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More on Limit Theorems for Iterates of Probability Measures on Semigroups and Groups

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Summary. In this paper, we continue earlier works of one of the authors on vague convergence of the sequence $\beta_{k,n}=\beta_{k+1}*\ldots*\beta_n$, where β_n is a sequence of probability measures on semigroups or groups. Typical results in this paper are: *Theorem.* Let S be a locally compact noncompact second countable group such that $S=\overline{\bigcup_{n=1}^{\infty} S_\beta^n}$, S_β being the support of a probability measure β on S. Suppose there exists an open set V with compact closure such that $x^{-1} Vx=V$ for every $x\in S$. Then for all compact sets K, $\sup\{\beta^n(Kx): x\in S\}\to 0$ as $n\to\infty$. *Theorem.* Let S be an at most countable discrete group. Let β_n be a sequence of probability measures on S. Then for all nonnegative integers k, the sequence $\beta_{k,n}$ converges vaguely to some probability measure if and only if there exists a finite subgroup G such that the series $\sum_{n=1}^{\infty}\beta_n(S-G)<\infty$ and for any proper subgroup G' of G and any choice of elements g_n in S, the series $\sum_{n=1}^{\infty}\beta_n(S-g_{n-1}G' g_n^{-1})=\infty$. A sufficient condition for the vague convergence of the sequence $\beta_{k,n}$ to a probability measure is that (i) there exists a finite subgroup G such that $\sum_{n=1}^{\infty}\beta_n(S-G)<\infty$ and (ii) $\beta_n(e)>s>0$ for all n, e being the identity.

1. Throughout this paper, S will denote a locally compact second countable semigroup (i.e. an algebraic semigroup with locally compact Hausdorff topology and jointly continuous multiplication).

By a measure on S, we mean a finite regular non-negative measure on the class of all Borel sets (generated by open sets) of S. Let $C(S)$ be the real-valued continuous functions with compact support. A net of measures (β_k) is said to converge in the weak*-sense (or vaguely) to a measure β if and only if for each $f\in C(S)$,

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$\int f d\beta_k$ converges to $\int f d\beta$. By Banach-Alaoglu's theorem, the set $B(S)$ of all measures β with $\beta(S) \leqq 1$ is compact in the weak*-topology. Also $B(S)$ is an algebraic semigroup under the usual convolution $(*)$ operation of measures.

Throughout we will use the following notation. For sets A and $B \subset S$ and any point x in S, we will write:

$$A x^{-1} = \{y: y x \in A\}; \quad x^{-1} A = \{y: xy \in A\};$$

$$A B^{-1} = \bigcup \{A x^{-1}: x \in B\} \quad \text{and} \quad A^{-1} B = \bigcup \{x^{-1} B: x \in A\}.$$

It is known that for any $\beta \in B(S)$ and any Borel set C, the mappings $x \to \beta(C x^{-1})$ and $x \to \beta(x^{-1} C)$ are measurable; also for β_1 and β_2 in $B(S)$, we have:

$$\beta_1 * \beta_2(C) = \int \beta_1(C x^{-1}) \beta_2(dx) = \int \beta_2(x^{-1} C) \beta_1(dx).$$

When S is compact, $B(S)$ as well as $P(S)$ (= the probability measures on S) is a compact topological semigroup with respect to convolution and the weak*-topology. But $P(S)$, despite being a topological semigroup, is compact if and only if S is compact while $B(S)$ although compact is not a topological semigroup since the convolution in $B(S)$ is not jointly continuous – for example, when $S = (0, \infty)$, $\delta_{\{n\}} \to 0$ vaguely, $\delta_{\{1/n\}} \to 0$ vaguely while $\delta_{\{n\}} * \delta_{\{1/n\}} = \delta_{\{1\}}$. When S is the semigroup $[0, \infty)$ under multiplication, the convolution in $B(S)$ is not even separately continuous; the reason is: one can show by using the Central Limit theorem that if μ is the normalized Lebesgue measure on $[0, e]$, $e = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then μ^n converge vaguely to the measure $\frac{1}{2} \cdot \delta_{\{0\}} \in B(S)$ while $\mu^n * \delta_{\{0\}} = \delta_{\{0\}}$ and $\frac{1}{2}\ \delta_{\{0\}} * \delta_{\{0\}} = \frac{1}{2} \cdot \delta_{\{0\}}$.

The purpose of this paper is to describe the limit behavior of the sequence $\beta_{0,n} = \beta_1 * \ldots * \beta_n$, where β_n is a sequence in $P(S)$, on semigroups S which are either compact or groups. So far the main work in this context seems to have been done by Kloss [6], Csiszár [1], and Tortrat [12]. See also [4, 7, 8, 9, 10]. Kloss [6] proved the following theorem

Theorem (Kloss). *Let S be a compact group. Then if (β_n) is a sequence in $P(S)$, there exist elements $a_n \in S$ such that the normalized sequence $\beta_1 * \beta_2 * \ldots * \beta_n * \delta_{a_n}$ converges vaguely as $n \to \infty$ to a probability measure on S.*

Kloss left the problem of extending this result to compact semigroups open. One of the results (Theorem 1) in this paper is this extension. Csiszár [1] proved the following theorem.

Theorem (Csiszár). *Let S be a locally compact group. Let (β_n) be a sequence in $P(S)$. Then either*

$$\sup \{\beta_1 * \beta_2 * \ldots * \beta_n(K x): x \in S\} \to 0 \quad \text{as } n \to \infty$$

for each compact set K or there exists $a_n \in S$ such that the sequence $\beta_1 * \beta_2 * \ldots * \beta_n * \delta_{a_n}$ converges weakly as $n \to \infty$ to some probability measure.

In [10], it was proved that in a non-compact locally compact second countable group S generated by the support of a probability measure $\beta \in P(S)$, the sequence β^n converges vaguely to zero. In this paper, we will discuss when we can expect a stronger limit theorem, namely when $\sup\{\beta^n(Kx): x\in S\}\to 0$ as n tends to infinity for each compact set K. In this paper, we also study the question of existence of constants a_n in a locally compact non-compact group such that the sequence $\beta^n * \delta_{a_n}$ converges vaguely to a probability measure on S, which is generated by the support of β. A good part of this paper (all of Section 4) is devoted to solving the problem of finding suitable necessary and sufficient conditions for the vague convergence of a sequence $\beta_{k,n}=\beta_{k+1} * \ldots * \beta_n$ to some probability measure for all positive integers k on at most countable discrete groups. This subject was first studied in details by Maksimov [7, 8] in the case of finite groups. Maksimov generalized the classical concept of variance for random variables taking values in a finite group. His methods and results are interesting, but do not seem to carry over to the case of infinite groups. Our results extend Maksimov's results to infinite groups, and our methods are more direct and different from those of Maksimov. We mainly use ideas of Csiszár and Tortrat. In this context, we should mention an interesting paper by Heyer [4]. He also considering finite groups to study this problem and presents a detailed proof of the following Maksimov theorem: if $\beta_n(e)>\delta>0$ for all n, then $\beta_{k,n}$ is vaguely convergent for all k. An extension of this result is obtained as a corollary to our characterization theorem (Theorem 16). We also present some new results in finite groups.

