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On the Global Limit Behaviour of Markov Chains and of General Nonsingular Markov Processes*

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0. Introduction

Limit theorems for null-recurrent and non-recurrent Markov chains are usually proved under two different assumptions: either probabilities of visits in single states $i, j, \ldots$ are considered and the initial distributions are allowed to be arbitrary, or else probabilities of visits in sets $A, B, \ldots$ of states are considered, but then the initial distribution is assumed to be concentrated in one state. In the latter case $A, B, \ldots$ usually have finite invariant or subinvariant measure. Theorems of both types are now also available for very general state-spaces, see e.g. [*19*], [*17*], [*18*], [*15*], [*3*]. Limit theorems which drop the *pointwise* assumptions both with respect to the initial distribution and the visited set will be called *global* limit theorems. We shall, roughly speaking, show that some theorems on ordinary and Cesàro-convergence do have global analogues, while global ratio limit theorems require crucial additional assumptions. A mutual generalization of the two pointwise generalizations of the Doeblin ratio limit theorem [*6*] does not hold.

In section 2 we prove a general stochastic ergodic theorem for arbitrary contractions in $\mathfrak{L}_1$ and a global limit theorem for the corresponding nonsingular Markov process. We also give a new counterexample to a conjecture of HUREWICZ. Our example, which is based on the example on Markov chains mentioned above, allows us to disprove the conjecture even for sets of finite invariant measure. (DOWKER and ERDÖS [*8*] showed that the conjecture is wrong for sets of possibly infinite measure.)

In section 3 an example is given showing that the known sufficient conditions for the strong ratio limit property do not imply such a property for probabilities of visits in sets of finite invariant measure. The ratios of n-step and $(n+1)$-step transition probabilities, however, satisfy a strong global ratio limit theorem. Section 3 is independent of section 2.

1. Markov Chains

We shall adopt the terminology and notation of CHUNG [*4*] and consider a Markov chain $\{X_n, n \geqq 0\}$ with countable state space I. For any subset $A \subset I$ let $p_{iA}^{(k)} = \sum_{j \in A} p_{ij}^{(k)}$ be the k-step transition probability from state i to the set A. Let I_P

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and I_N be the sets of positive states and null-states respectively. Within I_P the global limit behaviour is well known and, in fact, follows from the pointwise limit theorems. A measure $u = \{u_i, i \in I\}$ is called subinvariant if

$$u_k \geqq \sum_{i \in I} u_i p_{ik}$$

holds for all $k \in I$ and u is called invariant if equality holds. The following simple result is probably known:

Proposition 1.1. *Let $p = \{p_i, i \in I\}$ be any probability distribution on I, $u = \{u_i\}$ a subinvariant measure, $A \cap I_P = 0$, $A \subset \{i: u_i > 0\}$, $u(A) = \sum_{i \in A} u_i < \infty$. Then $\sum_{i \in I} p_i p_{iA}^{(n)}$ tends to* 0.

Proof. We may assume that p is concentrated in one state i_0 and that $u_{i_0} = p_{i_0} = 1$. If $\varepsilon > 0$ is given, choose finitely many states $a_1, \ldots, a_r \in A$ with $u(A - \{a_1, \ldots, a_r\}) < \varepsilon/2$. Then since $p \leqq u$ and $p_{i_0, a_k}^{(n)} \to 0 \, (k = 1, \ldots, r)$ it follows that $p_{i_0, A}^{(n)} < \varepsilon$ for n large enough.

Since the limit is 0, the question for ratio limit theorems arises: Is there a mutual generalization of the following two pointwise limit theorems (see e.g. [*15*]) ?:

Let the chain be recurrent and let I consist of only one ergodic class. Then:

(a) If $u = \{u_i\}$ is the (essentially unique) invariant measure (see Derman [*5*]) and if both $A \subseteqq I$ and $B \subseteqq I$ have finite invariant measure, then

$$\frac{\sum_{t=1}^{n} p_{iA}^{(t)}}{\sum_{t=1}^{n} p_{jB}^{(t)}} \to \frac{u(A)}{u(B)} \tag{1.1}$$

for all $i, j \in I$.

(b) If $p = \{p_i\}$ and $q = \{q_i\}$ are two probability distributions in I

$$\frac{\sum_{t=1}^{n} \sum_{i \in I} p_i p_{ij}^{(t)}}{\sum_{t=1}^{n} \sum_{i \in I} q_i p_{ik}^{(t)}} \to u_j u_k^{-1} \tag{1.2}$$

for all $j, k \in I$.

We shall show that no common generalization for (a)and (b) exists. Moreover we shall see rather exactly to what extent (1.1) can be generalized concerning more general initial distributions.

We shall frequently need taboo-probabilities: For $t \geqq 1$ ${}_H p_{iA}^{(t)} = \text{prob}\,\{X_t \in A; X_\nu \notin H \text{ for } \nu = 1, \ldots, t-1 \mid X_0 = i\}$, for $t = 0$ ${}_H p_{iA}^{(t)} = 0$ if $i \notin A$ and $= 1$ if $i \in A$. If $\{\alpha^{(t)}, t \geqq 0\}$ is a sequence of real numbers (eventually with some lower indices), then α^* shall denote $\sum_{t=1}^{\infty} \alpha^{(t)}$. For example ${}_{ih}p_{ii}^*$ is the probability of returning to i without passing through h, given a start at i.

Lemma 1.1. *Let I consist of only one ergodic class and let $u = \{u_i\}$ be a non-trivial subinvariant measure. If $A \subseteqq I$ satisfies $u(A) < \infty$ and if $p = \{p_i\}$ is a probability measure such that for some $h \in I$*

$$\sum_{i \neq h} p_i u_i^{-1} (1 - {}_{ih}p_{ii}^*)^{-1} < \infty \tag{1.3}$$

then

$$Q_N = \frac{\sum_{n=1}^{N} \sum_{i \in I} p_i p_{iA}^{(n)}}{\sum_{n=1}^{N} p_{hA}^{(n)}}$$

tends to a finite limit $Q(p, A) \neq 0$, *which is* 1 *if the chain is recurrent.*

Proof. For $n \geqq 1$ we have

$$p_{iA}^{(n)} = {}_h p_{iA}^{(n)} + \sum_{\nu=1}^{n-1} {}_h p_{ih}^{(\nu)} p_{hA}^{(n-\nu)}.$$

Multiplying with p_i, summing over all $i \in I$ and $n = 1, \ldots, N$, reordering the sums, and finally dividing by $\sum_{n=1}^{N} p_{hA}^{(n)}$ we obtain

$$(1.4) \qquad Q_N = \frac{\sum_{i \in I} \sum_{n=1}^{N} p_i\, {}_h p_{iA}^{(n)}}{\sum_{n=1}^{N} p_{hA}^{(n)}} + \frac{\sum_{n=1}^{N-1} p_{hA}^{(n)} \sum_{k=1}^{N-n} \sum_{i \in I} p_i\, {}_h p_{ih}^{(k)}}{\sum_{n=1}^{N} p_{hA}^{(n)}}.$$

