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Continuous State Branching Semigroups

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Summary. This paper is concerned with Markov processes with continuous creation where the phase space is a general separable compact metric space. The transition probabilities for such a process determine a semigroup of operators acting on a function space over the collection of bounded Borel measures on the phase space. Such a semigroup is characterized by a particular convolution condition and is called a *continuous state branching semigroup.* A connection is established between continuous state branching semigroups and certain semigroups of nonlinear operators and then this connection is exploited to establish an existence theorem for the former.

Introduction

In a previous paper [7] we studied a class of stochastic processes called branching Markov processes. These furnish a model for a collection of indistinguishable particles subject to both annihilation and creation and can be roughly characterized by the following two conditions:

0.1.1. The process as a whole is Markovian and homogeneous in time.

0.1.2. Conditioned on the past up to time t, the futures (including descendants) of all particles alive at time t are mutually independent.

The instantaneous configurations associated with the processes in [7] were represented by integral valued measures μ_t on the *phase space* $\mathbf{K}$ in which the trajectories of individual particles take values. The interpretation of μ_t was that for each x in $\mathbf{K}$ there are exactly $\mu_t(\{x\})$ particles located at x at time t. The conditional probabilities for the μ_t determined in a familiar manner a semigroup of Markovian operators $\mathbf{P}^t$ acting on functions over a collection of integral valued measures on $\mathbf{K}$. To be precise,

$$\mathbf{P}^t F(\mu)=\int p^t(\mu, d\nu) F(\nu) \tag{0.1}$$

where $p^t(\mu,\cdot)$ is the distribution of μ_t conditioned on the initial configuration $\mu_0=\mu$. Semigroups arising in this manner can be characterized by the following:

0.2. *Condition of convolution.* For each $t>0$ and for each pair μ, ν the transition probability $p^t(\mu+\nu,\cdot)$ is the convolution of $p^t(\mu,\cdot)$ and $p^t(\nu,\cdot)$. That is,

$$\int p^t(\mu+\nu, d\lambda) F(\lambda)=\int p^t(\mu, d\lambda_1)\int p^t(\nu, d\lambda_2) F(\lambda_1+\lambda_2). \qquad \#$$

In this paper we begin the study of processes with properties analogous to 0.1.1 and 0.1.2 but which provide a model for fluids whose total mass changes with time rather than for a collection of particles subject to annihilation and creation. The independence of individual particles is replaced by the independence of distinct portions of the fluid. One possible formulation is suggested by the second part

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of [7]. We imagine that each portion of the fluid present at time t can be traced back through all earlier times to a portion of the fluid present at time 0. Thus if $0<t_1<\cdots<t_n$ are given times and if $A_0, \ldots, A_n$ are given subsets of the phase space $\mathbf{K}$ in which the fluid "lives", then it is possible to speak of the portion of the fluid which is present at time t_n in the set A_n, which was in the set A_i at each earlier time t_i and which was initially in the set A_0. With this formulation, a typical sample points is a collection $\{\alpha_t: t\geqq 0\}$ with each α_t a measure on the collection of trajectories taking values in $\mathbf{K}$ and defined up to time t. The interpretation is that

$$\alpha_t(\{\omega: \omega(0)\in A_0, \omega(t_1)\in A_1, \ldots, \omega(t_n)\in A_n\})$$

is the mass of that portion of the fluid described above. In particular the *instantaneous configuration* μ_t, defined by

$$\mu_t(A)=\alpha_t(\{\omega: \omega(t)\in A\})$$

gives the amount of mass in each subset of $\mathbf{K}$ at time t. The transition operators $\mathbf{P}^t$ defined by (0.1) now act on functions defined on a collection of general measures on $\mathbf{K}$ (not necessarily integral valued) and continue to statisfy the Condition of convolution 0.2. In this paper we concentrate our attention on these semigroups which we call *continuous state branching semigroups.*

We also include an informal description of the underlying process for the sake of motivation. But we do not attempt to put the discussion of the underlying process on a rigorous basis, nor do we consider the problem of recovering this process from its associated continuous state branching semigroup. The reader is referred to [7] where these points are treated for branching Markov processes. The results obtained there can easily be modified to make sense in the present context, although a rigorous translation seems to raise certain technical problems.

