Z. Wahrscheinlichkeitstheorie verw. Gebiete
46, 237 – 246 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Pairwise Sufficiency

Eberhard Siebert

Mathematisches Institut der Universität, Auf der Morgenstelle 10, D-7400 Tübingen, Federal Republic of Germany

Summary. For an arbitrary experiment **E** we investigate the relation between its pairwise sufficient subalgebras and its sufficient sublattices in the M-space of **E** (in the sense of L. LeCam). By exhibiting an experiment without minimal pairwise sufficient subalgebra it is shown that this correspondence is in general not bijective. In view of this we introduce the rather large class of majorized experiments. They have a minimal pairwise sufficient subalgebra which can be described explicitely.

As a natural subclass of the majorized experiments appear the coherent experiments that are distinguished by the coincidence of sufficiency and pairwise sufficiency. It is shown that the coherent experiments are characterized by the fact that they admit a majorizing measure which is localizable. As a consequence we obtain that the class of coherent experiments coincides with classes of experiments previously introduced by T.S. Pitcher, D. Mußmann, M. Hasegawa and M.D. Perlman.

One of the fundamental concepts in mathematical statistics is the notion of sufficiency. Important contributions to its theory are due to P.R. Halmos and L.J. Savage [3], R.R. Bahadur [1] and D.L. Burkholder [2]. It turns out that for dominated experiments the class of sufficient subalgebras has very nice properties whereas for arbitrary experiments most of these properties break down. In particular there does not always exist a minimal sufficient subalgebra.

In order to preserve the nice sufficiency properties of dominated experiments generalizations of domination have been introduced. We mention the compact experiments of T.S. Pitcher [12], the weakly dominated experiments of D. Mußmann [10], and the coherent experiments of M. Hasegawa and M.D. Perlman [4]. A different approach was taken by L. LeCam [7]. In establishing his theory of comparison of experiments he introduced the concept of a sufficient sublattice (in the M-space of the experiment). The class of sufficient sublattices has the same nice properties as the class of sufficient subalgebras of dominated experiments. But LeCam's theory was formulated in terms of Banach lattices, so it remains to explore

0044-3719/79/0046/0237/$02.00

its connection with measure theoretic concepts. It turns out that the pairwise sufficient subalgebras (already introduced by Halmos and Savage in [3]) correspond to the sufficient sublattices [7, 8]. In fact it was claimed in [8] that this correspondence is bijective. As a consequence there would exist a minimal pairwise sufficient subalgebra for any experiment. But this is disproved in the present paper (cf. Section 2). Hence a more careful analysis of the relation between sufficient lattices and pairwise sufficient σ-algebras is necessary. In this paper we present some results in this direction.

The first section describes the connection between sufficient sublattices and pairwise sufficient subalgebras. The proofs (which in most cases are simple) are omitted. In Section 2 an example of an experiment is given that has no minimal pairwise sufficient subalgebra. This disproves results in [6] and [8]. In Section 3 we introduce the concept of a majorized experiment that generalizes the concept of a dominated experiment considerably. For majorized experiments we can prove the existence of a minimal pairwise sufficient subalgebra. Finally Section 4 is concerned with coherent experiments which are intermediate between dominated and majorized experiments. For coherent experiments sufficiency and pairwise sufficiency are essentially the same. It is shown that the class of coherent experiments coincides with the classes of Pitcher, Mußmann, Hasegawa and Perlman mentioned above.

With the exception of Proposition 4.1 this paper contains only results out of the Habilitationsschrift [14] of the author.

Preliminaries

Let X be a non-void set and $\mathfrak{A}$, $\mathfrak{A}'$ σ-algebras of subsets of X. Then $\mathfrak{A} \vee \mathfrak{A}'$ denotes the σ-algebra generated by $\mathfrak{A} \cup \mathfrak{A}'$. If $\mathfrak{A}'$ is contained in $\mathfrak{A}$ we call $\mathfrak{A}'$ a *subalgebra* of $\mathfrak{A}$ (though the correct term would be sub-σ-algebra). $\mathscr{L}(X, \mathfrak{A})$. denotes the space of bounded measurable real valued functions on $(X, \mathfrak{A})$. If A is a subset of X then 1_A is its indicator function. For a family $\mathscr{F}$ of functions on X we denote by $\mathfrak{A}(\mathscr{F})$ the σ-algebra on X generated by $\mathscr{F}$. If μ is a measure and $f \geqq 0$ a measurable function on $(X, \mathfrak{A})$ then $f \cdot \mu$ is the measure with μ-density f. $L^1(X, \mathfrak{A}, \mu)$ and $L^\infty(X, \mathfrak{A}, \mu)$ denote the spaces of (equivalence classes of) measurable functions on $(X, \mathfrak{A})$ integrable respectively essentially bounded with respect to μ. The measure μ is said to have the *finite subset property* if for every $A \in \mathfrak{A}$ such that $\mu(A) = \infty$ there exists a set $B \in \mathfrak{A}$ such that $B \subset A$ and $0 < \mu(B) < \infty$. The measure μ is said to be *localizable* if it has the finite subset property and if $L^\infty(X, \mathfrak{A}, \mu)$ is an order complete lattice (i.e. for any non-void majorized set in $L^\infty(X, \mathfrak{A}, \mu)$ its supremum exists in $L^\infty(X, \mathfrak{A}, \mu)$). Localizable measures are studied in some detail in [16].

