of the A_ρ's into Exact Components

Let $I=A_1\cup\cdots\cup A_s\cup B$ be a decomposition of T into ergodic components, and let $(A_\rho, T_\rho, \mu_\rho)$ be the corresponding dynamical systems for each $\rho\in\{1,\ldots,s\}$. We prove

Theorem 4.1. *For each $\rho\in\{1,\ldots,s\}$ there is a number $m(\rho)\in\mathbb{N}$ and a disjoint decomposition $A_\rho=A_{\rho 1}\cup\cdots\cup A_{\rho m(\rho)}$ such that $T^{m(\rho)}A_{\rho\sigma}=A_{\rho\sigma}$ $(\sigma=1,\ldots,m(\rho))$ and the transformation $T_\rho^*=T_\rho^{m(\rho)}$ is an exact endomorphism on each $A_{\rho\sigma}$. The $A_{\rho\sigma}$'s are permuted cyclically by T_ρ.*

Proof. (a) First assume the f_i's to satisfy the additional assumption $\sup|f_i'(x)|<\frac{1}{2}$. By Theorem 3.1. there exists a representation $A_\rho=\bigcup_{\nu=0}^{\infty} T^\nu Q_j$ for some interval Q_j. We have $T^N Q_j\supset Q_j$ for some $N\in\mathbb{N}$, the interval Q_j being itself a Q_j-set. We define sets S_α by $S_\alpha=\lim_{\nu\to\infty} T^{\nu N+\alpha}Q_j$ $(\alpha=0,\ldots,N-1)$. The sets S_α have the obvious properties $S_\alpha\subset A_\rho$, $\bigcup_{\alpha=0}^{N-1} S_\alpha=A_\rho$ and $T_\rho S_\alpha=S_{\alpha+1}(\mathrm{mod}\,N)$. Let $m(\rho)$ be the least positive integer such that $T^{m(\rho)}S_0=S_0$, hence $m(\rho)\leqq N$. It follows that $m(\rho)$ is the least positive integer such that $T^{m(\rho)}S_\alpha=S_\alpha$ for each $\alpha\in\{0,\ldots,N-1\}$. Now define $A_{\rho\sigma}$ by $A_{\rho\sigma}=S_{\sigma-1}$ $(\sigma=1,\ldots,m(\rho))$. By construction no two of the $A_{\rho\sigma}$ are identical. If we can prove the transformation T_ρ^* to be exact on each $A_{\rho\sigma}$, the sets $A_{\rho\sigma}$ must be mutually disjoint.

(b) Let $\varepsilon_0=\min_{\sigma\neq\tau} m(A_{\rho\sigma}\setminus A_{\rho\tau})$ and choose $\varepsilon>0$ such that $0<\varepsilon<\varepsilon_0$. By definition of $A_{\rho\sigma}$ there exists an integer $i(\varepsilon)\in\mathbb{N}$ with the property that $m(A_{\rho\sigma}\setminus T^{i\cdot m(\rho)+\sigma-1}Q_j)<\varepsilon\cdot m(A_{\rho\sigma})$. Let $D\subset A_{\rho\sigma}$ be an arbitrary set of positive Lebesgue measure. By the same argument as in Lemma 2.4. there exists, for every $\delta>0$, a Q_j-cylinder $c(\delta)$ (with rank $r(\delta)\geqq 1$) satisfying $m(D\cap c(\delta))\geqq(1-\delta)\,m(c(\delta))$.

It was shown in the proof of Lemma 2.4. that $m(T_\rho^{r(\delta)}(D\cap c(\delta)))>(1-K\delta)\cdot m(Q_j)$ with $K>0$ depending only on T_ρ.

We note that, for fixed values of i, $m(\rho)$, and σ, the transformation $T^{i\cdot m(\rho)+\sigma-1}$ is an absolutely continuous set function. Hence $m(A_{\rho\sigma}\setminus T^{i\cdot m(\rho)+\sigma-1+r(\delta)}D)<\varepsilon\cdot m(A_{\rho\sigma})$ for $\delta>0$ sufficiently small. The restriction $\varepsilon<\varepsilon_0$ implies $\sigma-1+r(\delta)\equiv 0 \bmod m(\rho)$. Thus $m(T_\rho^{*n}D)>(1-\varepsilon)\cdot m(A_{\rho\sigma})$ with $n=i+(\sigma-1+r(\delta))/m(\rho)$. By Rohlin's criterion T_ρ^* is exact on $A_{\rho\sigma}$.

(c) Now let τ be an arbitrary transformation satisfying conditions (C1) and (C2), and let $k\in\mathbb{N}$ be such that our additional assumption holds for $T=\tau^k$. For an arbitrary ergodic component C of τ there is a representation

$$C=A_{\rho_1}\cup\cdots\cup A_{\rho_l} \qquad (l\geqq 1),$$

the sets A_{ρ_i} being ergodic components of T. For each set A_{ρ_i} $(i=1,\ldots,l)$ we define the sets $A_{\rho_i\sigma}$ for the transformation T_{ρ_i} in the same way as in part (a).

Let m be the least common multiple of the numbers $m(\rho_i)$, and define the transformation T^* on C by $T^*=T^m$. Every power of an exact endomorphism is again exact, and all we have to show is that τ permutes the sets $A_{\rho_i\sigma}$ cyclically.

Fix a set $A_{\rho_i\sigma}$. For every integer $\nu\in\mathbb{N}$ the set $\tau^\nu A_{\rho_i\sigma}$ is invariant with respect to the transformation T^* because T^* and τ^ν commute. For the same reason T^* is exact on each $\tau^\nu A_{\rho_i\sigma}$. Thus every set $\tau^\nu A_{\rho_i\sigma}$ is again of the form $A_{\rho_i'\sigma'}$. Furthermore $\bigcup_{\nu=1}^{m\cdot k}\tau^\nu A_{\rho_i\sigma}=C$ since C does not contain a proper τ-invariant subset. This completes the proof.

References

1. Lasota, A., Yorke, J.A.: On the existence of invariant measures for piecewise monotonic transformations. Trans. Amer. Math. Soc. **186**, 481–488 (1973)
2. Fischer, R.: Mischungsgeschwindigkeit für Ziffernentwicklungen nach reellen Matrizen. Acta Arith. **23**, 5–12 (1973)
3. Rohlin, W.A.: Exact endomorphisms of a Lebesgue space. Isv. Akad. Nauk SSSR **25**, 499–530 (1960)
4. Wagner, G.: Zahlentheoretische Transformationen. Thesis, Stuttgart 1975

Received January 1, 1978; in revised form August 30, 1978