A well-known lemma on Nörlund-means (see e. g. CHUNG [*4*]) implies that the second term on the right hand side converges to $\sum_{k=1}^{\infty} \sum_{i \in I} p_i\, {}_h p_{ih}^{(k)}$, which is $= 1$ if the chain is recurrent. The lemma will be proved if we show that under our assumptions the nominator of the first term is finite. From

$${}_h p_{iA}^{(n)} = {}_{ih} p_{iA}^{(n)} + \sum_{t=1}^{n-1} {}_{ih} p_{ii}^{(t)}\, {}_h p_{iA}^{(n-t)} \quad (n \geqq 1)$$

it follows upon summing from 1 to ∞ that ${}_h p_{iA}^{*}$ satisfies the equation

$${}_h p_{iA}^{*} = {}_{ih} p_{iA}^{*} + {}_{ih} p_{ii}^{*}\, {}_h p_{iA}^{*}.$$

If $i \neq h$, then $(1 - {}_{ih} p_{ii}^{*}) \neq 0$. Hence for $i \neq h$ we have

$$(1.5) \qquad {}_h p_{iA}^{*} = {}_{ih} p_{iA}^{*} (1 - {}_{ih} p_{ii}^{*})^{-1}.$$

Clearly ${}_{ih} p_{iA}^{*} \leqq {}_i p_{iA}^{*}$. Since u is subinvariant the inequality

$$(1.6) \qquad u_i^{-1} u_j \geqq {}_i p_{ij}^{(1)} + \cdots + {}_i p_{ij}^{(n)}$$

follows by induction.

Altogether we obtain

$$\sum_{i \in I} p_i\, {}_h p_{iA}^{*} \leqq p_h\, {}_h p_{hA}^{*} + \sum_{i \neq h} p_i\, {}_{ih} p_{iA}^{*} (1 - {}_{ih} p_{ii}^{*})^{-1} \leqq$$
$$\leqq p_h u_h^{-1} u(A) + \sum_{i \neq h} p_i u_i^{-1} (1 - {}_{ih} p_{ii}^{*})^{-1} < \infty.$$

For any $i \neq h$ $(1 - {}_{ih} p_{ii}^{*})$ is the probability of visiting h before or at the time of the first return to i, given a start at i. This probability is denoted by b_{ih}. For $i = h$ we put $b_{ih} = 1$.

Proposition 1.2. *Let* I *consist of one ergodic class, which is recurrent. Let* $u = \{u_i\}$ *be the invariant measure (which is uniquely determined up to a constant factor),* $p = \{p_i\}$ *and* $q = \{q_i\}$ *two probability distributions in* I, *and* A, B *two subsets of* I

with $u(A) < \infty$ *and* $u(B) < \infty$. *If there exist two states* $h, k \in I$ *with*

$$\sum_{i \in I} p_i u_i^{-1} b_{ih}^{-1} < \infty \quad \textit{and} \quad \sum_{i \in I} q_i u_i^{-1} b_{ik}^{-1} < \infty \tag{1.7}$$

then

$$Q_N(A, B) = \frac{\sum_{n=1}^{N} \sum_{i \in I} p_i p_{iA}^{(n)}}{\sum_{n=1}^{N} \sum_{i \in I} q_i p_{iB}^{(n)}}$$

tends to $u(A)/u(B)$.

Proof. This is now an immediate consequence of (1.1) and lemma 1.1.

Example 1.1. The most crucial point of condition (1.7) is the occurrence of the factors b_{ih}^{-1}. They may be hard to compute. Since h and k do not appear in the result, it is a natural question to ask, whether b_{ih}^{-1} and b_{ik}^{-1} are only needed for the present proof. Unfortunately this is not so. We shall now exhibit a null-recurrent Markov chain $\{p_{ik}; i, k \in I\}$, a set $A \subsetneqq I$ with $u(A) < \infty$ and two probability distributions $p = \{p_i\}$, $q = \{q_i\}$ with $\sum_{i \in I} p_i u_i^{-1} < \infty$, $\sum_{i \in I} q_i u_i^{-1} < \infty$ for which $Q_N(A, A)$ diverges. In addition p and q will be bounded by u.

Let $I = \{0, \pm 1, \pm 2, \ldots\}$. We shall inductively define several strictly increasing sequences of nonnegative integers: $\{t_n\}$, $\{k_n\}$, $\{b_n\}$, $\{l_n\}$ $(n \geqq 0)$ starting with $t_0 = k_0 = b_0 = l_0 = 0$. Furthermore we shall need two sequences of real numbers $\{\alpha_n, n \geqq 1\}$ and $\{\beta_n, n \geqq 1\}$ with $0 < \alpha_n, \beta_n < 1$. The reader is invited to visualize the state space rearranged in the following way: The states $k_{n-1} + 1 = j_n, j_n + 1, \ldots, k_n$ form a column with top k_n and base j_n $(n \geqq 1)$. It will be called the n-th column to the right of 0. The states $-j_n$, $-(j_n + 1)$, $\ldots$, $-k_n$ form the n-th column to the left of 0, $-k_n$ is its top and $-j_n$ its base. For short let $C_{n*} = \{j_n, \ldots, k_n\}$, $C_{*n} = \{-j_n, \ldots, -k_n\}$ and $C_n = C_{n*} \cup C_{*n}$ be the right column, the left column and their union respectively. For $n \geqq 1$ let $b_n = j_n + t_n$, $l_n = b_n + t_{n-1} + 1$ and $k_n = l_n + t_n$. α_n, β_n and t_n will be specified at the n-th step of the construction. We shall put $A_n = \{b_0, \ldots, b_n\}$ and $A = \bigcup_{n=0}^{\infty} A_n$.

Now we define the transition probabilities in terms of α_n and β_n: Let $p_{j_n,0} = p_{-j_n,0} = 1$. In the left columns the mass moves downwards: for any j with $-k_n \leqq j < -j_n$ let $p_{j,j+1} = 1$. In the right columns it moves downwards with exception of b_n: for any j with $j_n < j < b_n$ or $b_n < j \leqq k_n$ let $p_{j,j-1} = 1$; finally put $p_{b_n,b_{n-1}} = 1 - \beta_n$, $p_{b_n,l_n} = \beta_n$, $p_{00} = 1/2$, $p_{0,k_n} = \alpha_n 2^{-(n+1)}$ and $p_{0,-k_n} = (1 - \alpha_n) 2^{-(n+1)}$. Then all other transition probabilities p_{ik} necessarily are 0.