Let I be a non-void index set. By $\mathscr{F}(I)$ and $\mathscr{A}(I)$ we denote the systems of finite respectively countable subsets $\neq \emptyset$ of I. By an *experiment* $\mathbf{E}$ with index set I we understand here a family $(P_i)_{i \in I}$ of probability measures on a measurable space $(X, \mathfrak{A})$. We write $\mathbf{E} = (X, \mathfrak{A}, (P_i)_{i \in I})$. For any subset J of I we put $\mathbf{E}_J = (X, \mathfrak{A}, (P_i)_{i \in J})$ and $\mathfrak{N}(J) = \{A \in \mathfrak{A}: P_i(A) = 0 \text{ for all } i \in J \text{ or } P_i(A) = 1 \text{ for all } i \in J\}$. If $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$ let $\tilde{\mathfrak{S}} = \mathfrak{S} \vee \mathfrak{N}(I)$. If μ is a measure on $(X, \mathfrak{A})$ we write $(P_i)_{i \in I} \approx \mu$ if $\mu(A) = 0$ is equivalent with $P_i(A) = 0$ for all $i \in I$ $(A \in \mathfrak{A})$. Let $\mathscr{M}^b(X, \mathfrak{A})$ be the L-space of bounded signed measures on $(X, \mathfrak{A})$ (with the norm of total variation). The band $L(\mathbf{E})$ generated by $\{P_i: i \in I\}$ in $\mathcal{M}^b(X, \mathfrak{A})$ is called the *L-space of the experiment* $\mathbf{E}$. Its (topological) dual space $M(\mathbf{E})$ is called the *M-space of the experiment* $\mathbf{E}$. The canonical bilinear form on $L(\mathbf{E}) \times M(\mathbf{E})$ is denoted by $\langle .,. \rangle$. $M(\mathbf{E})$ has a distinguished element $u_{\mathbf{E}}$ called the *norm functional* which is defined by $\langle \mu, u_{\mathbf{E}} \rangle = \|\mu^+\| - \|\mu^-\|$ for all $\mu \in L(\mathbf{E})$. For the theory of Banach lattices and especially of abstract L-spaces and M-spaces we refer to [13].

1. Sufficient Sublattices and Pairwise Sufficient Subalgebras

Let $\mathbf{E} = (X, \mathfrak{A}, (P_i)_{i \in I})$ be an experiment with L-space $L(\mathbf{E})$ and M-space $M(\mathbf{E})$ (with unit $u_{\mathbf{E}}$). A subspace H of $M(\mathbf{E})$ is called a *weak sublattice* if H is a $\sigma(M(\mathbf{E}), L(\mathbf{E}))$-closed sublattice containing $u_{\mathbf{E}}$.

Let H be a weak sublattice of $M(\mathbf{E})$. According to LeCam [7] H is said to be a *sufficient sublattice* for $\mathbf{E}$ if the (abstract) subexperiment of $\mathbf{E}$ induced by H is as informative as $\mathbf{E}$. The following assertions are equivalent ([7], Prop. 9 and 11):

(S 1) H is a sufficient sublattice for $\mathbf{E}$.
(S 2) There exists a positive linear projection Π of $M(\mathbf{E})$ onto H such that $\langle P_i, \Pi f \rangle = \langle P_i, f \rangle$ for all $f \in M(\mathbf{E})$ and for all $i \in I$.
(S 3) $\|rP_i - sP_j\| = \|rP_i - sP_j\|_H = \sup\{|\langle rP_i - sP_j, h \rangle|: h \in H, |h| \leqq u_{\mathbf{E}}\}$ for all pairs $(i,j) \in I \times I$ and $(r,s) \in \mathbb{R}_+ \times \mathbb{R}_+$.

The projection Π in (S 2) is uniquely determined by H. It is called the *sufficient projection* for H, and it has the property $\Pi(fg) = f(\Pi g)$ for all $f \in H$ and $g \in M(\mathbf{E})$ ([7], Prop. 9).

Let $\mathfrak{H}(\mathbf{E})$ be the class of sufficient sublattices for $\mathbf{E}$. There exists a smallest element $H(\mathbf{E})$ in $\mathfrak{H}(\mathbf{E})$ ([7], Prop. 10). Hence by (S 3) any weak sublattice H of $M(\mathbf{E})$ containing $H(\mathbf{E})$ is also in $\mathfrak{H}(\mathbf{E})$.

Remark 1. Applying further results from the theory of Banach lattices (cf. [13]) the following can be shown: There exists a bijection between $\mathfrak{H}(\mathbf{E})$ and the class $\mathfrak{B}(\mathbf{E})$ of closed sublattices of $L(\mathbf{E})$ containing $\{P_i: i \in I\}$. It is given in the following way: If $B \in \mathfrak{B}(\mathbf{E})$ then there exists a positive linear projection T of $L(\mathbf{E})$ onto B ([13], 120, Cor. 1). Let Π denote the adjoint of T (on $M(\mathbf{E})$). Then $\Pi M(\mathbf{E}) \in \mathfrak{H}(\mathbf{E})$, Π is the corresponding sufficient projection, and the topological dual space B' of B is isomorphic with $\Pi M(\mathbf{E})$. Moreover T is a transition (i.e. preserves the norm of the positive elements).

Conversely if $H \in \mathfrak{H}(\mathbf{E})$ with sufficient projection Π then $\{\mu \in L(\mathbf{E}): \langle \mu, f \rangle = 0$ for all f in the kernel of $\Pi\} \in \mathfrak{B}(\mathbf{E})$. Finally let $B(\mathbf{E})$ be the closed sublattice of $L(\mathbf{E})$ generated by $\{P_i: i \in I\}$. Then $B(\mathbf{E})$ corresponds to $H(\mathbf{E})$ under this bijection.

We are now going to describe the relation between the sufficient sublattices and the pairwise sufficient subalgebras for $\mathbf{E}$. If $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$ the subalgebra $\bar{\mathfrak{S}} = \bigcap \{\mathfrak{S} \vee \mathfrak{N}(J): J \in \mathscr{F}(I)\}$ is called the $\mathbf{E}$-*closure* of $\mathfrak{S}$. If $\bar{\mathfrak{S}} = \mathfrak{S}$ then $\mathfrak{S}$ is said to be $\mathbf{E}$-*closed*. It can be easily seen that $\mathscr{L}(X, \bar{\mathfrak{S}})$ is the $\sigma(\mathscr{L}(X, \mathfrak{A}), L(\mathbf{E}))$-closure of $\mathscr{L}(X, \mathfrak{S})$. Moreover one has the following elementary result:

Lemma 1. *There exists a bijection between the class of all* $\mathbf{E}$*-closed subalgebras* $\mathfrak{S}$ *of* $\mathfrak{A}$ *and the class of all* $\sigma(\mathscr{L}(X,\mathfrak{A}), L(\mathbf{E}))$*-closed vector sublattices* $\mathfrak{H}$ *of* $\mathscr{L}(X,\mathfrak{A})$ *such that* $1_X \in \mathfrak{H}$. *It is given by* $\mathscr{H}_{\mathfrak{S}} = \mathscr{L}(X,\mathfrak{S})$ and $\mathfrak{S}_{\mathscr{H}} = \{A \in \mathfrak{A}: 1_A \in \mathscr{H}\}$.