2. In this section, we extend Kloss's theorem to compact semigroups. Our method is different from that of Kloss (Kloss' method actually does not extend to semigroups) and our proof is very much like that of Mukherjea [9]. For completeness, we include the proof.

Theorem 1. *Let S be a compact Hausdorff second countable semigroup and β_n be a sequence in $P(S)$. Then there exist a sequence of constants a_n in S such that for all nonnegative integers k, the sequence $\beta_{k,n} * \delta_{a_n}$ converges vaguely to a probability measure as n tends to infinity.*

Proof. Since S is a compact semigroup, $P(S)$ is a compact semigroup. By Theorem 2.1 in [1], there exists a subsequence n_k of positive integers such that for each nonnegative integer j, we have:

$$\beta_{j,n_k}\to\pi_j \quad \text{as } k \text{ tends to infinity},$$
$$\pi_{n_k}\to\pi_\infty=\pi_\infty^2, \qquad \pi_j * \pi_\infty=\pi_\infty.$$

Let G_n's be open sets such that $S_{\pi_\infty}=\bigcap_{n=1}^{\infty} G_n$. We choose a subsequence (p_k) of (n_k) such that

$$\beta_{p_k,p_{k+1}}\to\pi_\infty \quad \text{as } k \text{ tends to infinity}$$

and

$$\beta_{p_k,p_{k-i}}(G_k)>1-\frac{1}{k} \quad \text{for each } k \text{ and for each } i\geq 1.$$

Let m be any positive integer such that $p_k < m \leqq p_{k+1}$. Then

$$\begin{aligned}\beta_{p_k, p_{k+2}}(G_k) &= \beta_{p_k, m} * \beta_{m, p_{k+2}}(G_k) \\ &= \int \beta_{p_k, m}(G_k y^{-1}) \, \beta_{m, p_{k+2}}(dy) \\ &\leqq \beta_{p_k, m}(G_k z_m^{-1}) \text{ (for some } z_m \text{ in } S).\end{aligned}$$

Now let z be any element in the support of π_∞. Then we claim that the sequence $\beta_{k,n} * \delta_{z_n z}$ converges vaguely to the probability measure $\pi_k * \delta_z$ as $n \to \infty$. To prove this claim, we consider a cluster point Q_k of the sequence $\beta_{k,n} * \delta_{z_n z}$. Then there exists a subsequence m_j of positive integers such that

$$\beta_{k, m_j} * \delta_{z_{m_j} z} \to Q_k \quad \text{as} \quad j \to \infty.$$

We now replace the sequence m_j by a suitable subsequence (and still calling this subsequence m_j's such that we can choose a subsequence (p_{k_j}) of the sequence (p_k) such that $p_{k_j} < m_j \leqq p_{k_{j+1}}$. The sequence $\beta_{p_{k_j}, m_j} * \delta_{z_{m_j}}$ has cluster points of the form $\lambda * \delta_{z_0}$, where z_0 is a cluster point of the z_{m_j}'s and λ is a cluster point of the sequence $\beta_{p_{k_j}, m_j}$. Because of the way the sequence z_m's have been chosen, it follows that the support of $\lambda * \delta_{z_0}$ is contained in the support of π_∞; otherwise there would be an open set V with $V \cap G$ empty (G an open set containing S_{π_∞}) and V intersecting the support of $\lambda * \delta_{z_0}$, which means that $\lambda * \delta_{z_0}(V) > 0$ and for infinitely many j, $\beta_{p_{k_j}, m_j} * \delta_{z_{m_j}}(V)$ is greater than a fixed positive number which clearly contradicts the way the z_m's have been chosen. Now by [11], we have for any Borel set B and for any y in the support of π_∞ that

$$\pi_\infty(B z^{-1} y^{-1}) = \pi_\infty(B z^{-1}).$$

Then

$$\begin{aligned}\pi_\infty * \lambda * \delta_{z_0 z}(B) &= \pi_\infty * \lambda * \delta_{z_0}(B z^{-1}) \\ &= \int \pi_\infty(B z^{-1} y^{-1}) \, \lambda * \delta_{z_0}(dy) \\ &= \int \pi_\infty(B z^{-1}) \, \lambda * \delta_{z_0}(dy) = \pi_\infty * \delta_z(B).\end{aligned}$$

Since $\beta_{k, m_j} * \delta_{z_{m_j} z} = \beta_{k, p_{k_j}} * (\beta_{p_{k_j}, m_j} * \delta_{z_{m_j} z})$, it is clear that

$$\begin{aligned}Q_k &= \pi_k * (\lambda * \delta_{z_0 z}) = \pi_k * \pi_\infty * (\lambda * \delta_{z_0 z}) \\ &= \pi_k * (\pi_\infty * \delta_z) = \pi_k * \delta_z.\end{aligned}$$

This proves the claim and the theorem follows.

3. In this section, *S will always denote a locally compact non-compact topological group.* It is known [10] that if $\mu \in P(S)$ and S is the smallest closed group containing S_μ, then $\mu^n \to 0$ vaguely as $n \to \infty$. In this section we show that actually a stronger form of this result is valid in many topological groups.

Proposition 2. *Let $\mu \in P(S)$. Suppose there exist $a_n \in S$ such that the sequence $\mu^n * \delta_{a_n}$ converges vaguely as $n \to \infty$ to a probability measure $Q \in P(S)$. Then for some $z \in S$, $\mu * Q = Q * \delta_z$.*

Proof. If $\mu^n * \delta_{a_n} \to Q$ as $n \to \infty$, then $\mu^{n+1} * \delta_{a_n} \to \mu * Q$ as $n \to \infty$. But $\mu^{n+1} * \delta_{a_n} = \mu^{n+1} * \delta_{a_{n+1}} * \delta^{-1}_{a_{n+1} a_n}$. This means that if $a_{n+1}^{-1} a_n \to \infty$, then $\mu^{n+1} * \delta_{a_n} \to 0$ as $n \to \infty$ which is impossible. Therefore, $a_{a+1}^{-1} a_n \nrightarrow \infty$ and consequently, this sequence has some limit point $z \in S$. It follows easily that $\mu * Q = Q * \delta_z$.