The initial distributions $p = \{p_i\}$ and $q = \{q_i\}$ are defined by the equations: For $n \geqq 2$ even $p_{l_n} = 2^{-2n}$ and $q_{l_n} = 0$, for $n \geqq 1$ odd $p_{l_n} = 0$ and $q_{l_n} = 2^{-2n}$, furthermore $p_{l_0} = 1 - \sum_{n=1}^{\infty} p_{l_n}$; $q_{l_0} = 1 - \sum_{n=1}^{\infty} q_{l_n}$, and $p_i = q_i = 0$ for all $i \notin \{l_n, n \geqq 0\}$.

n-th step: Let $G_n = \bigcup_{\nu \geqq n} C_\nu$ and $F_n = I - G_n$. At this point α_ν, β_ν, t_ν, l_ν, and k_ν are well-defined for $\nu = 1, \ldots, n-1$. Therefore the set G_n and the taboo-probabilities ${}_{G_n}p_{ik}^{(t)}$ are well-defined for all $i, k \in F_n$. Necessarily $p_{0, G_n} = 2^{-n}$

and this implies

$$y_n^{(s)} = \sup_{j \in F_n} {}_{G_n}p_{j,A_{n-1}}^{(s)} \to 0 \quad (s \to \infty). \tag{1.8}$$

Let $s_n \geqq t_{n-1}$ be so large that $s \geqq s_n$ implies $y_n^{(s)} < 2^{-(2n+1)} n^{-1} (t_{n-1} + 2)^{-1}$. Now it is possible to choose natural numbers t_n and m_n of the form $t_n = m_n(t_{n-1} + 2)$ such that

$$m_n \cdot 2^{-2n} \geqq 1 + n(s_n + m_n \cdot 2^{-(2n+1)} n^{-1}).$$

This determines k_n, l_n, b_n as well as t_n.

Now $3/4 < \beta_n < 1$ is chosen so close to 1 that

$$m_n \cdot \beta_n^{m_n} \cdot 2^{-2n} > n(s_n + m_n 2^{-(2n+1)} n^{-1}).$$

Finally $0 < \alpha_n < 1$ is determined in such a way that the invariant measure assigns probability 2^{-n} both to l_n and b_n. We shall show that this can be achieved by putting $\alpha_n = 2(1 - \beta_n)$. Now the Markov chain is completely defined.

It follows from the definitions of the transition probabilities that it is recurrent and has one ergodic class only. Hence there exists an invariant measure $u = \{u_i\}$ which is uniquely determined if we put $u_0 = 1$. It is easily checked, that $u_{k_n} = u_{k_n - 1} = \cdots = u_{l_n+1} = \alpha_n 2^{-(n+1)}$, $u_{l_n} = u_{l_n - 1} = u_{b_n} = \alpha_n 2^{-(n+1)} + \beta_n 2^{-n} = 2^{-n}$, $u_{b_n - 1} = \cdots = u_{j_n} = \alpha_n 2^{-(n+1)}$ and $u_i = (1 - \alpha_n) 2^{-(n+1)}$ for $i \in C_{*n}$. This implies $u(A) = \sum_{n=0}^{\infty} u_{b_n} = \sum_{n=0}^{\infty} 2^{-n} < \infty$, $\sum_{i \in I} p_i u_i^{-1} = \sum_{n=0}^{\infty} p_{l_n} u_{l_n}^{-1} \leqq \sum_{n=0}^{\infty} 2^{-2n} 2^n < \infty$ and $\sum_{i \in I} q_i u_i^{-1} < \infty$ by the same inequality. It remains to show, that $Q_N(A, A)$ diverges.

For this we consider first the case $N = t_n$ with $n \geqq 2$ even: For $r \geqq 0$ and $k = (t_{n-1} + 1) + r(t_{n-1} + 2)$ we have $p_{l_n, b_n}^{(k)} \geqq \beta_n^r$. Hence

$$\begin{aligned} \sum_{t=0}^{t_n} \sum_{i \in I} p_i p_{iA}^{(t)} &\geqq \sum_{t=0}^{t_n} p_{l_n} p_{l_n, b_n}^{(t)} \geqq \sum_{r=0}^{m_n - 1} 2^{-2n} \beta_n^r \geqq \\ &\geqq m_n \beta_n^{m_n} 2^{-2n} > n(s_n + m_n 2^{-(2n+1)} n^{-1}). \end{aligned} \tag{1.9}$$

Next we estimate the denominator: For any $n_1 > n$ we have $l_{n_1} - b_{n_1} > t_n$, so that no mass coming from l_{n_1} can reach A before time $t_n + 1$. This and $q_{l_n} = 0$ together imply

$$\sum_{t=0}^{t_n} \sum_{i \in I} q_i p_{iA}^{(t)} = \sum_{t=0}^{t_n} \sum_{\nu=0}^{n-1} q_{l_\nu} p_{l_\nu, A}^{(t)}. \tag{1.10}$$

Furthermore for any $\nu \geqq n$ we have $k_\nu - b_\nu > t_n$, so that no mass entering G_n from state 0 can reach A before time $t_n + 1$. Hence

$$\sum_{t=0}^{t_n} \sum_{\nu=0}^{n-1} q_{l_\nu} p_{l_\nu, A}^{(t)} = \sum_{t=0}^{t_n} \sum_{\nu=0}^{n-1} q_{l_\nu} {}_{G_n}p_{l_\nu, A_{n-1}}^{(s)}. \tag{1.11}$$

From the choice of s_n we have

$$\sum_{t=0}^{t_n} \sum_{\nu=0}^{n-1} q_{l_\nu} {}_{G_n}p_{l_\nu, A_{n-1}}^{(t)} \leqq s_n + \sum_{t=s_n}^{t_n} y_n^{(t)} \leqq s_n + m_n 2^{-(2n+1)} n^{-1}. \tag{1.12}$$

The inequalities (1.9)—(1.12) imply $Q_{t_n} > n$. In the same way it follows for $n \geqq 1$ odd, that $Q_{t_n} < n^{-1}$. Hence $\liminf Q_t = 0$ and $\limsup Q_t = \infty$. This also implies that the Markov chain is null-recurrent, since $Q_N(A, A)$ converges for chains with positive states.

Remarks. (1) One might also be interested in simpler examples, even if they show somewhat less. Some such examples can be obtained using similar ideas. If it shall only be shown, that (1.1) and (1.2) do not possess a mutual generalization, one can dispense with the numbers β_n and put $p_{b_n, b_n-1} = 1$. $A \cap C_{n^*}$ must then consist of sufficiently many states $\{b_n, \ldots, b_n - x_n\}$ with $x_n \leqq t_n$ and both x_n and t_n sufficiently large. α_n must then be determined in such a way, that $u(A \cap C_{n^*}) < < 2^{-n}$. This construction is not appreciably much simpler. It has, however, another advantage: the numbers b_{i0} are then all $= 1$ and it follows that also the factor u_i^{-1} in (1.7) is essential. (2) The method indicated in remark 1 can also be used in order to show that $u(A) < \infty$ is an indispensable condition in lemma 1.1, even if

$$\sum_{i \in I} p_i u_i^{-1} b_{ih}^{-1} < \infty \quad \text{and} \quad \sum_{i \in I} q_i u_i^{-1} b_{ih}^{-1} < \infty .$$

This is of interest in view of the fact that for recurrent chains

$$\frac{\sum_{n=1}^{N} p_{iA}^{(n)}}{\sum_{n=1}^{N} p_{kA}^{(n)}} \to 1 \tag{1.13}$$

for *all* $A \subseteqq I$. (1.13) is a consequence of the inequality

$$\liminf_{N \to \infty} \left(\frac{\sum_{n=1}^{N} \sum_{i \in I} p_i p_{iA}^{(n)}}{\sum_{n=1}^{N} p_{kA}^{(n)}} \right) \geqq 1 \tag{1.14}$$

which follows from (1.4).