A subalgebra $\mathfrak{S}$ of $\mathfrak{A}$ is called *pairwise sufficient* for $\mathbf{E}$ if $\mathfrak{S}$ is sufficient for all experiments $\mathbf{E}_J$, $J \in \mathscr{F}(I)$. By $\mathfrak{P}(\mathbf{E})$ we denote the class of all subalgebras of $\mathfrak{A}$ that are pairwise sufficient for $\mathbf{E}$. If $\mathfrak{S} \in \mathfrak{P}(\mathbf{E})$ we have $\mathfrak{T} \in \mathfrak{P}(\mathbf{E})$ for any σ-algebra $\mathfrak{T}$ such that $\mathfrak{S} \subset \mathfrak{T} \subset \mathfrak{A}$; especially $\tilde{\mathfrak{S}} \in \mathfrak{P}(\mathbf{E})$. Conversely $\tilde{\mathfrak{S}} \in \mathfrak{P}(\mathbf{E})$ implies $\mathfrak{S} \in \mathfrak{P}(\mathbf{E})$.

Let ρ denote the canonical mapping of $\mathscr{L}(X,\mathfrak{A})$ into $M(\mathbf{E})$. If $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$ and if $H_{\mathfrak{S}}$ denotes the $\sigma(M(\mathbf{E}), L(\mathbf{E}))$-closure of $\rho(\mathscr{L}(X,\mathfrak{S}))$ in $M(\mathbf{E})$ then $H_{\mathfrak{S}}$ obviously is a weak sublattice of $M(\mathbf{E})$. Now one has the following *result*:

(1) $\mathfrak{S}$ is pairwise sufficient for $\mathbf{E}$ if and only if $H_{\mathfrak{S}}$ is a sufficient sublattice of $M(\mathbf{E})$ ([8], 321, Prop. 8).

But as we shall see in the next section not every sufficient sublattice can be obtained in this way. One can gain more information about this relationship by proceeding along the following lines: $L(\mathbf{E})$ is the inductive limit of the L-spaces $L(\mathbf{E}_J)$, $J \in \mathscr{A}(I)$. Hence the inductive limit $L_0(\mathbf{E})$ of the spaces $L(\mathbf{E}_J)$, $J \in \mathscr{F}(I)$, is dense in $L(\mathbf{E})$. Thus $M(\mathbf{E})$ is also the topological dual space of $L_0(\mathbf{E})$, i.e. $M(\mathbf{E})$ is (algebraically) the projective limit of the system $\{M(\mathbf{E}_J): J \in \mathscr{F}(I)\}$.

Let ε_J denote the inclusion mapping of $L(\mathbf{E}_J)$ into $L(\mathbf{E})$ and π_J its adjoint mapping of $M(\mathbf{E})$ into $M(\mathbf{E}_J)$. If ρ_J denotes the canonical mapping of $\mathscr{L}(X,\mathfrak{A})$ into $M(\mathbf{E}_J)$ then we have $\rho_J = \pi_J \circ \rho$. Now let H be a weak sublattice of $M(\mathbf{E})$. Then $\rho_J^{-1}(\pi_J(H))$ is a $\sigma(\mathscr{L}(X,\mathfrak{A}), L(\mathbf{E}_J))$-closed sublattice of $\mathscr{L}(X,\mathfrak{A})$ containing 1_X. By Lemma 1 there corresponds a subalgebra $\mathfrak{S}_H(J)$ of $\mathfrak{A}$ to it ($J \in \mathscr{F}(I)$). From these observations we obtain easily the following *results*:

(2) We have $\mathfrak{S}_H(J) = \mathfrak{S}_H(J) \vee \mathfrak{N}(J)$. If $\mathfrak{S}_H = \mathfrak{S}_{\rho^{-1}(H)} = \{A \in \mathfrak{A}: \rho(1_A) \in H\}$ then $\mathfrak{S}_H(J) \downarrow \mathfrak{S}_H$ if J runs through $\mathscr{F}(I)$.

(3) If $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$ and if $H := H_{\mathfrak{S}}$ (= the $\sigma(M(\mathbf{E}), L(\mathbf{E}))$-closure of $\rho(\mathscr{L}(X,\mathfrak{S}))$ in $M(\mathbf{E})$) then $\mathfrak{S}_H(J) = \mathfrak{S} \vee \mathfrak{N}(J)$ and $\mathfrak{S}_H = \tilde{\mathfrak{S}}$.

(4) The following assertions are equivalent:

(i) H is a sufficient sublattice for $\mathbf{E}$.

(ii) $\mathfrak{S}_H(J)$ is a pairwise sufficient subalgebra for $\mathbf{E}$ for any $J \in \mathscr{F}(I)$.

(5) If H is a sufficient sublattice for $\mathbf{E}$ such that $\overline{\rho(\rho^{-1}(H))} = H$ then $\mathfrak{S}_H$ is pairwise sufficient for $\mathbf{E}$, and we have $\mathfrak{S}_H(J) = \mathfrak{S}_H \vee \mathfrak{N}(J)$ for all $J \in \mathscr{F}(I)$.

2. Minimal Pairwise Sufficient Subalgebras

Let $\mathbf{E} = (X, \mathfrak{A}, (P_i)_{i \in I})$ be an experiment. $\mathfrak{S} \in \mathfrak{P}(\mathbf{E})$ is said to be a *minimal pairwise sufficient subalgebra* for $\mathbf{E}$ if $\mathfrak{S} \subset \tilde{\mathfrak{T}}$ for any $\mathfrak{T} \in \mathfrak{P}(\mathbf{E})$. In view of the results in Section 1 a subalgebra $\mathfrak{S}$ of $\mathfrak{A}$ is minimal pairwise sufficient for $\mathbf{E}$ if and only if $H_{\mathfrak{S}} = H(\mathbf{E})$. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are minimal pairwise sufficient subalgebras for $\mathbf{E}$ then one has $\tilde{\mathfrak{S}}_1 = \tilde{\mathfrak{S}}_2$. But in general there exists no minimal pairwise sufficient subalgebra.

