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Moments of Ladder Variables for Driftless Random Walks*

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1. Introduction

Let X, X_1, $X_2, \ldots$ be i.i.d. random variables such that $EX=0$ and $E|X|>0$. Let $S_n = X_1 + \ldots + X_n$ and define

$$N = \inf\{n \geqq 1: S_n \leqq 0\}. \tag{1}$$

If EX were negative, then ES_N would be finite and $ES_N = (EX)(EN)$ by Wald's lemma. However, since $EX=0$ and $EN=\infty$, what can be said about ES_N? In 1960, Spitzer [3] proved the following remarkable result concerning ES_N:

$$EX^2 < \infty \Rightarrow E|S_N| = (\tfrac{1}{2} EX^2)^{\frac{1}{2}} \exp\left\{\sum_1^\infty n^{-1}(P[S_n > 0] - \tfrac{1}{2})\right\} < \infty. \tag{2}$$

Spitzer's method is analytic in nature and involves certain generating functions associated with the fluctuation theory of the random walk $\{S_n\}$ and a Tauberian argument. His approach was recently extended by Lai [2] to prove that for $k = 1, 2, \ldots$,

$$E|X|^{k+1} < \infty \Rightarrow E|S_N|^k < \infty. \tag{3}$$

(An explicit expression for $E|S_N|^k$ was also obtained in [2], but it is quite complicated for large k.) In this paper, we shall use a probabilistic approach to obtain the following theorem, which is considerably sharper than (3).

Theorem 1. *Let X, X_1, $X_2, \ldots$ be i.i.d. random variables such that $EX=0$. Let $S_n = X_1 + \ldots + X_n$ and define N as in* (1). *Then for every $p > 0$,*

$$E(X^-)^{p+1} < \infty \Rightarrow E|S_N|^p < \infty. \tag{4}$$

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While (3) only considers integral moments of the (weak descending) ladder variable S_N, p in Theorem 1 need not be an integer. Moreover, it is natural to think that the finiteness of $E|S_N|^p$ should be related to the negative tail rather than the positive tail of the distribution of X, and therefore (4) is a significant improvement over (3). The proof of Theorem 1 will be given in Sect. 3. Our method is to replace the driftless random walk $\{S_n\}$ by the negative-drift random walk $\{S_n - \varepsilon n\}$ whose descending ladder variable Z_ε is much easier to handle than the ladder variable S_N. In Sect. 2 we shall obtain some simple estimates for the moments of Z_ε and of a related quantity involving the maximum $M_\varepsilon = \max_{n \geqq 0} (S_n - \varepsilon n)$. These estimates are then applied in Sect. 3 where Theorem 1 is proved by letting $\varepsilon \downarrow 0$.

2. The Maximum and the Descending Ladder Variable of a Negative-drift Random Walk

Throughout this section we shall use the following notations. Let $Y, Y_1, Y_2, \ldots$ be i.i.d. with a common distribution function $F(y)$ such that $EY = -\varepsilon < 0$. Let $U_n = \sum_1^n Y_i$ $(U_0 = 0)$ and define

$$M = \max_{n \geqq 0} U_n, \qquad \tau_- = \inf\{n \geqq 1 : U_n \leqq 0\}, \tag{5}$$

$\#(x)$ = number of strict ascending ladder points (for the random walk $\{U_n\}$) in the interval $(0, x]$,

$$\psi(x) = 1 + E\,\#(x), \qquad x \geqq 0.$$

Since $EY < 0$, $E\tau_- < \infty$. The following properties, which can be proved by relatively simple arguments, are basic results in fluctuation theory:

$$P[M=0] = 1/E\tau_- > 0, \tag{6}$$

$$P[M \leqq x] = P[M=0]\,\psi(x), \qquad x \geqq 0, \tag{7}$$

and

$$P[U_{\tau_-} \leqq x] = \int_{[0,\infty)} F(x-y)\,\psi(dy), \qquad x \leqq 0, \tag{8}$$

(cf. (2.3), (2.6), and (3.7a) in Chapter XII of [1]). An important observation in our proof of Theorem 1 is the following