(3) For recurrent *random walks* all sets of finite invariant measure are finite, so that all results concerning sets A with $u(A) < \infty$ follow from the pointwise results. The ratios $Q_N(A, A)$ need not converge for arbitrary sets $A \subseteqq I$, and probability measures $\{p_i\}$ and $\{q_i\}$. An example may be given using the coin-tossing random walk. It is omitted, since its description is almost half as complicated as that of example 1.1 and example 1.1 shows so much more for Markov chains.

2. Markov Processes

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space. All sets and functions introduced are assumed to be measurable. Sets as well as functions are identified if they coincide almost everywhere. A^c denotes the complement of a set $A \subseteqq X$, 1_A is the indicator function of A. $\Phi = \Phi(\mu)$ denotes the Banach space of finite signed measures φ, absolutely continuous with respect to μ: $\varphi \ll \mu$. For any function f on X the set $\{x \in X\colon f \neq 0\}$ will be called the support of f (supp(f)) and for $\varphi \in \Phi$ we put supp(φ) = supp$(d\varphi/d\mu)$. For any function f and any set function φ define

$$\begin{aligned} \boldsymbol{I}_A f &= 1_A f, \\ \varphi \boldsymbol{I}_A(B) &= \varphi(A \cap B). \end{aligned} \tag{2.1}$$

Let $\boldsymbol{P}$ denote a positive contraction in $\Phi(\mu)$, i.e. a linear operator in Φ with $\varphi \boldsymbol{P} \geqq 0$ for all $\varphi \geqq 0$ and with $\|\boldsymbol{P}\| = \sup_{\|\varphi\| \leqq 1} \|\varphi \boldsymbol{P}\| \leqq 1$. The dual $\boldsymbol{P}^*$ of $\boldsymbol{P}$ is a positive contraction in $\mathfrak{L}_\infty(\mu): f \to \boldsymbol{P}^* f$. A substochastic kernel $p(x, A)$ is called nonsingular if $\mu(A) = 0$ implies $p(x, A) = 0$. It then induces a positive contraction $\boldsymbol{P}$ in Φ by

$$(2.2) \qquad (\varphi \boldsymbol{P})(A) = \int p(x, A)\, d\varphi \quad (A \in \mathscr{F},\ \varphi \in \Phi)$$

and its dual by

$$(\boldsymbol{P}^* f)(x) = \int f(y)\, p(x, dy) \quad (f \in \mathfrak{L}_\infty).$$

In particular: A measurable point-mapping $\boldsymbol{T}$ in X is called nonsingular, if $\varphi(A) = 0$ $(\varphi \in \Phi)$ implies $\varphi \boldsymbol{P}(A) = \varphi(\boldsymbol{T}^{-1}A) = 0$ $(A \in \mathscr{F},\ \varphi \in \Phi)$. $\boldsymbol{P}$ is then induced by $p(x, A) = 1_{\boldsymbol{T}^{-1}A}$. A σ-finite measure $\lambda \ll \mu$ is called (sub)-invariant or $\boldsymbol{P}$-(sub)-invariant if $\int \boldsymbol{P}^* f\, d\lambda (\leqq) \int f\, d\lambda$ for all $f \in \mathfrak{L}_\infty^+ = \{0 \leqq f \in \mathfrak{L}_\infty\}$. A set $A \in \mathscr{F}$ is called invariant if for any $0 \leqq \varphi \in \Phi$ with $\varphi(A^c) = 0$ also $\varphi \boldsymbol{P}(A^c) = 0$. As usual C and D denote the conservative and dissipative part of X, see e.g. [*10*], [*17*], [*19*]. If $X = C$ the operator is called conservative. $\boldsymbol{P}$ is called ergodic if for any invariant set A, $\mu(A) = 0$ or $\mu(A^c) = 0$. P denotes the positive part of the state space X, i.e. the maximal carrier of finite invariant measures, see NEVEU [*20*], KRENGEL [*16*].

$\Phi(\mu)$ is isomorphic to $\mathfrak{L}_1(\mu)$ by the Radon-Nikodym theorem and $\boldsymbol{P}$ thereby induces an isomorphic operator in $\mathfrak{L}_1(\mu)$, which also will be denoted by $\boldsymbol{P}$ and acts from the right side.

Theorem 2.1. *Let $\lambda \geqq 0$ with $\lambda \ll \mu$ be a σ-finite $\boldsymbol{P}$-subinvariant measure, $S = \operatorname{supp}(\lambda)$, $A \subseteqq S$, $\lambda(A) < \infty$ and $A \cap P = 0$. Then for any $\varphi \in \Phi$ we have*

$$(2.3) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \boldsymbol{P}^k(A) = 0\,.$$

For the proof the following simple lemma is needed:

Lemma 2.1. *If $0 \leqq \lambda \ll \mu$ is a σ-finite subinvariant measure, then $S = \operatorname{supp}(\lambda)$ is invariant.*

Proof. Let $0 \leqq \psi \in \Phi$ and $\operatorname{supp}(\psi) \subseteqq S$. Further let $f \geqq 0$ be the density of ψ with respect to λ and let $K_\varepsilon \geqq 0$ be so large that $\int (f - (f \wedge K_\varepsilon))\, d\lambda < \varepsilon > 0$. Then

$$\begin{aligned} 0 &\leqq \int 1_{S^c}\, d(\psi \boldsymbol{P}) = \int (\boldsymbol{P}^* 1_{S^c})\, d\psi = \\ &= \int (\boldsymbol{P}^* 1_{S^c})(f \wedge K_\varepsilon)\, d\lambda + \int (\boldsymbol{P}^* 1_{S^c})(f - (f \wedge K_\varepsilon))\, d\lambda \leqq \\ &\leqq K_\varepsilon \int 1_{S^c}\, d(\lambda \boldsymbol{P}) + \varepsilon \leqq K_\varepsilon \int 1_{S^c}\, d\lambda + \varepsilon = \varepsilon\,. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary it follows that $\operatorname{supp}(\psi \boldsymbol{P}) \subseteqq S$.