Example 1. Let $X = [0, 2[$, $\mathfrak{A} = \mathfrak{B}([0, 2[)$, $P_i = \frac{1}{2}(\varepsilon_i + \varepsilon_{i+1})$ and $\mathfrak{S}_i = \{A \in \mathfrak{A}: P_i(A) = 0$ or $P_i(A) = 1\}$ for all $i \in [0, 1[$. Moreover let P_1 and P_2 be the Lebesgue measures on $[0, 1[$ and $[1, 2[$ respectively. Let $I = [0, 1] \cup \{2\}$. We consider the experiment $\mathbf{E} = (X, \mathfrak{A}, (P_i)_{i \in I})$. Since $\mathfrak{S}_i = \mathfrak{N}(\{i\})$ we have $\tilde{\mathfrak{S}}_i = \bar{\mathfrak{S}}_i = \mathfrak{S}_i$ for $i \in [0, 1[$. Now $\mathfrak{S}_i$ is sufficient for $\mathbf{E}$ if $i\in[0,1[$. [For any $A\in\mathfrak{A}$ the function $1_{A\setminus\{i,i+1\}}+\frac{1}{2}(1_A(i)+1_A(i+1))\,1_{\{i,i+1\}}$ is a version of the conditional probability $P_j(A|\mathfrak{S}_i)$ for every $j\in I$.]

$$\mathfrak{S}=\bigcap\{\mathfrak{S}_i\colon\ i\in[0,1[\}=\{A\in\mathfrak{A}\colon\ 1+(A\cap[0,1[)=A\cap[1,2[\}$$

is not pairwise sufficient for $\mathbf{E}$.

[Taking into account the translation invariance of the Lebesgue measure λ on $\mathbb{R}$ we obtain for any $A\in\mathfrak{S}$ the relation $P_1(A)=\lambda(A\cap[0,1[)=\lambda(1+(A\cap[0,1[))=\lambda(A\cap[1,2[)=P_2(A)$, i.e. P_1 and P_2 coincide on $\mathfrak{S}$. Since $P_1\neq P_2$ the subalgebra $\mathfrak{S}$ cannot be pairwise sufficient for $\mathbf{E}$.]

If there existed a minimal pairwise sufficient subalgebra $\mathfrak{T}$ for $\mathbf{E}$ we would have $\mathfrak{T}\subset\mathfrak{S}_i$ for all $i\in[0,1[$ and hence $\mathfrak{T}\subset\mathfrak{S}$. But then also $\mathfrak{S}$ had to be pairwise sufficient for $\mathbf{E}$. Thus there exists no minimal pairwise sufficient subalgebra for $\mathbf{E}$.

Remarks 1. Example 1 is a modification of an example of Pitcher [11] proving that in general there exists no minimal sufficient subalgebra for an experiment.

2. H. Luschgy has informed the author that he has found the same counterexample [9].

3. Let $\mathbf{E}=(X,\ \mathfrak{A},\ (P_i)_{i\in I})$ be an experiment. For any $i\in I$ we denote by $\hat{\mathfrak{A}}^{P_i}$ the completion of $\mathfrak{A}$ with respect to P_i. Then any P_i can be extended to a probability measure $\hat{P}_i$ on $\hat{\mathfrak{A}}=\bigcap\{\hat{\mathfrak{A}}^{P_i}\colon\ i\in I\}$. The experiment $\hat{\mathbf{E}}=(X,\ \hat{\mathfrak{A}},\ (\hat{P}_i)_{i\in I})$ is called the *completion* of $\mathbf{E}$.

Example 1 is also valid for the completion of $\mathbf{E}$. [$\hat{\mathfrak{A}}^{P_i}$ is the power set of X for all $i\in[0,1[$. Hence $\hat{\mathfrak{A}}$ is the completion of the Borel-σ-algebra $\mathfrak{B}([0,2[)$ with respect to the Lebesgue measure $\lambda_{[0,2[}$ on $[0,2[$. Since the extension of $\lambda_{[0,2[}$ to $\hat{\mathfrak{A}}$ is also translation invariant we can argue as above.]

4. In view of Example 1 and Remark 3 the assertions in [6] and [8], 322 concerning the existence of a minimal pairwise sufficient subalgebra can only hold under appropriate additional conditions.

5. Let $\mathbf{E}=(X,\ \mathfrak{A},\ (P_i)_{i\in I})$ be as in Example 1. In view of the results in section 1 there cannot exist an $\mathfrak{S}\in\mathfrak{P}(\mathbf{E})$ such that $H_{\mathfrak{S}}=H(\mathbf{E})$. Hence the correspondence $\mathfrak{S}\to H_{\mathfrak{S}}$ between $\mathfrak{P}(\mathbf{E})$ and $\mathfrak{H}(\mathbf{E})$ is in general not bijective.

Obviously a minimal pairwise sufficient subalgebra $\mathfrak{S}$ for an experiment $\mathbf{E}$ has the following property:

(M) For any $\mathfrak{T}\in\mathfrak{P}(\mathbf{E})$ such that $\mathfrak{T}\subset\bar{\mathfrak{S}}$ it holds $\bar{\mathfrak{T}}=\bar{\mathfrak{S}}$.

Conversely a subalgebra $\mathfrak{S}\in\mathfrak{P}(\mathbf{E})$ with the property (M) is minimal pairwise sufficient for $\mathbf{E}$.

[Let $\mathfrak{T}\in\mathfrak{P}(\mathbf{E})$ and $J\in\mathscr{F}(I)$. Then for any $K\in\mathscr{F}(I)$ such that $K\supset J$ we have $\mathfrak{N}(K)\subset\mathfrak{N}(J)$. Hence $\mathfrak{U}_J=\bar{\mathfrak{S}}\cap(\mathfrak{T}\vee\mathfrak{N}(J))$ is sufficient for all $\mathbf{E}_K$, $K\supset J$ (since $\mathbf{E}_K$ is dominated) and thus $\mathfrak{U}_J\in\mathfrak{P}(\mathbf{E})$. Since $\mathfrak{U}_J$ is $\mathbf{E}$-closed we have $\mathfrak{U}_J=\bar{\mathfrak{S}}$ by (M) and thus $\bar{\mathfrak{S}}\subset\mathfrak{T}\vee\mathfrak{N}(J)$ for all $J\in\mathscr{F}(I)$. The definition of $\bar{\mathfrak{T}}$ yields $\bar{\mathfrak{S}}\subset\bar{\mathfrak{T}}$.]