Theorem 3. *Suppose $\mu \in P(S)$ and $S = \overline{\bigcup_{n=1}^{\infty} S_\mu^n}$. Suppose there exists an open set V with compact closure such that for every $x \in S$, $x^{-1} V x = V$. Then for every compact K, $\sup\{\mu^n(K x) \colon x \in S\} \to 0$ as $n \to \infty$.*

Proof. Suppose the conclusion of the theorem is false. Then by Csiszár's theorem (see Section 1), $\mu^n * \delta_{a_n} \to Q \in P(S)$ as $n \to \infty$. With no loss of generality, we can assume that $S_Q \cap V \neq \emptyset$. Now by Proposition 2, $\mu * Q = Q * \delta_z$ for some $z \in S$. Then $\mu^n * Q = Q * \delta_{z^n}$.

The function $x \to Q(x^{-1} K)$, $K = \bar{V}$, is upper semi-continuous and therefore, attains its maximum at some $x = x_0$ in S. Then

$$\begin{aligned} Q(x_0^{-1} K) &= \mu^n * Q(x_0^{-1} K z^n) \\ &= \int \mu^n(x_0^{-1} K z^n y^{-1}) \, Q(dy) \\ &= \int \mu^n(x_0^{-1} z^n y^{-1} K) \, Q(dy) = \int_{z^{-n} x_0} (\mu^n)(y^{-1} K) \, Q(dy) \end{aligned}$$

where for any measure λ, $\int_x \lambda(B) = \lambda(x^{-1} B)$.

Let us write: $\nu = \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)_{z^{-n} x_0} (\mu^n)$. Then ν is a probability measure and we have:

$$\begin{aligned} Q(x_0^{-1} K) &= \sum_{n=1}^{\infty} \frac{1}{2^n} Q(x_0^{-1} K) = \int \nu(y^{-1} K) \, Q(dy) \\ &= \int \nu(K y^{-1}) \, Q(dy) = \nu * Q(K) = \int Q(y^{-1} K) \, \nu(dy). \end{aligned}$$

This means that

$$\int [Q(x_0^{-1} K) - Q(y^{-1} K)] \, \nu(dy) = 0.$$

Since $Q(x_0^{-1} K) \geq Q(y^{-1} K)$ for all $y \in S$, it follows that $Q(x_0^{-1} K) = Q(y^{-1} K)$ for almost all $y(\nu)$ in S_ν. By the upper semicontinuity of the function $x \to Q(x^{-1} K)$, $Q(x_0^{-1} K) = Q(y^{-1} K)$ for *all* $y \in S_\nu$. This means that S_ν is compact. To see this, suppose there exist infinitely many $y_n \in S$ such that the sequence y_n doesn't have a limit point. Then for each n, $Q(y_n^{-1} K) = Q(x_0^{-1} K) > 0$; also since $y_n^{-1} K \cap y_1^{-1} K \neq \emptyset$ for all n implies that $y_n \in K(y_1^{-1} K)^{-1}$ (which is a compact set) for all n, there exists $n_1 > 1$ such that $y_n^{-1} K \cap y_1^{-1} K = \emptyset$. Again, since $y_n^{-1} K \cap (y_{n_1}^{-1} K \cup y_1^{-1} K) \neq \emptyset$ for all $n > n_1$ implies that the y_n's (for all $n > n_1$) lie in a compact set, there exists $n_2 > n_1$ such that $y_{n_2}^{-1} K \cap (y_{n_1}^{-1} K \cup y_{n_1}^{-1} K) = \emptyset$. In this way, we can show the existence of infinitely many pairwise disjoint sets $y_{n_i}^{-1} K$, each having the same positive Q-measure. This contradicts that Q is a probability measure, proving that S_ν is compact.

Now we claim that $H=\overline{\bigcup_{k=1}^{\infty} S_\mu^{-k} S_\mu^k}$ is a compact normal subgroup of S. To see this, we notice that $S_\nu=\overline{\bigcup_{k=1}^{\infty} z^{-k} x_0 S_\mu^k}$, and $S_\nu^{-1} S_\nu \supset H$ so that H is compact. Since by hypothesis, $S=\overline{\bigcup_{k=1}^{\infty} S_\mu^k}$, it is clear that for any $x \in S$, $x^{-1} H x \subset H$, this inclusion being easily valid for all $x \in S_\mu^n$ (n any positive integer). This means that for all $x \in S$, $x H = H x$. It follows that H is a subsemigroup since for any two positive integers m and n,

$$\begin{aligned}(S_\mu^{-m} S_\mu^m)(S_\mu^{-n} S_\mu^n) \subset H(S_\mu^{-n} S_\mu^n) &= (H S_\mu^{-n}) S_\mu^n \\ &= (S_\mu^{-n} H) S_\mu^n \subset H.\end{aligned}$$

Since $H=H^{-1}$, H is a compact normal subgroup of S as claimed.

Since $S_\mu^{-1} S_\mu \subset H$, it follows immediately that $S_\mu \subset H \cdot x$ for some x. This $x \notin H$, since otherwise $S_\mu \subset H = a$ compact subgroup, contradicting that the smallest closed subgroup containing S_μ is non-compact. Since $S_\mu^n \subset H x^n$, by the normality of H, $S=\overline{\bigcup_{n=1}^{\infty} S_\mu^n}=\overline{\bigcup_{n=1}^{\infty} H x^n}$. Now we notice that the set $\bigcup_{n=1}^{\infty} H x^n$ is closed. The reason is: let y be not contained in, but a limit point of $\bigcup_{n=1}^{\infty} S_\mu^n$, then there are elements $u_{n_i} \in S_\mu^{n_i}$ such that $u_{n_i} \to y$ as $i \to \infty$. Notice that $S_\nu=\overline{\bigcup_{k=1}^{\infty} z^{-k} x_0 S_\mu^k}$ is compact. Hence there is a subsequence n_{i_j} such that $z^{-n_{i_j}} \cdot x_0 u_{n_{i_j}} \to w \in S$. But this means that $z^{n_{i_j}} \to x_0 \cdot y w$ as $j \to \infty$. Since by Proposition 2, $\mu^{n_{i_j}} * Q = Q * \delta_{z^{n_{i_j}}}$, this means that $\mu^{n_{i_j}} * Q \to Q * \delta_{x_0 y w}$ as $j \to \infty$. This is a contradiction, since $\mu^n \to 0$ vaguely as $n \to \infty$.

It is now clear that $S=\bigcup_{n=1}^{\infty} H x^n$. Let $a \in S_\mu \subset H x$. Then $a^{-1} \in H x^n$ for some positive integer n and therefore, $H x^n = H x^{-1}$ or $x^{n+1} \in H$. This means that $S=\bigcup_{k=1}^{n} H x^k$, which is a contradiction since S is non-compact. The theorem now follows.