Proof of theorem 2.1. We may and do assume $\varphi \geqq 0$. We define inductively

$$\begin{aligned} \varphi_0 &= \varphi \boldsymbol{I}_S, & \varphi_0^* &= \varphi \boldsymbol{I}_{S^c}, \\ \varphi_{k+1} &= \varphi_k^* \boldsymbol{P} \boldsymbol{I}_S, & \varphi_{k+1}^* &= \varphi_k^* \boldsymbol{P} \boldsymbol{I}_{S^c}. \end{aligned}$$

Then

$$(2.4) \qquad \varphi \boldsymbol{P}^N = \varphi_0 \boldsymbol{P}^N + \cdots + \varphi_{N-1} \boldsymbol{P} + \varphi_N + \varphi_N^*$$

follows by induction and $\|\varphi_{k+1}\| + \|\varphi_{k+1}^*\| \leqq \|\varphi_k^*\|$ implies $\sum_{k=0}^{\infty} \|\varphi_k\| \leqq \|\varphi\|$. Let $\varepsilon > 0$ be given. Choose $N = N_\varepsilon$ so large that $\sum_{k=N+1}^{\infty} \|\varphi_k\| < \varepsilon/4$. We shall consider

$\psi = \varphi \boldsymbol{P}^N \boldsymbol{I}_S$. Since λ is equivalent to $\mu \boldsymbol{I}_S$ we have

$$\| \psi - (\psi \wedge K_\varepsilon \lambda) \| < \varepsilon/8$$

for sufficiently large K_ε. Let $\psi_1 = \psi \wedge K_\varepsilon \lambda$ and $\psi_2 = \psi - \psi_1$. By theorem 1 of [*16*] P^c is a countable disjoint union $P^c = \bigcup_{k=1}^{\infty} X_k$ of sets X_k with the property $n^{-1} \sum_{i=0}^{n-1} \varphi \boldsymbol{P}^i (X_k) \to 0$. Let the integer r be so large that $\lambda(A \cap (X_i \cup \cdots \cup X_r)^c) < < \varepsilon/16\, K_\varepsilon$. Let $M = M_\varepsilon$ be $\geqq N_\varepsilon$ and so large that $N_\varepsilon \| \varphi \| M_\varepsilon^{-1} < \varepsilon/16$ and for $n \geqq M_\varepsilon$

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi \boldsymbol{P}^i (X_k) < \varepsilon/16 r \quad (k = 1, \ldots, r).$$

It follows from the invariance of S that

$$\| \varphi \boldsymbol{P}^N \boldsymbol{I}_{S^c} \boldsymbol{P}^k \boldsymbol{I}_S \| = \left\| \sum_{j=0}^{k-1} \varphi_{N+k-j} \boldsymbol{P}^j \right\| \leqq \sum_{j=N+1}^{\infty} \| \varphi_j \| < \varepsilon/4$$

for all $k \geqq 0$. Finally we observe that $\varrho \leqq \lambda$ implies $\varrho \boldsymbol{P} \leqq \lambda$; in fact, denoting by f_ϱ the density of ϱ with respect to μ, we have:

$$\begin{aligned} \varrho \boldsymbol{P}(A) = \int 1_A\, d(\varrho \boldsymbol{P}) = \int 1_A (f_\varrho \boldsymbol{P})\, d\mu = \int (\boldsymbol{P}^* 1_A) f_\varrho\, d\mu \leqq \\ \leqq \int \boldsymbol{P}^* 1_A\, d\lambda \leqq \int 1_A\, d\lambda = \lambda(A). \end{aligned}$$

Let $A_r = A \cap (X_1 \cup \cdots \cup X_r)^c$. For $n \geqq M_\varepsilon$ the following inequalities holds:

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} \varphi \boldsymbol{P}^i (A) &\leqq r \cdot \varepsilon/16 r + \frac{1}{n} \sum_{i=0}^{n-1} \varphi \boldsymbol{P}^k (A_r) \leqq \\ &\leqq \varepsilon/16 + \frac{1}{n} \sum_{i=0}^{N-1} \| \varphi \| + \frac{1}{n} \sum_{i=0}^{n-N-1} \varphi \boldsymbol{P}^N \boldsymbol{P}^i (A_r) \leqq \\ &\leqq \varepsilon/8 + \frac{1}{n} \sum_{i=0}^{n-N-1} \psi \boldsymbol{P}^i (A_r) + \frac{1}{n} \sum_{i=0}^{n-N-1} \varphi \boldsymbol{P}^N \boldsymbol{I}_{S^c} \boldsymbol{P}^i (A_r) \leqq \\ &\leqq \varepsilon/8 + \frac{1}{n} \sum_{i=0}^{n-N-1} \| \psi_2 \boldsymbol{P}^k \| + \frac{1}{n} \sum_{i=0}^{n-N-1} \psi_1 \boldsymbol{P}^i (A_r) + \varepsilon/4 \leqq \\ &\leqq \varepsilon/8 + \varepsilon/8 + K_\varepsilon \cdot \varepsilon/16\, K_\varepsilon + \varepsilon/4 < \varepsilon. \end{aligned}$$

An alternative proof of theorem 2.1 may be given using pointwise ergodic theory and a uniform integrability argument. Let f_φ be the density of φ with respect to μ. It follows from a very general ergodic theorem of Chacon [*2*] (or by an extension of the Hopf-Dunford-Schwartz ergodic theorem [*12*], [*9*]) that $n^{-1} \sum_{k=0}^{n-1} f_\varphi \boldsymbol{P}^k$ converges a.e. in S. Proposition 1 of [*16*] implies that the limit is 0 in P^c. One uses the following lemma:

Lemma 2.2. *Let $\lambda \geqq 0$ with $\lambda \ll \mu$ be a σ-finite $\boldsymbol{P}$-sub invariant measure. Let $A \subsetneqq \operatorname{supp}(\lambda)$ and $\lambda(A) < \infty$. Then for any $\varphi \in \Phi$ the sequence $\{1_A (f_\varphi \boldsymbol{P}^k)\}$ is uniformly integrable.*

(We call a sequence $\{f_k\}$ uniformly integrable, if for any $\varepsilon > 0$ there is a $g_\varepsilon \in \mathfrak{L}_1^+(\mu)$ such that $\int |f_k| - (|f_k| \wedge g_\varepsilon)\, d\mu < \varepsilon$ for all k. This definition is equivalent to the standard one if μ is finite, but even in that case it simplifies the proof.) The proof of lemma 2.2 makes use of the fact that $\varrho \leqq \lambda$ implies $\varrho \boldsymbol{P} \leqq \lambda$. It is similar to the proof of lemma 4 in [*16*], which treats the special case of a finite invariant measure λ, and is therefore omitted.

We now prove a theorem on comparative averaging of *two* measures. $\mathscr{I}$ denotes the σ-field of invariant subsets of C.