This shows that the order theoretic concepts "minimal" and "smallest" coincide for the set $\{\bar{\mathfrak{T}}\colon\ \mathfrak{T}\in\mathfrak{P}(\mathbf{E})\}$ ordered with respect to inclusion.

3. Majorized Experiments

We present a sufficient condition for the existence of a minimal pairwise sufficient subalgebra. The experiment $\mathbf{E}=(X,\ \mathfrak{A},\ (P_i)_{i\in I})$ is said to be *majorized* if there exists a positive measure λ on $(X, \mathfrak{A})$ and for any $i\in I$ a function $f_i\in L^1(X, \mathfrak{A}, \lambda)$ such that $P_i = f_i \cdot \lambda$. In this case λ is called a majorizing measure for $\mathbf{E}$ and $\mathbf{E}$ is said to be λ-majorized. Of course any dominated experiment is majorized.

Proposition 1. *Let $\mathbf{E}=(X, \mathfrak{A}, (f_i\cdot\lambda)_{i\in I})$ be a λ-majorized experiment. Without loss of generality we may assume $0\leqq f_i<+\infty$ for all $i\in I$. Then the subalgebra $\mathfrak{S}$ of $\mathfrak{A}$ generated by the family $\{(f_i/\sum_{j\in J} f_j)\, 1_{[f_i>0]}: i\in J, J\in\mathscr{F}(I)\}$ is minimal pairwise sufficient for* $\mathbf{E}$.

Proof. For any $J\in\mathscr{F}(I)$ let $P_J=\sum_{i\in J} P_i$ and $f_J=\sum_{i\in J} f_i$. For any $i\in I$ let $B(i)=[f_i>0]$. For any $J\in\mathscr{F}(I)$ and $i\in J$ let $g_{i,J}=(f_i/f_J)\, 1_{B(i)}$. Finally let $\mathscr{F}_J=\{g_{i,K}: i\in K, K\in\mathscr{F}(I)$ such that $J\subset K\}$. Then we have $\mathfrak{A}(\mathscr{F}_J)=\mathfrak{A}(\mathscr{F}_{J'})$ for all $J, J'\in\mathscr{F}(I)$.

[Without loss of generality we may assume $J\subset J'$. Then we have $\mathscr{F}_{J'}\subset\mathscr{F}_J$ and hence $\mathfrak{A}(\mathscr{F}_{J'}))\subset\mathfrak{A}(\mathscr{F}_J)$. For proving the converse inclusion it is sufficient to show that any $f\in\mathscr{F}_J$ is $\mathfrak{A}(\mathscr{F}_{J'})$-measurable. Now for $f\in\mathscr{F}_J$ there exist $K\in\mathscr{F}(I)$, $J\subset K$ and $i\in K$ such that $f=g_{i,K}$. Let $K'=K\cup J'$. Then we have $g_{i,K}=(f_i/f_{K'})\,(f_{K'}/f_K)\, 1_{B(i)} = g_{i,K'}/(\sum_{j\in K} g_{j,K'})$. Since $J'\subset K'$ this proves the $\mathfrak{A}(\mathscr{F}_{J'})$-measurability of f.]

We put $\mathfrak{S}=\mathfrak{A}(\mathscr{F}_J)$ for an arbitrary $J\in\mathscr{F}(I)$. $\mathfrak{S}$ is pairwise sufficient for $\mathbf{E}$.

[Let $J\in\mathscr{F}(I)$. Since $P_i=f_i\cdot\lambda=(f_i 1_{B(i)})\cdot\lambda$ and $P_J=f_J\cdot\lambda$ we have $P_i=g_{i,J}\cdot P_J$ for all $i\in J$. Thus by the Halmos-Savage theorem ([3], Theorem 1 or [5], 33, Satz 5.2) $\mathfrak{S}$ is sufficient for $\mathbf{E}_J$.]

$\mathfrak{S}$ is minimal pairwise sufficient for $\mathbf{E}$.

[Let $\mathfrak{T}\in\mathfrak{P}(\mathbf{E})$ and $J\in\mathscr{F}(I)$. By the Halmos-Savage theorem there exist $\mathfrak{T}$-measurable functions $g_i\geqq 0$ such that $P_i=g_i\cdot P_J$ for all $i\in J$. It follows $g_i=g_{i,J}$ a.e. (P_J), hence the functions $g_{i,J}$, $i\in J$, are $\mathfrak{T}\vee\mathfrak{N}(J)$-measurable. But for $K\in\mathscr{F}(I)$, $J\subset K$, we have $\mathfrak{N}(K)\subset\mathfrak{N}(J)$. Hence the functions $g_{i,K}$, $i\in K$, are $\mathfrak{T}\vee\mathfrak{N}(J)$-measurable too. This proves $\mathfrak{S}=\mathfrak{A}(\mathscr{F}_J)\subset\mathfrak{T}\vee\mathfrak{N}(J)$. Since $J\in\mathscr{F}(I)$ was arbitrary we get $\mathfrak{S}\subset\bigcap\{\mathfrak{T}\vee\mathfrak{N}(J): J\in\mathscr{F}(I)\}=\bar{\mathfrak{T}}$.]

Hence the proposition is completely proved. ⌋

Remarks. 1. If $\mathbf{E}=(X, \mathfrak{A}, (f_i\cdot\lambda)_{i\in I})$ is a λ-majorized experiment then the σ-algebra $\mathfrak{A}(\{f_i: i\in I\})$ is pairwise sufficient for $\mathbf{E}$. [This is immediate by Proposition 1.]

2. The σ-algebra $\mathfrak{S}$ of proposition 1 is also generated by the families $\{(f_i/f_j)\, 1_{[f_j>0]}: (i,j)\in I\times I\}$ and $\{(f_i/(f_i+f_j))\, 1_{[f_i>0]}: (i,j)\in I\times I\}$.