The following corollary now follows immediately.

Corollary 4. *Let S be discrete, $\mu \in P(S)$ and $S=\bigcup_{k=1}^{\infty} S_\mu^k$. Then for every compact set K, $\sup\{\mu^n(K x)\colon x \in S\} \to 0$ as $n \to \infty$.*

Remark 5. It is *not* clear how we can prove Theorem 3, if, instead of assuming $S=\overline{\bigcup_{k=1}^{\infty} S_\mu^k}$ (i.e. the group S is generated by S_μ as a semigroup), we assumed $S=\overline{\bigcup_{k=1}^{\infty} (S_\mu \cup S_\mu^{-1})^k}$ (i.e. the group S is generated by S_μ as a group). The difficulty is in showing (in the proof of Theorem 3) that $H=\overline{\bigcup_{k=1}^{\infty} S_\mu^k S_\mu^{-k}}$ is a semigroup. However, we can prove the following theorem when S is abelian or μ is symmetric.

Theorem 6. *Suppose $\mu\in P(S)$ and $S=\overline{\bigcup_{k=1}^{\infty}(S_\mu\cup S_\mu^{-1})^k}$. Suppose also that either S is abelian or μ is symmetric*[1] (i.e. $\mu(B)=\mu(B^{-1})$). *Then there exist elements $a_n\in S$ such that the sequence $\mu^n*\delta_{a_n}$ converges vaguely as $n\to\infty$ to some $Q\in P(S)$ if and only if there exists a compact normal subgroup H such that $S_\mu\subset H\cdot x$ for some $x\notin H$. Also, the limiting measure Q (when it exists) is the translate of some Haar measure on a compact group.*

Proof. The proof of the 'if' part follows from that of Theorem 3. We will prove only the "only if" part.

First, suppose that S is abelian. If $a_n\in S$ such that the sequence $\mu^n*\delta_{a_n}$ converges to some $Q\in P(S)$ as $n\to\infty$, then the sequence

$$\nu_n=(\mu^n*\delta_{a_n})*(\delta_{a_n^{-1}}*\bar\mu^n)=\mu^n*\bar\mu^n \text{ (here, } \bar\mu(B)=\mu(B^{-1}))$$

converges, as $n\to\infty$, to $Q*\bar Q=\lambda$, say, where for any measure ν, $\bar\nu$ denotes the measure $\bar\nu(B)=\nu(B^{-1})$. This means that $\nu_n=(\mu*\bar\mu)^n$ and therefore, since $\lambda=\lim_{n\to\infty}(\mu*\bar\mu)^n$, $\lambda=\lambda^2$ is the normal Haar measure on its support $H(=S_\lambda)$, a compact subgroup. Also, $\mu*\bar\mu*\lambda=\lambda$ and therefore, $S_\mu\cdot S_\mu^{-1}\subset H$ and $H=\overline{\bigcup_{n=1}^{\infty}(S_\mu\cdot S_\mu^{-1})^n}$. It is clear that if $x\in S_\mu\cup S_\mu^{-1}$, $x^{-1}Hx\subset H$. It follows that for all $x\in S$, $x^{-1}Hx\subset H$ and consequently, H is a compact normal subgroup and $S_\mu\subset H\cdot x$ for some $x\notin H$.

In case, S is not abelian and $\mu=\bar\mu$, the sequence ν_n equals μ^{2n} and $\lambda=\lim_{n\to\infty}\mu^{2n}$. Again, $S_\lambda=H$ is a compact subgroup since $\lambda=\lambda^2$. Since $\mu=\bar\mu$, $S_\mu=S_\mu^{-1}$. It is clear that $H=\bigcup_{n=1}^{\infty}S_\mu^{2n}$ and as above, H is normal and $S_\mu\subset H\cdot x$, $x\notin H$. Note that in this case, S has to be compact, a contradiction.

Finally, about the structure of the limiting measure Q (when it exists), let S be abelian. With no loss of generality, we assume that $e\in S_Q$. By Proposition 2, there exists z in S such that $\mu*Q=Q*\delta_z$ and therefore, by the abelian property of S, $((\delta_{z^{-1}})*\mu)*Q=Q$. It follows from [12] that $Q(Bx^{-1})=Q(B)$ for all Borel sets B and all x in H_1, the group generated by the support $z^{-1}\cdot S_\mu$ of the measure $(\delta_{z^{-1}})*\mu$. Since S_Q contains e, it is clear that $S_Q\subset H$. Since the group generated by $z^{-1}\cdot S_\mu$ contains H, it follows that Q is the Haar measure on S_Q.

Before we present our next results in the context of the question when $\sup\{\mu^n(Kx): x\in S\}\to 0$ as $n\to\infty$ for every compact set K in a non-compact group S, we consider the following example due to Harry Kesten (private communication to the second-named author). This example shows, among other things, the importance of Theorem 3 in the non-connected case and the interesting nature of Theorem 6 and also, Theorems 8 and 9 to follow.

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[1] It follows easily that for a symmetric μ there don't exist elements a_n such that the sequence $\mu^n*\delta_{a_n}$ converges vaguely to a probability measure

Example 7. Let S be the group of matrices of the form

$$\begin{pmatrix} d^k & r \\ 0 & 1 \end{pmatrix}$$

where $0<d<1$ and d is a fixed real number, and k and r range over the integers and real numbers respectively. We assume that the support of μ, a probability measure on S, is the set

$$\left\{ \begin{pmatrix} d & r \\ 0 & 1 \end{pmatrix} : r \text{ is a real number} \right\}$$

and such that $E(|r|)<\infty$. Then a product

$$\begin{pmatrix} d & r_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & r_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} d & r_n \\ 0 & 1 \end{pmatrix}$$

of n independent identically distributed (according to μ) matrices in S is given by

$$\begin{pmatrix} d^n & s_n \\ 0 & 1 \end{pmatrix}, \qquad s_n = r_1 + d r_2 + d^2 r_3 + \ldots + d^{n-1} r_n.$$

It is clear that s_n has a limit distribution since $0<d<1$ and $E(|r|)<\infty$, and therefore, the sequence

$$\begin{pmatrix} d^n & s_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-n} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s_n \\ 0 & 1 \end{pmatrix}$$

has also a limit distribution. Hence, in this case the sequence $\mu^n * \delta_{a_n}$, where $a = \begin{pmatrix} d^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, converges vaguely to a probability measure as $n \to \infty$.

Theorem 8. *Let S be nilpotent, $\mu \in P(S)$ and* $S = \overline{\bigcup_{n=1}^{\infty} (S_\mu \cup S_\mu^{-1})^n}$. *Suppose $e \in S_\mu$. Then for any compact set K,*

$$\sup\{\mu^n(Kx):\ x \in S\} \to 0$$

as $n \to \infty$.