Theorem 2.2*. *Let* $\boldsymbol{P}$ *be a positive conservative contraction in* Φ *and* $\varphi, \psi \in \Phi$. *If (and clearly only if)* $\varphi(F) = \psi(F)$ *for all* $F \in \mathscr{I}$ *then*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} (\varphi - \psi)\boldsymbol{P}^k \right\| = 0 . \tag{2.5}$$

Remark. For point-mappings theorem 4 was proved by Mrs. DOWKER [*7*]. The present elegant proof is due to NEVEU. (The authors original proof was based on lemma 1 of [*3*].)

Proof. Let $\boldsymbol{I}$ denote the identity operator. $\left\| n^{-1} \sum_{k=0}^{n-1} \varrho \boldsymbol{P}^k \right\|$ tends to 0 if ϱ belongs to the closure of the image $\Phi(\boldsymbol{P} - \boldsymbol{I})$ of Φ under $(\boldsymbol{P} - \boldsymbol{I})$. This is the case iff ϱ is orthogonal to all $\boldsymbol{P}$-invariant elements of $\mathfrak{L}_\infty$. However, the $\boldsymbol{P}$-invariant elements in $\mathfrak{L}_\infty$, by a theorem of NEVEU [*19*, p. 179] are just those, which are measurable with respect to $\mathscr{I}$. By the usual approximation the condition: $(\varphi - \psi)(A) = 0$ for all $A \in \mathscr{I}$ implies $\int f d\varphi = \int f d\psi$ for all $f \in \mathfrak{L}_\infty(\mathscr{I})$.

Remark. For aperiodic, recurrent and ergodic Markov chains a theorem of OREY [*21*] proves (2.5) even without averaging; see also BLACKWELL and FREEDMAN [*1*].

The question for ratio limit theorems has already been negatively answered in section 1. Here we want to point out that one can use example 1.1 to give examples also in the deterministic case of a point-mapping. This considerably strengthens a counterexample of DOWKER and ERDÖS [*8*] to a conjecture of HUREWICZ.

Let $\boldsymbol{T}$ be a point-mapping in the σ-finite measure space $(X, \mathscr{F}, \mu)$. Assume $\boldsymbol{T}$ to be $1 - 1$, onto, $\boldsymbol{T}$ and $\boldsymbol{T}^{-1}$ measurable, and nonsingular, $\boldsymbol{T}$ ergodic and conservative, μ is assumed to be nonatomic. HUREWICZ asked, whether for any two normalized measures φ_1, φ_2, which are equivalent to μ the ratio

$$\frac{\sum_{k=0}^{n} \varphi_1(\boldsymbol{T}^{-k} F)}{\sum_{k=0}^{n} \varphi_2(\boldsymbol{T}^{-k} F)} \tag{2.6}$$

tends to 1 for all $F \in \mathscr{F}$. DOWKER and ERDÖS showed that this is only so, if $\boldsymbol{T}$ possesses a finite invariant measure, which is equivalent to μ. We shall show, that (2.6) may diverge even if φ_1 and φ_2 are bounded by a σ-finite invariant measure μ and $\mu(F) < \infty$.

We shall only indicate the method: One first shows that in example 1.1 $p = \{p_i\}$ and $q = \{q_i\}$ may be modified in such a way that all p_i and q_i are strictly positive. Let X^* be the unilateral infinite product space with coordinate spaces $I = \{0, \pm 1, \ldots\}$ = state space from example 1.1. The measures p, q and u induce measures φ_1^*, φ_2^* and μ^* by the theorem of C. IONESCU-TULCEA [*13*]. Some argu-

* Theorem 2 has independently been proved by SUCHESTON (to appear).

ments of Moy [17, sec. IV] show that φ_1^*, φ_2^* and μ^* are equivalent to each other. Let f_1^* and f_2^* be the densities of φ_1^* and φ_2^* with respect to μ^*. Let X be the bilateral product space and μ the measure induced by u in X. If $\boldsymbol{T}$ is the shift, then $\boldsymbol{T}$ is conservative and ergodic by a theorem of Harris and Robbins [11]. The densities f_1 and f_2 are defined by $f_i(\ldots x_{-k}, \ldots, x_{-1}, x_0, x_1, \ldots) = f_i^*(x_0, x_1, \ldots)$. The corresponding measures φ_i with $d\varphi_i = f_i d\mu$ are then bounded by μ since p and q were bounded by u. The example is completed if we take $F = \{x = (\ldots, x_0, x_1, \ldots): x_0 \in A\}$, where A is the set constructed in example 1.1. In fact

$$\varphi_1(\boldsymbol{T}^{-k}F) = \sum_{i\in I} p_i p_{iA}^{(k)} \text{ and } \quad \varphi_2(\boldsymbol{T}^{-k}F) = \sum_{i\in I} q_i p_{iA}^{(k)}.$$

The end of this section is devoted to a proof of a pointwise rather than global limit theorem:

A sequence $\{f_k\}$ of measurable functions on X is said to converge stochastically to a function g if $\mu(A \cap \{x: |f_k - g| > \varepsilon\}) \to 0$ for any $\varepsilon > 0$ and any set A with $\mu(A) < \infty$. (This is one of two standard definitions which are equivalent if μ is finite.) Let $\boldsymbol{P}$ be an arbitrary contraction in $\mathfrak{L}_1(\mu)$, not necessarily positive, and let $|\boldsymbol{P}|$ be its modulus (see [27]). The maximal carrier of $|\boldsymbol{P}|$-invariant μ-integrable functions in X is denoted by P. We may call P the positive part of X and $N = X - P$ the null part of X (with respect to $\boldsymbol{P}$).

Theorem 2.3. *(Stochastic ergodic theorem)*[1]. *If $\boldsymbol{P}$ is a linear operator in $\mathfrak{L}_1(\mu)$ with $\|\boldsymbol{P}\| \leqq 1$ then for any $f \in \mathfrak{L}_1$ the sequence $f_n = \frac{1}{n}\sum_{k=0}^{n-1} f\boldsymbol{P}^k$ converges stochastically. The limit vanishes in the null-part N of X.*

Remark. The sequence f_n need not converge almost everywhere as shown by Chacon [28], see also A. Ionescu-Tulcea [29]. However, for any contraction $\boldsymbol{P}$ in $\mathfrak{L}_1$ there exists a matrix summation method M, which is stronger than the Cesàro method and which enforces almost everywhere M-convergence of $\{f\boldsymbol{P}^k\}$ for all $f \in \mathfrak{L}_1$. This was shown for $\boldsymbol{P} \geqq 0$ by the author [16]. (The assumption $\boldsymbol{P} \geqq 0$ used in [16] is unnecessary if one uses $|\boldsymbol{P}|$ in the proof.)