3. Let $\mathbf{E}=(X, \mathfrak{A}, (f_i\cdot\lambda)_{i\in I})$ be a λ-majorized experiment. Then $L(\lambda)=\{f\cdot\lambda: f\in L^1(X, \mathfrak{A}, \lambda)\}$ is a norm closed ideal and hence a band in $\mathscr{M}^b(X, \mathfrak{A})$ ([13], 113, 8.3 (ii)). Obviously $L(\mathbf{E})\subset L(\lambda)$. Moreover the following holds: "$(f_i\cdot\lambda)_{i\in I}\approx\lambda$" $\Leftrightarrow$ "$L(\mathbf{E})=L(\lambda)$ and λ has the finite subset property". ["$\Rightarrow$" $L(\lambda)$ is the ordered direct sum of $L(\mathbf{E})$ and a band L in $L(\lambda)$ ([13], 113, Prop. 8.3). Let $f\cdot\lambda\in L_+$ and $N=[f>0]$. Since $f\cdot\lambda$ and $f_i\cdot\lambda$ are orthogonal it is $(f_i\cdot\lambda)(N)=0$ (all $i\in I$) and hence $\lambda(N)=0$ by assumption. Thus $f\cdot\lambda=0$ i.e. $L=\{0\}$. Concerning the finite subset property cf. [10], Lemma 2.9, (1). "$\Leftarrow$" Let us assume that there exists an $N\in\mathfrak{A}$ such that $(f_i\cdot\lambda)(N)=0$ for all $i\in I$ and $\lambda(N)>0$. Since λ has the finite subset property we may assume without loss of generality $\lambda(N)<\infty$. Then $L=\{f\cdot\lambda\in L(\lambda): (|f|\cdot\lambda)(N)=0\}$ is a band

in $L(\lambda)=L(\mathbf{E})$ containing $\{f_i \cdot \lambda : i \in I\}$. Hence $L(\lambda)=L$. This implies $1_N \cdot \lambda \in L$. But this contradicts $(1_N \cdot \lambda)(N)=\lambda(N)>0$.]

Example 1. Let G be a locally compact group, $\mathfrak{B}(G)$ the σ-algebra of Borel sets in G and μ as well as λ regular Borel measures on $(G, \mathfrak{B}(G))$ such that $\mu(G)=1$. Let the translation experiment $\mathbf{E}(\mu)=(G, \mathfrak{B}(G), (\mu * \varepsilon_x)_{x \in G})$ be λ-majorized. Then $\mathbf{E}(\mu)$ is also majorized by the right Haar measure ω_G on G. [If G is σ-compact this is well known ([15], 1384). For general G this follows from the fact that G is the union of its open σ-compact subgroups.]

Let now be especially $\mu = 1_B \cdot \omega_G$ where $B \in \mathfrak{B}(G)$ is such that $\omega_G(B)=1$. Then it follows from Proposition 1 and Remark 2 that the σ-algebra generated by $\{Bx: x \in G\}$ is minimal pairwise sufficient for $\mathbf{E}(\mu)$. If moreover B is open we have $(\mu * \varepsilon_x)_{x \in G} \approx \omega_G$.

4. Coherent Experiments

Let $\mathbf{E}=(X, \mathfrak{A}, (P_i)_{i \in I})$ be an experiment, ρ the canonical mapping of $\overline{\mathscr{L}(X, \mathfrak{A})}$ into $M(\mathbf{E})$ and H a sufficient sublattice of $M(\mathbf{E})$. Generally we don't have $\overline{\rho(\rho^{-1}(H))}=H$, for otherwise $\mathfrak{S}_{H(\mathbf{E})}$ would be a minimal pairwise sufficient subalgebra for $\mathbf{E}$ (by (5) of Section 1). But we always have the relation $\rho(\rho^{-1}(H))=H \cap \rho(\mathscr{L}(X, \mathfrak{A}))$. Thus if ρ is surjective then for any sufficient sublattice H of $M(\mathbf{E})$ the σ-algebra $\mathfrak{S}_H = \{A \in \mathfrak{A}: \rho(1_A) \in H\}$ is pairwise sufficient for $\mathbf{E}$ (by (5) of Section 1).

The experiment $\mathbf{E}$ is said to be *coherent* if $\rho(\mathscr{L}(X, \mathfrak{A}))=M(\mathbf{E})$.

Examples. 1. An experiment $\mathbf{E}=(X, \mathfrak{A}, (P_i)_{i \in I})$ is said to be *weakly dominated* [10] if there exists a localizable measure λ on $(X, \mathfrak{A})$ such that $\mathbf{E}$ is λ-majorized. The measure λ can be chosen such that $(P_i)_{i \in I} \approx \lambda$ ([10], Lemma 2.9, (2)). Then by remark 3.3 the L-spaces $L(\mathbf{E})$ and $L^1(X, \mathfrak{A}, \lambda)$ are isomorphic. Hence $M(\mathbf{E})$ can be identified with $L^\infty(X, \mathfrak{A}, \lambda)$. Thus any weakly dominated experiment is coherent.

Especially the experiment of example 3.1 is weakly dominated (since ω_G is localizable) and hence coherent.

2. An experiment is coherent if and only if it is coherent in the sense of Hasegawa and Perlman ([4], 1051). Especially any compact experiment [12] is coherent.

Proposition 1. *For an experiment* $\mathbf{E}=(X, \mathfrak{A}, (P_i)_{i \in I})$ *the following assertions are equivalent:*

(i) $\mathbf{E}$ *is coherent.*

(ii) $\mathbf{E}$ *is weakly dominated.*

Proof. In view of Example 1 we only have to show "(i)$\Rightarrow$(ii)". By the representation theorem of Kakutani for L-spaces ([13], 114, Theorem 8.5) there exist a locally compact space Y which is the topological sum of a family $(K_\alpha)_{\alpha \in A}$ of compact spaces, a strictly positive regular Baire measure μ on $(Y, \mathfrak{B}_0(Y))$ (where $\mathfrak{B}_0(Y)$ is the σ-algebra of Baire sets in Y) and an isomorphism φ of the L-space $L^1(Y, \mathfrak{B}_0(Y), \mu)$ onto the L-space $L(\mathbf{E})$.

Let $\nu_\alpha = \varphi(1_{K_\alpha})$ for all $\alpha \in A$ and $\nu = \sup\{\sum_{\alpha \in F} \nu_\alpha : F \in \mathscr{F}(A)\}$. Then ν is a positive measure on $(X, \mathfrak{A})$. We prove that ν is localizable and majorizes $\mathbf{E}$.