Proof. Since S is nilpotent, by definition there is a finite sequence of closed normal subgroups $(Z_i)_{i=1}^n$ such that

$$\{e\} = Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$$

and the quotient group Z_{i+1}/Z_i is the center of the quotient group S/Z_i for $i=0, 1, \ldots, n-1$. We make an induction argument on n. If $n=1$, S is non-compact abelian and the theorem follows by Csiszár's theorem and Theorem 6. Suppose the theorem is true for all non-compact nilpotent groups S whose central ascending series (as above) has length less than n. It is clear that the quotient group S/Z (where Z = the center of S) has length $n-1$ for its central ascending series, if the corresponding series for S has length n. If S/Z is compact, then by [2], the quotient S/S' (S' = the topological commutator subgroup) is non-compact and abelian with S' compact; therefore, it follows easily by Csiszár's theorem that the conclusion of the theorem follows for S. Now the only case left to be considered is when the quotient S/Z is non-compact. Then by induction-hypothesis, the conclusion of the theorem holds for the group S/Z, which is non-compact, nilpotent and has length $n-1$ for its central ascending series. Let us define the measure λ on the Borel subsets of S/Z by $\lambda(B)=\mu(\Phi^{-1}(B))$, where Φ is the natural map from S onto S/Z. Then for any compact set $K\subset S$, $\Phi(K)$ is compact and

$$\sup_{x\in S} \lambda^k(\Phi(K)\,\Phi(x)^{-1})\to 0$$

as $k\to\infty$, since S_λ contains the identity of S/Z. Since for each k,

$$\lambda^k(B)=\mu^k(\Phi^{-1}(B)),$$

it follows that the theorem holds for S. The induction argument is complete. The theorem now follows.

Our last theorem in this section gives a complete picture of when there exist elements a_n such that $\mu^n * \delta_{a_n}$ converges weakly in the case of non-compact abelian groups.

Theorem 9. *Let S be a locally compact non-compact abelian group. Let $\mu\in P(S)$ and $S=\overline{\bigcup_{n=1}^{\infty}(S_\mu\cup S_\mu^{-1})^n}$. Then there exist elements $a_n\in S$ such that the sequence $\mu^n*\delta_{a_n}$ converges vaguely as $n\to\infty$ to a probability measure if and only if the following conditions hold:*

(i) *S is topologically isomorphic to $Z\times H_0$, where Z is the discrete group of integers and H_0, a compact abelian group;*

(ii) *$S_\mu=\{1\}\times A$, where A is some compact subset of H such that $\overline{\bigcup_{n=1}^{\infty}(A\cup A^{-1})^n}=H$.*

Proof. Suppose there exist $a_n\in S$ such that $\mu^n*\delta_{a_n}\to Q\in P(S)$ as $n\to\infty$. Then by Theorem 6, $S_\mu\subset Hx$, $x\notin H$ and H is a compact subgroup. Hence S_μ is compact and consequently, S is compactly generated. By [3, p. 90] S is topologically isomorphic to the direct product $R^n\times Z^m\times H$, where R is the additive group of reals, Z is the additive group of integers and H_0 is a compact abelian group, and n, m are non-negative integers. If m and n are both positive, then since $S_\mu\subset H\,x$ and H is a compact subgroup of S, it is clear after identifying S with $R^n\times Z^m\times H_0$ that

$$S_\mu\subset\{x_1\}\times\{x_2\}\times H_0$$

where $x=(x_1,x_2,x_3)\in S$. But if $n>0$, then S_μ cannot generate the group S and consequently, $n=0$. If $m>1$, then also S_μ cannot generate the group S since a single element cannot generate Z^m, $m>1$. Hence, $m=1$ and S is topological isomorphic with $Z\times H_0$. Since for S_μ to generate $Z\times H_0$, x_2 above must be the integer 1, it is clear that $S_\mu=\{1\}\times A$, $A\subset H_0$.

The converse is clear by Theorem 6.

We end this section with the following conjecture.

Conjecture. Let S be a locally compact non-compact connected group and $\mu \in P(S)$ such that $S = \bigcup_{n=1}^{\infty} (S_\mu \cup S_\mu^{-1})^n$. Then for every compact set K,

$$\sup\{\mu^n(K x):\ x \in S\} \to 0$$

as $n \to \infty$ (or equivalently, there don't exist elements a_n such that the sequence $\mu^n * \delta_{a_n}$ is weakly convergent).

4. In this section, we will consider the weak convergence of the sequence $\beta_{k,n} = \beta_{k+1} * \beta_{k+2} * \dots * \beta_n$, where $\beta_i \in P(S)$ and S is an at most countable group with discrete topology. In the case of compact groups where the β_i's are all the same, we know from the Kawada-Ito theorem [5] that the weak convergence of $\beta_{k,n}$ depends upon the support of the measure. This is not so when the measures are different as can easily be seen. For instance, let $S = \{e, x\}$, the cyclic group of order two. Then if $\beta_{2n+1} = \delta_e$ and $\beta_{2n} = \delta_x$, the sequence $\beta_{k,n}$ does not converge. Also, by defining here $\beta_n = x_n \delta_e + (1 - x_n) \delta_x$, we see that for proper choice of the x_n's, we can have $\beta_{0,2n+1}(e) < 3/8$ and $\beta_{0,2n}(e) > 5/8$. Our goal in this section is to obtain necessary and sufficient conditions for the weak convergence of the sequence $\beta_{k,n}$. While this problem is solved in the case of discrete groups here, it remains unsolved in the non-discrete situation; however, there are two very useful (and significant) results available in this context in the case of *general locally compact second countable groups*, the first one being due to I. Csiszár [1] and A. Tortrat [12] independently and the second one due to I. Csiszár [1]. To state the results, let $X_1, X_2, \dots$ be independent random variables with values in S with distributions $\beta_1, \beta_2, \dots$ respectively. Since the natural mapping θ: $S \to S/G$, where S/G is the set of all left cosets of G with the quotient topology, is continuous and therefore, $\theta(X_n)$ is a random variable with values in S/G. By saying that the sequence X_n converges mod G, we will mean the convergence of the corresponding sequence $\theta(X_n)$ in S/G. The first of the two results mentioned above is stated in Section 1. The second one can be stated as follows:

Theorem 10. *Assume that the product $X_k X_{k+1} \dots X_n$ (for all $k \geqq 1$) have limiting distributions as $n \to \infty$. Then there exists a unique compact subgroup $G \subset S$ such that all the above limiting distributions are G-uniform and the product $X_1 X_2 \dots X_n$ converges* mod G *with probability one.*

Though the problem of weak convergence is completely unsolved in the non-discrete situation, it is rather easy to give a necessary and sufficient condition for the vague convergence to zero of the sequence $\beta_{0,n}$ in the special case when the sequence β_n is a periodic sequence (i.e. for all k, $\beta_k = \beta_{k+p}$ for some fixed positive integer p). Before we state this, we may observe the following: The vague cluster points of the sequence $\beta_{k,n}$ can only be zero or probability measures; this follows because in case of a non-zero cluster point of the sequence $\beta_{k,n}$, by Csiszár's theorem there exist constants a_n such that $\beta_{k,n} * a_n$ converge vaguely to some probability measure β'_k and it is then clear that for any non-zero cluster point Q of $\beta_{k,n}$, $Q * \delta_b = \beta'_k$ for some cluster point b of the sequence a_n, implying that Q is a probability measure.