Proof of theorem 2.3. Let $\bar{f} \in \mathfrak{L}_1^+$ be $|\boldsymbol{P}|$-invariant and $\operatorname{supp}(\bar{f}) = P$, and let $f \in \mathfrak{L}_1$, $\varepsilon > 0$, $\delta > 0$ and A with $\mu(A) < \infty$ be given. Corollary 2 of Chacon-Krengel [27] implies that $\sum_{k=0}^{n-1} f\boldsymbol{P}^k / \sum_{k=0}^{n-1} \bar{f}|\boldsymbol{P}|^k = f_n/\bar{f}$ converges a. e. in P, hence f_n converges stochastically in P. Next we show stochastic convergence of f_n to 0 in N. Since $|f_n| \leqq \frac{1}{n}\sum_{k=0}^{n-1}|f|\,|\boldsymbol{P}|^k$ we may assume $f \geqq 0$ and $\boldsymbol{P} \geqq 0$. Again we use the fact that N is a countable disjoint union $N = \bigcup_{i=1}^{\infty} X_i$ with the property

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi\boldsymbol{P}^k(X_i) = 0 \qquad (\varphi \in \Phi,\ i = 1, 2, \ldots).$$

Choose n_0 so large that $\mu\left(A \cap \bigcup_{i=n_0+1}^{\infty} X_i\right) < \delta/2$. Let $N_0 = \bigcup_{i=1}^{n_0} X_i$ and let φ be the

[1] If μ is σ-finite, then it may happen that $X = N$ and for some $\varepsilon > 0$ $(\mu\{x: |f_n| > \varepsilon\})$ does not tend to 0. Thus the theorem would be wrong for the stronger version of stochastic convergence (convergence in measure).

measure with Radon-Nikodym derivative f. For n_1 sufficiently large and $n \geqq n_1$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi \boldsymbol{P}^k(N_0) < \varepsilon \cdot \delta/2 .$$

Hence

$$\frac{\varepsilon \cdot \delta}{2} \geqq \int_{N_0 \cap \{|f_n| \leqq \varepsilon\}} f_n \, d\mu + \int_{N_0 \cap \{|f_n| > \varepsilon\}} f_n \, d\mu \geqq \varepsilon \cdot \mu(N_0 \cap \{|f_n| > \varepsilon\} .$$

Hence

$$\mu(A \cap N \cap \{|f_n| > \varepsilon\}) \leqq \mu\left(A \cap \bigcup_{i=n_0+1}^{\infty} X_i\right) + \mu(N_0 \cap \{|f_n| > \varepsilon\}) < \delta .$$

Since $\delta > 0$ was arbitrary the theorem follows.

Remarks. (1) For $\boldsymbol{P} \geqq 0$ the limit can be identified also in P, see e.g. [*16*]. (2) The application of corollary 2 of [*27*] can be avoided. (This is desirable since that corollary depends on a very deep theorem of CHACON [*2*]). Using $\bar{f}$ one may reduce the proof of almost everywhere convergence in P to the proof of the ergodic theorem in the case $\|\boldsymbol{P}\| \leqq 1$, $\|\boldsymbol{P}\|_\infty \leqq 1$; for this technique see HOPF [*12*], NEVEU [*19*].

3. On the Strong Ratio Limit Property

Following KINGMAN and OREY [*23*] we say that a recurrent, aperiodic and irreducible Markov chain has the *strong ratio limit property* if there exist positive numbers $\{u_i, i \in I\}$ such that

$$\lim_{t \to \infty} \frac{p_{ij}^{(t+m)}}{p_{kh}^{(t)}} = \frac{u_j}{u_h}, \quad m = 0, 1, \ldots, i \in I,\ j \in I,\ k \in I,\ h \in I . \tag{3.1}$$

(We shall not need the more general definition of PRUITT [*25*], since we restrict ourselves to the recurrent case.) KINGMAN and OREY, and OREY [*25*] showed that both the following conditions are sufficient for the strong ratio limit property to hold:

(I) There exists an N and an $\varepsilon > 0$ such that

$$\inf_{i \in I} \sum_{n=1}^{N} p_{ii}^{(n)} \geqq \varepsilon .$$

(II) The chain is *reversible*, i.e.: there exists a sequence $\{\varrho_i, i \in I\}$ of positive constants such that $\varrho_i p_{ij} = \varrho_j p_{ji}$ for every i and j. (It follows immediately that $\varrho = \{\varrho_i\}$ is an invariant measure.)

We are interested in the question, whether in (3.1) the probabilities of visits in single states may be replaced by probabilities of visits in sets of finite invariant measure. The answer will be negative, though the crucial part of the implication "(I) implies (3.1)" remains valid for probabilities of visits in arbitrary sets and for arbitrary initial distributions.

Proposition 3.1. *Let (p_{ij}) be the matrix of transition probabilities of a recurrent, aperiodic and irreducible Markov chain. If it satisfies condition*

$$\inf_{i \in I} p_{ii} > 0 , \tag{I*}$$

then for any $A \subseteqq I$ *and any initial distribution* $p = \{p_i\}$ $(p_i \geqq 0, \sum_{i \in I} p_i = 1)$ *it follows that*

$$\lim_{t \to \infty} \frac{\sum_{i \in I} p_i p_{iA}^{(t+1)}}{\sum_{i \in I} p_i p_{iA}^{(t)}} = 1.$$

Proof. The proof of Kingman and Orey [23] of the special case "$A = \{j\}$, $\{p_i\}$ = point-mass in one point" remains valid. It follows the lines of the proof of Chung and Erdös [22] of the strong ratio limit property for recurrent random walks.

Corollary 3.1. *Under the assumptions of proposition 3.1*

$$\lim_{t \to \infty} \frac{p_{iA}^{(t)}}{p_{jA}^{(t)}} = 1 \tag{3.1}$$

for all $i, j \in I$ *and* $A \subseteqq I$.

The proof is obtained in a straightforward way from the usual first-entrance decomposition.

Corollary 3.2. *Let* $p = \{p_i\}$ *be any initial distribution, then, under the assumptions of proposition* 3.1 *we have*

$$\lim_{t \to \infty} \frac{\sum_{i \in I} p_i p_{ij}^{(t)}}{\sum_{i \in I} p_i p_{ik}^{(t)}} = \frac{u_j}{u_k}$$

for all $j, k \in I$.

Proof. The fact that the limes inferior of the above ratios is $\geqq u_j/u_k$ follows using the decomposition $p_{ij}^{(t)} = {}_k p_{ij}^{(t)} + \sum_{s=1}^{t-1} p_{ik}^{(t-s)} \, {}_k p_{kj}^{(s)}$. The other inequality follows by symmetry.

Example 3.1. We now describe a null-recurrent, irreducible Markov chain with the properties (I*) and (II) (and hence (I)) and such that

$$Q_t(i, A) = \frac{p_{iA}^{(t)}}{p_{ii}^{(t)}}$$

diverges for some state i and some set A of finite invariant measure.