1. ν has the finite subset property.

[We denote by φ' the adjoint mapping to φ of $M(\mathbf{E})$ onto $L^\infty(Y, \mathfrak{B}_0(Y), \mu)$ and by ψ the inverse mapping to φ'. Then $(1_Y - 1_{K_\alpha}) \wedge 1_{K_\alpha} = 0$ implies $(\psi(1_Y) - \psi(1_{K_\alpha})) \wedge \psi(1_{K_\alpha}) = 0$. Since $\mathbf{E}$ is coherent there exists a $g_\alpha \in \mathscr{L}(X, \mathfrak{A})$ such that $\psi(1_{K_\alpha}) = \rho(g_\alpha)$. But $(1_X - g_\alpha) \wedge g_\alpha = 0$ a.e. (P_i) for all $i \in I$ yields the existence of a $B_\alpha \in \mathfrak{A}$ such that $\tilde{1}_{B_\alpha} = \rho(1_{B_\alpha}) = \rho(g_\alpha) = \psi(1_{K_\alpha})$ $(\alpha \in A)$.

Let $B \in \mathfrak{A}$, $\tilde{1}_B = \rho(1_B)$ and $\alpha, \beta \in A$. Then

$$\begin{aligned} \nu_\alpha(B \cap B_\beta) &= \langle \nu_\alpha, \tilde{1}_B \wedge \tilde{1}_{B_\beta} \rangle = \langle 1_{K_\alpha}, \varphi'(\tilde{1}_B) \wedge 1_{K_\beta} \rangle \\ &= \int 1_{K_\alpha} [\varphi'(\tilde{1}_B) \wedge 1_{K_\beta}]\, d\mu = 0 \quad \text{if } \alpha \neq \beta \end{aligned}$$

and

$$\begin{aligned} &= \int 1_{K_\beta} \varphi'(\tilde{1}_B)\, d\mu = \langle 1_{K_\beta}, \varphi'(\tilde{1}_B) \rangle = \langle \varphi(1_{K_\beta}), \tilde{1}_B \rangle \\ &= \langle \nu_\beta, \tilde{1}_B \rangle = \nu_\beta(B) \quad \text{if } \alpha = \beta. \end{aligned}$$

Hence we finally have:

$$\nu(B \cap B_\beta) = \sup\{\sum_{\alpha \in F} \nu_\alpha(B \cap B_\beta) \colon F \in \mathscr{F}(A)\} = \nu_\beta(B \cap B_\beta)$$

and

$$\begin{aligned} \nu(B) &= \sup\{\sum_{\alpha \in F} \nu_\alpha(B) \colon F \in \mathscr{F}(A)\} = \sup\{\sum_{\alpha \in F} \nu_\alpha(B \cap B_\alpha) \colon F \in \mathscr{F}(A)\} \\ &= \sup\{\sum_{\alpha \in F} \nu(B \cap B_\alpha) \colon F \in \mathscr{F}(A)\}. \end{aligned}$$

If $\nu(B) > 0$ then there exists an $\alpha \in A$ such that $\nu(B \cap B_\alpha) > 0$. On the other hand we have $\nu(B \cap B_\alpha) = \nu_\alpha(B \cap B_\alpha) < \infty$ since $\nu_\alpha \in L(\mathbf{E})$. Hence ν has the finite subset property.]

2. $P_i \ll \nu$ for all $i \in I$.

[There exist $f_i \in L^1(Y, \mathfrak{B}_0(Y), \mu)$, $0 \leqq f_i < \infty$ such that $\varphi(f_i) = P_i$. We put $P_i^{(\alpha)} = \varphi(f_i \cdot 1_{K_\alpha})$ $(\alpha \in A)$. Then $f_i = \sup\{\sum_{\alpha \in F} f_i \cdot 1_{K_\alpha} \colon F \in \mathscr{F}(A)\}$ implies $P_i(B) = \lim_{F \in \mathscr{F}(A)} \sum_{\alpha \in F} P_i^{(\alpha)}(B)$ for all $B \in \mathfrak{A}$ ([13], 113, Theorem 8.3 (i)). Since $f_i \cdot 1_{K_\alpha} \wedge n 1_{K_\alpha} \uparrow f_i 1_{K_\alpha}$ as $n \to \infty$ we have $P_i^{(\alpha)} \wedge n \nu_\alpha \uparrow P_i^{(\alpha)}$ as $n \to \infty$. Hence $P_i^{(\alpha)} \ll \nu_\alpha (\alpha \in A)$. But then we also have $P_i \ll \nu$ $(i \in I)$.

3. $(P_i)_{i \in I} \approx \nu$.

[Let $N \in \mathfrak{A}$ such that $P_i(N) = 0$ for all $i \in I$. $L = \{\lambda \in L(\mathbf{E}) \colon |\lambda|(N) = 0\}$ is a band in $L(\mathbf{E})$ ([13], 78 and 113, Theorem 8.3 (ii)). Since $P_i \in L$ for all $i \in I$ we have $L = L(\mathbf{E})$. Hence $\nu_\alpha \in L$ for all $\alpha \in A$. This shows $\nu(N) = 0$.]

4. ν is localizable.

[By our assumption $\mathscr{L}(X, \mathfrak{A})/\rho^{-1}(0)$ and $M(\mathbf{E})$ are isomorphic. By 3. we conclude $\rho^{-1}(0) = \{f \in \mathscr{L}(X, \mathfrak{A}) \colon P_i([f \neq 0]) = 0$ for all $i \in I\} = \{f \in \mathscr{L}(X, \mathfrak{A}) \colon \nu([f \neq 0]) = 0\}$. Hence $\mathscr{L}(X, \mathfrak{A})/\rho^{-1}(0) = L^\infty(X, \mathfrak{A}, \nu)$. But then $L^\infty(X, \mathfrak{A}, \nu)$ is order complete ([13], 72, Prop. 4.2). Taking into account 1. the measure ν is therefore localizable.]

5. $\mathbf{E}$ is weakly dominated.