Theorem 11. *Let (β_n) be a periodic sequence (with period p) of probability measures on S, a locally compact group. Then the sequence $\beta_{0,n}$ converges to zero vaguely as n tends to infinity if and only if for all s with $1 \leqq s \leqq p$, the set $S_{\beta_{s-1,s+p-1}}$ is not contained in a compact group.*

We omit the proof.

In what follows, S is always an at most countable group with discrete topology. We provide necessary and sufficient conditions for the weak convergence of $\beta_{k,n}$ for all positive integers k. Maksimov [7, 8] and Heyer [4] studied this problem for S a finite group. Maksimov's methods of proofs are based on a generalized concept of variance that he introduces for group-valued random variables. His proofs do not seem to carry over to the case of infinite groups. Our proofs are different and hold for groups which are not necessarily finite. Also Maksimov seems to have studied the convergence problem in [7] in the case where the limit measures π_k are not of the form $\pi_k * w_G$, where $G \neq \{e\}$. On the other hand, Heyer introduces a concept of "filling" and then proves a number of results on the convergence problem based on this concept. He also considers only finite groups. An extension of Maksimov's main result is presented here as a corollary to our characterization theorem (Theorem 16).

Lemma 12. *If for each positive integer k, the sequence $\beta_{k,n}$ converges vaguely to a probability measure $\pi_k \in P(S)$, then there is a finite subgroup G such that π_k converges vaguely to w_G as k tends to ∞ and $\pi_k = \pi_k * w_G$. Furthermore there exists a positive integer N such that for any $k \geqq N$, $\pi_k = \pi_k * w_H$ (H a finite subgroup of S) implies that $H \subset G$. Here w represents the uniform distribution.*

Proof. This is precisely the assertion 3.31 on page 291 in [3]. The last assertion follows easily by noting that $\pi_k(G) > 1/2$ for k sufficiently large, and that $\pi_k = \pi_k * w_H$ implies that for $x \in H$, $\pi_k(G) = \pi_k(G x^{-1})$ so that $G \cap G x^{-1} \neq \emptyset$.

It is easy to see that the sequence $\beta_{0,n}$ converges vaguely to w_H, H a finite subgroup if $\beta_1 = w_H$ and for $k > 1$, $S_{\beta_k} \subset H$ although the sequence $\beta_{k,n}$ (for $k \geqq 1$) may not converge. Our next result describes when this kind of situation is not possible.

Proposition 13. *Suppose for some nonnegative integer k, the sequence $\beta_{k,n}$ converges vaguely to $\pi_k \in P(S)$ and π_k is not of the form $\pi_k * w_G$ where G is a finite subgroup distinct from identity. Then for every nonnegative integer k, the sequence $\beta_{k,n}$ converges vaguely to a probability measure.*

Proof. If $\beta_{k,n}$ converges to π_k for some k, then $\beta_{p,n}$ converges for all $p \leqq k$. We now claim the following:

$$\lim_{n \to \infty} \inf_{m > n} \beta_{n,m}(e) = 1. \tag{1}$$

If (1) is not true, then we can find subsequences n_i, m_i with $n_i < m_i$ of positive integers such that β_{n_i,m_i} vaguely converges to some β' as $i \to \infty$ and $\beta'(e) < 1$. But since

$$\beta_{k,n_i} * \beta_{n_i,m_i} = \beta_{k,m_i}$$

we have:

$$\pi_k * \beta' = \pi_k$$

so that $\beta' \in P(S)$. It is clear that $S_{\beta'}$ is contained in a finite subgroup G of S, since otherwise β'^n converges vaguely to the zero measure and this means that $\pi_k = \pi_k * \beta'^n \to 0$ as $n \to \infty$, which is, of course, a contradiction. Since $(1/n) \sum_{i=1}^{n} \beta'^i \to w_G$ as $n \to \infty$, it is clear that $\pi_k = \pi_k * w_G$. By the hypothesis of our proposition, $G = \{e\}$ and thus, $\beta' =$ the unit mass at e. This proves (1). Now let $p > k$. Let π_{1p} and π_{2p} be any two vague cluster points of $\beta_{p,n}$. Then we choose subsequences n_i and n_i' of positive integers such that $n_i < n_i'$ and

$$\beta_{p,n_i} \to \pi_{1p}, \qquad \beta_{p,n_i'} \to \pi_{2p}.$$

Since we have (1) and the following equality

$$\beta_{p,n_i'} = \beta_{p,n_i} * \beta_{n_i,n_i'},$$

it follows that $\pi_{2p} = \pi_{1p} * \delta_e = \pi_{1p}$. The proposition now follows.

Now we will present two lemmas which will be needed to establish our main theorem (Theorem 16).

Lemma 14. *Let G be a finite subgroup of S, $\beta_n \in P(S)$, and $X_1, X_2, \dots$ be independent random variables with values in S and distributions $\beta_1, \beta_2, \dots$. Then the product $X_1 X_2 \dots X_n$ converges* mod G *if and only if $\sum_{n=1}^{\infty} \beta_n(S-G) < \infty$. Also, in this case* $\lim_{n\to\infty} \inf_{m>n} \beta_{n,m}(G) = 1$.

Proof. The first assertion of the lemma follows easily by the Borel-Cantelli lemma. For the second assertion, note that if $\sum_{n=1}^{\infty} \beta_n(S-G) < \infty$, then given any positive number t, there exists a positive integer N such that $n > N$ implies that

$$P\left(\bigcap_{k=N}^{n} [X_k \in G]\right) > 1 - t,$$

which means that

$$P(X_N X_{N+1} \dots X_n \in G) > 1 - t.$$

It follows now that $\lim_{n\to\infty} \inf_{m>n} \beta_{n,m}(G) = 1$.