Lemma 1. *For* $p > 0$,

$$E((M+Y)^-)^p = P[M=0]\,E|U_{\tau_-}|^p, \tag{9}$$

$$P[M=0]\,E(Y^-)^p \leqq E((M+Y)^-)^p \leqq P[M<1]\,\{E(Y^-)^p + E(Y^-)^{p+1}\}. \tag{10}$$

Proof. From (7) and (8), for $x \leqq 0$,

$$P[M=0]\,P[U_{\tau_-} \leqq x] = \int_{[0,\infty)} F(x-y)\,P[M \in dy] = P[M+Y \leqq x].$$

Hence (9) follows. Since $(M+Y)^- \geqq Y^- I_{[M=0]}$, the lower bound in (10) is obvious. To establish the upper bound in (10), we need only note that for $y>0$,

$$\begin{aligned} E[((M+Y)^-)^p \mid Y=-y] &= E((M-y)^-)^p \\ &\leqq \int_{[0,y)} (y-z)^p P[M\in dz] \\ &\leqq y^p \sum_{n=0}^{[y]} P[n\leqq M<n+1] \\ &\leqq y^p(1+y) P[M<1]. \end{aligned}$$

The last relation above follows from

$$P[n\leqq M<n+1] \leqq P[\tau(n)<\infty,\ U_{\tau(n)}<n+1]\, P[M<1],$$
$$\text{where } \tau(n)=\inf\{j\colon U_j\geqq n\} \quad (\inf\emptyset=\infty). \tag{11}$$

3. Proof of Theorem 1

For notational convenience we shall sometimes use Vinogradov's symbol $\ll$ instead of Landau's 0 notation. Making use of (7), we obtain

Lemma 2. *Let $X, X_1, X_2, \ldots$ be i.i.d. with $EX=0$ and $E|X|>0$. Let $S_n=X_1+\ldots+X_n$ $(S_0=0)$. For $\varepsilon>0$, define*

$$M_\varepsilon=\sup_{n\geqq 0}(S_n-\varepsilon n). \tag{12}$$

Then for every fixed $x\geqq 0$, as $\varepsilon\downarrow 0$,

$$P[M_\varepsilon\leqq x]\ll P[M_\varepsilon=0]. \tag{13}$$

Proof. Choose $a>0$ such that $P[X>2a]=\delta>0$. By an argument similar to that in (11), we obtain that

$$\begin{aligned} P[M_\varepsilon\leqq x] &\leqq \sum_{n=0}^{[x/a]} P[na\leqq M_\varepsilon\leqq(n+1)a] \\ &\leqq (1+x/a) P[M_\varepsilon\leqq a]. \end{aligned} \tag{14}$$

Let $U_n(\varepsilon)=S_n-\varepsilon n$ and let $\tau=\inf\{n\colon U_n(\varepsilon)>0\}$ $(\inf\emptyset=\infty)$. Let $\#_\varepsilon$ denote the number of strict ascending ladder points in the interval $(0,a]$ for the random walk $\{U_n(\varepsilon)\}$. We now show that

$$E\#_\varepsilon=0(1) \quad \text{as } \varepsilon\downarrow 0. \tag{15}$$

To prove (15), we obtain by the independence of the successive ladder heights that for $n=1,2,\ldots,$

$$P[\#_\varepsilon\geqq n]\leqq P^n[\tau<\infty,\ U_\tau(\varepsilon)\leqq a]\leqq\{1-P[\tau<\infty,\ U_\tau(\varepsilon)>a]\}^n. \tag{16}$$

By the choice of a, for $0<\varepsilon<a$,

$$P[\tau<\infty, U_\tau(\varepsilon)>a] \geqq P[X_1-\varepsilon>a] \geqq P[X>2a]=\delta>0. \tag{17}$$

From (16) and (17), (15) follows.

By (7) and (15), $P[M_\varepsilon \leqq a] \ll P[M_\varepsilon=0]$. Therefore in view of (14), the desired conclusion (13) follows immediately.

We now make use of Lemmas 1 and 2 to give a simple probabilistic proof of Theorem 1.