[Since a localizable measure has the Radon-Nikodym property ([16], 265, Theorem 3) there exist $f_i \in L^1(X, \mathfrak{A}, \nu)$ such that $P_i = f_i \cdot \nu$ $(i \in I)$. Hence $\mathbf{E}$ is ν-majorized.] $\lrcorner$

Remarks. 1. In [7] (Theorem 5) LeCam states a similar result for the completed experiment $\hat{\mathbf{E}}$. But his proof has a lacuna since a localizable measure space need not be strictly localizable (i.e. a direct sum of finite measure spaces).

2. If $\mathbf{E}$ is coherent then also $\hat{\mathbf{E}}$ is coherent. [$L(\hat{\mathbf{E}})$ and $L(\mathbf{E})$ are obviously isomorphic hence also $M(\hat{\mathbf{E}})$ and $M(\mathbf{E})$.] But there exist non-coherent experiments $\mathbf{E}$ with coherent $\hat{\mathbf{E}}$ ([7], 1448f).

3. If $\mathbf{E} = (X, \mathfrak{A}, (P_i)_{i \in I})$ is a coherent experiment and if $\mathfrak{S}$ is an $\mathbf{E}$-closed subalgebra of $\mathfrak{A}$ then also $\mathbf{E}^{\mathfrak{S}} = (X, \mathfrak{S}, (P_i | \mathfrak{S})_{i \in I})$ is a coherent experiment.

[Since $\mathfrak{S} = \bar{\mathfrak{S}}$ and since ρ is surjective we have $\rho(\mathscr{L}(X, \mathfrak{S})) = H_{\mathfrak{S}}$. But $H_{\mathfrak{S}}$ can be identified with $M(\mathbf{E}^{\mathfrak{S}})$.]

4. If $\mathbf{E}$ is a coherent experiment and if $\mathfrak{S} \in \mathfrak{P}(\mathbf{E})$ such that $\bar{\mathfrak{S}} = \mathfrak{S}$ then $\mathfrak{S}$ is even sufficient for $\mathbf{E}$.

[$H_{\mathfrak{S}}$ is a sufficient sublattice of $M(\mathbf{E})$ ((1) of Section 1). Let Π be the corresponding sufficient projection. Since $\rho(\mathscr{L}(X, \mathfrak{S})) = H_{\mathfrak{S}}$ there exists for $f \in \mathscr{L}(X, \mathfrak{A})$ a function $h \in \mathscr{L}(X, \mathfrak{S})$ such that $\Pi \rho(f) = \rho(h)$. Hence for any $g \in \mathscr{L}(X, \mathfrak{S})$ we have $\int f g \, dP_i = \langle P_i, \rho(f) \, \rho(g) \rangle = \langle P_i, \Pi(\rho(f) \, \rho(g)) \rangle = \langle P_i, \rho(g) \, \Pi \, \rho(f) \rangle = \langle P_i, \rho(g) \, \rho(h) \rangle = \int h \, g \, dP_i$ (all $i \in I$).]

5. Any coherent experiment $\mathbf{E}$ admits a minimal sufficient subalgebra. This result which is due to Hasegawa and Perlman [4] (for coherent experiments) and to Mußmann [10] (for weakly dominated experiments) can now be easily derived.

[Since $\mathbf{E}$ is majorized there exists a minimal pairwise sufficient subalgebra $\mathfrak{S}$ for $\mathbf{E}$ (Prop. 3.1). Without loss of generality we may assume $\bar{\mathfrak{S}} = \mathfrak{S}$. Then $\mathfrak{S}$ is sufficient for $\mathbf{E}$ by remark 4. Let $\mathfrak{T}$ be a sufficient subalgebra for $\mathbf{E}$. Then we have $\tilde{\mathfrak{T}} = \bar{\tilde{\mathfrak{T}}}$ and $\tilde{\mathfrak{T}} \in \mathfrak{P}(\mathbf{E})$. Hence $\mathfrak{S} \subset \tilde{\mathfrak{T}}$. Thus $\mathfrak{S}$ is minimal sufficient.]

References

1. Bahadur, R.R.: Statistics and Subfields. Ann. Math. Statist. **26**, 490–497 (1955)
2. Burkholder, D.L.: Sufficiency in the undominated case. Ann. Math. Statist. **32**, 1191–1200 (1961)
3. Halmos, P.R., Savage, L.J.: Applications of the Radon-Nikodym theorem to the theory of sufficient statistics. Ann. Math. Statist. **20**, 225–241 (1949)
4. Hasegawa, M., Perlman, M.D.: On the existence of a minimal sufficient subfield. Ann. Statist. **2**, 1049–1055 (1974). Correction: Ann. Statist. **3**, 1371–1372 (1975)
5. Heyer, H.: Mathematische Theorie statistischer Experimente. Berlin-Heidelberg-New York: Springer 1973
6. LeBihan, M.-F., Littaye-Petit, M., Petit, J.-L.: Exhaustivité par paire. C.R. Acad. Sci. Paris Sér. A **270**, 1753–1756 (1970)
7. LeCam, L.: Sufficiency and approximate Sufficiency. Ann. Math. Statist. **35**, 1419–1455 (1964)
8. Littaye-Petit, M., Piednoir, J.-L., van Cutsem, B.: Exhaustivité. Ann. Inst. H. Poincaré V, 289–322 (1969)

9. Luschgy, H.: Sur l'existence d'une plus petite sous-tribu exhaustive par paire. Preprint 1977
10. Mußmann, D.: Vergleich von Experimenten im schwach dominierten Fall. Z. Wahrscheinlichkeitstheorie verw. Gebiete **24**, 295–308 (1972)
11. Pitcher, T.S.: Sets of measures not admitting necessary and sufficient statistics or subfields. Ann. Math. Statist. **28**, 267–268 (1957)
12. Pitcher, T.S.: A more general property than domination for sets of probability measures. Pacific J. Math. **15**, 597–611 (1965)
13. Schaefer, H.H.: Banach Lattices and Positive Operators. Berlin-Heidelberg-New York: Springer 1974
14. Siebert, E.: Klasseneigenschaften statistischer Experimente und ihre Charakterisierung durch Kegelmaße. Habilitationsschrift, Tübingen 1976
15. Torgersen, E.N.: Comparison of translation experiments. Ann. Math. Statist. **43**, 1383–1399 (1972)
16. Zaanen, A.C.: Integration. Amsterdam: North Holland 1967

Received August 13, 1976; in revised form July 25, 1978