Lemma 15. *Let G be a finite subgroup of S, $\beta_n \in P(S)$, and $X_1, X_2, \dots$ be a sequence of independent random elements on S with distributions $\beta_1, \beta_2, \dots$. Suppose that the product $X_1 X_2 \dots X_n$ converges* mod G *with probability one, but there do not exist constants a_n such that $X_1 X_2 \dots X_n a_n$ converges* mod G' *with probability one for any proper subgroup G' of G. Then for every positive integer k, the sequence $\beta_{k,n}$ vaguely converges to some $\pi_k \in P(S)$.*

Proof. Suppose $X_1 X_2 \dots X_n$ converges mod G with probability one. Then by Csiszár's theorem, there exist constants a_n such that $X_1 X_2 \dots X_n a_n$ converges mod G' for some finite subgroup G', the sequence $\beta_{k,n} * \delta_{a_n}$ vaguely converges to some $\pi_k \in P(S)$, $\pi_k \to w_{G'}$ and $\pi_k = \pi_k * w_{G'}$. It is clear from Lemma 14 that every vague cluster point of the sequence $\beta_{k,n}$ is a probability measure (because of the validity of the "lim inf" condition there). This means that every subsequence of the sequence a_n has a cluster point; for, otherwise for some subsequence n_i, $\delta_{a_{n_i}} \to 0$ vaguely and by the joint continuity of convolution as a mapping from $P(S) \times B(S)$ into $B(S)$, it will follow that $\pi_k = 0$, a contradiction. Now we choose a subsequence n_s of positive integers such that $\beta_{n_s, n_{s+1}} * \delta_{a_{n_{s+1}}} \to w_{G'}$ vaguely. If b is a cluster point of the subsequence $(a_{n_{s+1}}^{-1})$, then $w_{G'} * \delta_b$ is a vague cluster point of the double sequence $\beta_{n,m}$. This fact alongwith the lim inf condition in Lemma 14 (which is also valid here) implies that $G' b \subset G$ or $G' \subset G b^{-1}$. This means that $G' \subset G$ and $b \in G$. By our assumption in the statement of the lemma, $G' = G$. It follows that every cluster point of the sequence a_n lies in G. It is clear then that for each positive integer k, $\beta_{k,n} \to \pi_k$ vaguely.

Theorem 16. *For all nonnegative integers k the sequence $\beta_{k,n}$ converges vaguely to some $\pi_k \in P(S)$ as $n \to \infty$ if and only if there exists a finite subgroup G such that $\sum_{n=1}^{\infty} \beta_n(S-G) < \infty$ and for any proper subgroup G' of G and any selection of elements* g_n in S, $n = 0, 1, 2, \ldots$, *the series* $\sum_{n=1}^{\infty} \beta_n(S - g_{n-1} G' g_n^{-1}) = \infty$.

Proof. Note that if $X_1, X_2, \ldots$ are random variables with distributions $\beta_1, \beta_2, \ldots$, then the sequence $g_{n-1} X_n g_n^{-1}$ has distributions given by the sequence $\beta_n(g_{n-1}^{-1} \cdot g_n)$ and by Lemma 14, it follows that the sequence $X_1 X_2 \ldots X_n g_n^{-1} = g_0^{-1}(g_0 X_1 g_1^{-1}) \cdot (g_1 X_2 g_2^{-1}) \ldots (g_{n-1} X_n g_n^{-1})$ converges mod G' with probability one if and only if $\sum_{n=1}^{\infty} \beta_n(g_{n-1}^{-1}(S-G') g_n) < \infty$. The 'if' part of the theorem now follows from Lemma 15.

For the 'only if' part, the proof goes as follows. Suppose that for all $k \geqq 0$, the sequence $\beta_{k,n}$ converges to $\pi_k \in P(S)$. Then by Theorem 10 and Lemma 14, there is a finite subgroup G such that $\sum_{n=1}^{\infty} \beta_n(S-G) < \infty$. Suppose there is a proper subgroup G' of G and constants g_n such that $\sum_{n=1}^{\infty} \beta_n(S - g_{n-1} G' g_n^{-1}) < \infty$. Then since G has only finitely many proper subgroups, we can assume with no loss of generality that G' does not have any proper subgroup with the above property. Now we define the measures β'_n by

$$\beta'_n = \delta_{g_{n-1}^{-1}} * \beta_n * \delta_{g_n}.$$

Then we have

$$\sum_{n=1}^{\infty} \beta'_n(S-G') < \infty,$$

and for all proper subgroups G'' of G' and elements $a_n \in S$, $n = 0, 1, 2, \ldots$, the series $\sum_{n=1}^{\infty} \beta'_n(S - a_{n-1} G' a_n^{-1}) = \infty$. By the 'if' part of the theorem, it follows that for all

$k=0,1,2,\ldots$, the sequence

$$\beta'_{k,n}=\beta'_{k+1}*\ldots*\beta'_n\to\pi'_k \text{ vaguely as } n\to\infty,$$

$\pi'_k\to w_{G'}$, $\pi'_k=\pi'_k*w_{G'}$ and for k sufficiently large,

$$\pi'_k=\pi'_k*w_H \text{ implies that } H\subset G'.$$

Since $\beta'_{k,n}=\delta_{g_{k-1}^{-1}}*\beta_{k,n}*\delta_{g_n}$ and $\beta_{k,n}$ as well as $\beta'_{k,n}$ converges vaguely to probability measures, it is clear that the sequence g_n has a cluster point g in S. This means that

$$\pi'_k=\delta_{g_{k-1}^{-1}}*\pi_k*\delta_g$$

and

$$\begin{aligned}\pi'_k*w_{g^{-1}Gg}&=\delta_{g_{k-1}^{-1}}*\pi_k*w_G*\delta_g\\&=\delta_{g_{k-1}^{-1}}*\pi_k*\delta_g \quad (\text{since } \pi_k=\pi_k*w_G)\\&=\pi'_k.\end{aligned}$$

This means that $g^{-1}\,G\,g\subset G'$, which is a contradiction since G' is a proper subgroup of the finite group G. The proof of the theorem is now complete.

We now present three interesting corollaries of Theorem 16.

Our first corollary is an extension of the main result of Maksimov [8] and Heyer [4].