The construction is done inductively. At the n-th step we define two natural numbers a_n and $k_n > a_n$ and a real number α_n with $0 < \alpha_n < 1/2$. The state space I will be the union of $J_0 = \{0, 1, 2, \ldots\}$ and the set J_1 of all ordered pairs (i, j) with $i \in \{1, 2, \ldots\}$ and $0 \leqq j \leqq k_i$. Let

$$p_{00} = 2^{-1}, \; p_{0n} = p_{0,(n,0)} = 2^{-(n+2)} \qquad (n \geqq 1).$$

Further for $n \geqq 1$ let:

$$p_{n0} = \alpha_n, \; p_{nn} = (1 - \alpha_n), \; p_{(n,0),0} = 2^{-2}, \; p_{(n,k_n),(n,k_n)} = 3/4,$$
$$p_{(n,j),(n,j)} = 2^{-1} (0 \leqq j < k_n), \; p_{(n,j),(n,j+1)} = 2^{-2} (0 \leqq j < k_n),$$
$$p_{(n,j),(n,j-1)} = 2^{-2} (0 < j \leqq k_n).$$

Then all other transition probabilities necessarily are 0. A shall be the union of all states (n, a_n) with $n \geqq 1$. Since $\alpha_n \leqq 1/2$ we have $p_{ii} \geqq 1/2$ for all $i \in I$, so that (I*) is satisfied. Condition (II) is easily checked: One defines $\varrho_0 = 1$ and finds all other ϱ_i from the transition probabilities. It follows that $\sum_{i \in A} \varrho_i = 1$, so that A has finite invariant measure.

We shall consider the ratios $Q_t(0, A)$ for certain values $t = t_n$, where $0 = t_0 < t_1 < \cdots$. Before describing the n-th step of the construction we recall some facts about the familiar coin-tossing random walk: Let $S_r = Y_1 + Y_2 + \cdots + Y_r$, where $Y_k (k = 1, 2, \ldots)$ are independent random variables with $\text{Prob}\{Y_k = 1\} = \text{Prob}\{Y_k = -1\} = 1/2$. Let $\tilde{p}(t,l) = \text{Prob}\ \{S_{2t} = 2l, \min_{1 \leqq r \leqq 2t} S_r \geqq 0\}$. The reflection principle yields for all integers $l \geqq 0$

$$\tilde{p}(t, l) = 2^{-2t}\left\{\binom{2t}{t+l} - \binom{2t}{t+l+1}\right\} = 2^{-2t}\binom{2t}{t+l+1} \cdot \frac{2l+1}{t-l}.$$

Let l_t be a sequence of nonnegative integers with $l_t \sim t^{\frac{1}{2}}$ (in the sense that $l_t t^{-\frac{1}{2}} \to 1$). An application of Stirling's formula and two applications of the formula $\lim_{s \to \infty} \left(1 + \frac{\alpha}{s}\right)^s = e^\alpha$ yield

$$\tilde{p}(t, l_t) \sim 2\pi^{-\frac{1}{2}} e^{-1} t^{-1}.$$

Hence

$$\sum_{t=1}^{\infty} \tilde{p}(t, l_t) = \infty. \tag{3.2}$$

We are now ready to describe the n-th step of the construction ($n \geqq 1$): Let $F_n = \{n, n+1, \ldots\}$. At this point α_ν and k_ν are defined for all $\nu < n$. Therefore the taboo-probabilities ${}_{F_n}p_{00}^{(t)}$ are well defined for $t \leqq \min_{\nu \geqq n} k_\nu$. Since we do not know k_ν for $\nu \geqq n$ we let $p_n(t)$ be that value of ${}_{F_n}p_{00}^{(t)}$ which would arise if k_ν were larger than t for all $\nu \geqq n$. Since $p_{0, F_n} = 2^{-(n+1)} > 0$ it follows that

$$\sum_{t=1}^{\infty} p_n(t) < \infty. \tag{3.3}$$

Now observe that the transition probabilities p_{ij} for $i, j \in \{(n, s), 0 \leqq s < k_n\}$ are the two-step transition probabilities of the coin-tossing random walk. Hence, assuming k_n sufficiently large,

$${}_0p_{(n,0),(n,l)}^{(t)} = \text{Prob}\{S_{2t} = 2l, \min_{1 \leqq r \leqq 2t} S_r \geqq -1\} \geqq \tilde{p}(t, l). \tag{3.4}$$

Because of (3.2) and (3.3) we may choose an integer $t_n > t_{n-1}$ so large that $t_n > l_{t_n - 1}$ and

$$\tilde{p}(t_n - 1, l_{t_n - 1}) \geqq 2^{n+3} n\, p_n(t_n). \tag{3.5}$$

Let $a_n = l_{t_n - 1}$ and $k_n = t_n$. Finally for $\nu \geqq n$ we choose real numbers $\alpha_{n\nu}$ with $0 < \alpha_{n\nu} < 1/2$ so small that

$$t_n \sum_{\nu = n}^{\infty} \alpha_{n\nu} < p_n(t_n), \tag{3.6}$$

and we put $\alpha_n = \min_{1 \leqq k \leqq n} \alpha_{kn}$. This completes the construction. Clearly all α_n are strictly positive. Therefore the chain is recurrent. No mass coming from 0 reaches any state (ν, k_ν) with $\nu \geqq n$ before time t_n. This and (3.6) imply

$$p_{00}^{(t_n)} \leqq p_n(t_n) + t_n \sum_{\nu = n}^{\infty} \alpha_\nu < 2\, p_n(t_n). \tag{3.7}$$

The inequalities (3.4), (3.5) and (3.7) yield

$$p^{(t_n)}_{0,(n,a_n)} \geqq p_{(0,(n,0)}\, p^{(t_n-1)}_{(n,0),(n,a_n)} \geqq 2^{-(n+2)}\, \tilde{p}\,(t_n - 1,\, l_{t_n-1}) \geqq 2\, n\, p_n(t_n) > n\, p^{(t_n)}_{00}$$

and hence $Q_t(0, A) > n$. Hence $\limsup_{t\to\infty} Q_t(0, A) = \infty$.

Remarks. (1) Since the chain in example 3.1 is reversible it follows that $p^{(t)}_{0A} = \sum_{a\in A} \varrho_0 p^{(t)}_{0a} = \sum_{a\in A} \varrho_a p^{(t)}_{a0}$. Let $\{p_i\}$ be defined by $p_i = \varrho_i (i \in A)$, $p_i = 0 (i \notin A)$. Then $\{p_i\}$ is an initial distribution which is dominated by the invariant measure. However $\sum_{i\in I} p_i p^{(t)}_{i0}/p^{(t)}_{00}$ diverges.

(2) One may ask whether condition (I) or (II) are strong enough to imply a global ratio limit theorem as discussed in section 1. The author has checked, that example 1.1 can be modified in such a way that it satisfies condition (I*) and hence (I). It seems plausible and easy that the same can be done concerning condition (II), and with (I) and (II) simultaneously. The resulting point mapping as constructed in section 2 in either case is strongly mixing in the sense of KRICKEBERG[2] [24].

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