Proof of Theorem 1. If $X=0$ a.s., then $N=1$ and $S_N=0$ a.s. Now assume that $P[X \neq 0]>0$ and that $E(X^-)^{p+1}<\infty$. For $\varepsilon>0$, let

$$U_n(\varepsilon)=S_n-\varepsilon n, \qquad \tau_- = \tau_-(\varepsilon)=\inf\{n: U_n(\varepsilon) \leqq 0\}, \tag{18}$$

and define M_ε by (12) and set $Y_\varepsilon=X-\varepsilon$. Obviously, as $\varepsilon \downarrow 0$,

$$U_{\tau_-}(\varepsilon) \to S_N \text{ a.s.} \tag{19}$$

Therefore by Fatou's lemma,

$$E|S_N|^p \leqq \liminf_{\varepsilon \downarrow 0} E|U_{\tau_-}(\varepsilon)|^p. \tag{20}$$

Since $P[M_\varepsilon=0]>0$ by (6), it follows from (9) and the upper bound in (10) that

$$\begin{aligned} E|U_{\tau_-}(\varepsilon)|^p &\leqq \frac{P[M_\varepsilon<1]}{P[M_\varepsilon=0]}\{E(Y_\varepsilon^-)^p+E(Y_\varepsilon^-)^{p+1}\} \\ &\ll P[M_\varepsilon<1]/P[M_\varepsilon=0], \qquad \text{since } E(X^-)^{p+1}<\infty, \\ &=O(1), \qquad \text{by Lemma 2.} \end{aligned} \tag{21}$$

From (20) and (21), the finiteness of $E|S_N|^p$ follows.

4. Remarks and Ramifications

The upper bound in (10) and the estimate (13) have played a key role in the preceding proof of Theorem 1. In this section we shall show that these estimates are asymptotically sharp and so is the lower bound in (10), and we shall also give some interesting consequences of these estimates.

With the notations of Section 2, we note that by Wald's lemma,

$$\varepsilon E\tau_- = -EU_{\tau_-} \geqq EY^-, \tag{22}$$

and therefore by (6),

$$P[M=0] \leqq \varepsilon/EY^-. \tag{23}$$

Applying this result to Lemma 2 and then to the upper bound in (10), we obtain the following interesting result on the order of magnitude of $P[M_\varepsilon \leqq x]$ and of $E((M_\varepsilon+Y_\varepsilon)^-)^p$.

Corollary 1. *With the same notations and assumptions as in Lemma 2, let* $Y_\varepsilon = X - \varepsilon$. *Then for every fixed* $x \geqq 0$,

$$P[M_\varepsilon \leqq x] \ll \varepsilon \quad \text{as } \varepsilon \downarrow 0. \tag{24}$$

Moreover, if $E(X^-)^{p+1} < \infty$ *for some* $p > 0$, *then*

$$E((M_\varepsilon + Y_\varepsilon)^-)^p \ll \varepsilon \quad \text{as } \varepsilon \downarrow 0. \tag{25}$$

In the proof of Theorem 1, we have shown that $EU_{\tau_-}(\varepsilon)$ is bounded as $\varepsilon \downarrow 0$ under the assumptions $EX = 0$, $E|X| > 0$, and $E(X^-)^2 < \infty$ (see (21) with $p = 1$). Since $EU_{\tau_-}(\varepsilon) = -\varepsilon E\tau_-(\varepsilon)$ by Wald's lemma, the boundedness of $EU_{\tau_-}(\varepsilon)$ implies that $E\tau_-(\varepsilon) \ll \varepsilon^{-1}$, which by (6) in turn implies that $P[M_\varepsilon = 0] \gg \varepsilon$. Putting this result into the lower bound of (10) completes the proof of the following

Corollary 2. *With the same notations and assumptions as in Lemma 2, assume further that* $E(X^-)^2 < \infty$. *Then as* $\varepsilon \downarrow 0$,

$$P[M_\varepsilon = 0] \gg \varepsilon, \tag{26}$$

and

$$E((M_\varepsilon + Y_\varepsilon)^-)^p \gg \varepsilon \quad \text{for every fixed } p > 0, \tag{27}$$

where $Y_\varepsilon = X - \varepsilon$.

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