Corollary 17. *Let S be an at most countable discrete group. Suppose there is a finite subgroup G such that $\sum_{n=1}^{\infty}\beta_n(S-G)<\infty$. Then a sufficient condition for the vague convergence of $\beta_{k,n}$ to some π_k in $P(S)$ for all nonnegative integers k is that for all positive integers k, $\beta_k(e)\geqq s$ for some fixed positive number s.*

Proof. Let p be the smallest integer with the property that there is a finite subgroup G' with cardinality p such that $\sum_{n=1}^{\infty}\beta_n(S-G')<\infty$. Such a p exists clearly because of our assumption in the corollary. Suppose that $\beta_m(e)\geqq s>0$ for all m, and for some positive integer k, the sequence $\beta_{k,n}$ does not converge vaguely to a probability measure. Then by Theorem 16, there exist elements g_n in S and a proper subgroup H of G' such that $\sum_{n=1}^{\infty}\beta_n(S-g_{n-1}\,H\,g_n^{-1})<\infty$. This means that there exists a positive integer N such that for all $n\geqq N$, the identity e belongs to $g_{n-1}\,H\,g_n^{-1}$, or $g_{n-1}\,H=g_n\,H$ and $H\,g_{n-1}^{-1}=H\,g_n^{-1}$. This means that $g_{N-1}\,H\,g_{N-1}^{-1}=g_{N-1}\,H\,g_N^{-1}=g_{n-1}\,H\,g_n^{-1}$ and $\sum_{n=1}^{\infty}\beta_n(S-g_{N-1}\,H\,g_{N-1}^{-1})$ is convergent. Since $g_{N-1}\,H\,g_{N-1}^{-1}$ has fewer number of elements than G', this contradicts the minimality of p. The corollary now follows.

Corollary 18. *Suppose S is an at most countable discrete group and has no nontrivial proper finite subgroup. Let $\beta_n\in P(S)$ and suppose that $\beta_n(e)=1-r_n$. Then the following statements are valid:*

(a) *If* $\sum_{n=1}^{\infty} r_n < \infty$, *then the sequence* $\beta_{k.n}$ *converges vaguely to some* $\pi_k \in P(S)$ *for all nonnegative integers k.*

(b) *Suppose S is infinite and* $\sum_{n=1}^{\infty} r_n = \infty$. *Then the sequence* $\beta_{k,n}$ *does not converge vaguely to a probability measure for any positive integer k.*

(c) *Suppose that S is finite and* $\sum_{n=1}^{\infty} r_n = \infty$. *Then*

(i) *if* $\sum_{n=1}^{\infty} [1 - \sup\{\beta_n(x)\colon\ x \in S\}] < \infty$, *then for some positive integer k, the sequence* $\beta_{k,n}$ *does not converge vaguely to a probability measure;*

(ii) *if* $\sum_{n=1}^{\infty} [1 - \sup\{\beta_n(x)\colon\ x \in S\}] = \infty$, *then for all nonnegative integers k, the sequence* $\beta_{k.n} \to w_S$ *vaguely as* $n \to \infty$.

Proof. The statement (a) follows from Theorem 16. To prove part (b), notice that S is assumed to have no proper finite subgroup other than $\{e\}$, and therefore by Theorem 16, $\sum_{n=1}^{\infty} r_n = \infty$ implies that for some k, the sequence $\beta_{k,n}$ does not converge vaguely to a probability measure. Then by Proposition 13, assertion (b) follows. Let us now prove (c). So we assume that S is finite and $\sum_{n=1}^{\infty} r_n = \infty$. Note that the condition that for any choice of constants g_n in S, the series $\sum_{n=1}^{\infty} \beta_n(S - g_{n-1}\, e\, g_n^{-1})$ is divergent is equivalent to saying that

$$\sum_{n=1}^{\infty} \inf\{\beta_n(S - x)\colon\ x \in S\} = \sum_{n=1}^{\infty} [1 - \sup\{\beta_n(x)\colon\ x \in S\}] = \infty.$$

Since $\{e\}$ is the only proper subgroup and S itself is the only finite subgroup G such that $\sum_{n=1}^{\infty} \beta_n(S - G) < \infty$, it is clear that assertion (c) holds by Theorem 16. Note that in case of convergence, the limit probability measures π_k satisfies $\pi_k = \pi_k * w_S = w_S$.

Our last corollary shows what is needed to obtain a theorem like Theorem 1 in the non-compact situation. We only look at the discrete case. Csiszár's results, of course, somewhat describes this situation even in the non-discrete situation.

Corollary 19. *Let S be an at most countable discrete group. Let* β_n *be in* $P(S)$. *Then there exist constants* a_n *in S such that the sequence* $\beta_{k,n} * \delta_{a_n}$ *converges vaguely to some probability measure as* $n \to \infty$ *for all nonnegative integers k if and only if there exists a finite subgroup G such that* $\sum_{n=1}^{\infty} \beta_n(S - g_{n-1}\, G\, g_n^{-1}) < \infty$ *for some selection of elements* g_n *in S with* $g_0 = e$.

Proof. To prove the 'if' part, let define the sequence

$$\beta_n' = \delta_{g_{n-1}^{-1}} * \beta_n * \delta_{g_n}.$$

Then $\sum_{n=1}^{\infty} \beta'_n(S-G) < \infty$ for some finite subgroup G, which means that

$$\lim_{n\to\infty} \inf_{m>n} \beta'_{n,m}(G) = 1$$

(by Lemma 14). By Csiszár's theorem, there are constants b_n in S such that for all k, $\beta'_{k,n} * \delta_{b_n}$ converges vaguely to some probability measure π'_k. It follows that with $a_n = g_n b_n$ and $\pi_k = \delta_{g_{k-1}} * \pi'_k$, the sequence $\beta_{k,n} * \delta_{a_n}$ converges vaguely to π_k. The proof for the 'only if' part is also similar.

We end this paper with an interesting proposition which can also be used to give straightforward and elementary proofs of some special cases of Theorem 16.

Proposition 20. *Let S be an at most countable discrete group and G a subgroup of S. Let $K \subset S$. Then for nonnegative integers k, p and m with $k+1 \leqq p$ and $m \geqq 1$, we have:*

$$\beta_{k,p}(KG) - \sum_{n=p+1}^{p+m} \beta_n(S-G) \leqq \beta_{k,p+m}(KG)$$
$$\leqq \beta_{k,p}(KG) + \sum_{n=p+1}^{p+m} \beta_n(S-G).$$

Proof. Using definition of convolution, we have:

$$\begin{aligned}\beta_{k,p+1}(KG) &= \sum_{x\in S} \beta_{k,p}(KGx^{-1})\,\beta_{p+1}(x)\\ &\leqq \beta_{k,p}(KG) + \sum_{x\notin G} \beta_{k,p}(KGx^{-1})\,\beta_{p+1}(x)\\ &\leqq \beta_{k,p}(KG) + \beta_{p+1}(S-G).\end{aligned}$$

Repeating this process m times gives us the right-hand inequality of the proposition. For the other inequality,

$$\begin{aligned}\beta_{k,p+1}(KG) &\geqq \sum_{x\in G} \beta_{k,p}(KGx^{-1})\,\beta_{p+1}(x)\\ &= \beta_{k,p}(KG)[1-\beta_{p+1}(S-G)] \geqq \beta_{k,p}(KG) - \beta_{p+1}(S-G).\end{aligned}$$

Another m times produces the left-hand inequality of the proposition.

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