The special case when the phase space $\mathbf{K}$ reduces to a single point gives much insight into the general situation. The collection of bounded measures over $\mathbf{K}$ is the half line $[0, \infty)$ and so the operators $\mathbf{P}^t$ act on functions defined on $[0, \infty)$. The condition of convolution 0.2 requires that for fixed $t>0$ the probabilities $p^t(x,\cdot)$ form a semigroup with respect to convolution as x varies. In general such a semigroup can be quite pathological (see [2, p. 296]), but it is known that if the $p^t(x,\cdot)$ satisfy reasonable regularity conditions as x varies (for example, if they are continuous in law), then they are all roots of a single infinitely divisible distribution. (See [2, Chap. 9].) In this case it follows from well known results of Paul Lévy on nonnegative infinitely divisible distributions (see for example [2, Chap. XIII, Sect. 7]) that the spatial Laplace transforms can be represented

$$\begin{aligned}\int p^t(x, dy)\exp\{-y\alpha\}\\ =\exp\{-x\psi^t(\alpha)\}\end{aligned} \tag{0.2}$$

with each ψ^t a function of $\alpha>0$ having the form

$$\psi^t(\alpha)=\psi_0^t\,\alpha+\int\psi_*^t(dl)(1-\exp\{-l\alpha\}) \tag{0.3}$$

with $\psi_0^t\geqq 0$ and with ψ_*^t a measure on $(0, \infty)$ satisfying

$$\int\psi_*^t(dl)\frac{l}{1+l}<\infty.$$

Lamperti has shown in [4] that the underlying Markov process can always be recovered via a random time change from a process with independent increments which cannot jump to the left and which has been stopped upon first reaching 0. From this it follows readily (see [8, Sect. 4]) that the functions ψ^t satisfy an equation of the form

$$\frac{d}{dt}\psi^t(\alpha)=\Omega[\psi^t(\alpha)]; \qquad \psi^0(\alpha)=\alpha \tag{0.4.1}$$

where Ω is a function which can be represented

$$\Omega(\alpha)=a\alpha-b\alpha^2+\int_{0^+}^{1^+}\pi(dl)(1-\exp\{-l\alpha\}-l\alpha)+\int_{1^+}^{\infty}\pi(dl)(1-\exp\{-l\alpha\}). \tag{0.4.2}$$

Here π, the "Lévy measure," is concentrated on the half line $(0, \infty)$ and satisfies

$$\int\pi(dl)\frac{l^2}{1+l^2}<\infty;$$

a, the "deterministic rate," can be any real number; and $b\geqq 0$ measures the Gaussian contribution.

More generally, the phase space $\mathbf{K}$ can be any separable, compact metric space. A continuous state branching semigroup is a semigroup of Markovian operators $\mathbf{P}^t$ acting on functions defined over the collection $\mathscr{M}(\mathbf{K})$ of bounded regular Borel measures on $\mathbf{K}$ and such that the corresponding transition probabilities $p^t(\mu,\cdot)$ satisfy a condition analogous to (0.2) where now the functions ψ^t are replaced by operators $\overline{\boldsymbol{\Psi}}^t$ acting on nonnegative functions defined on $\mathbf{K}$. This condition is equivalent to the Condition of convolution 0.2 plus a regularity condition. We will prove that the individual operators $\overline{\boldsymbol{\Psi}}^t$ necessarily admit a representation analogous to (0.3). The study of continuous state branching semigroups $\mathbf{P}^t$ is essentially equivalent to the study of semigroups of (in general nonlinear) operators $\overline{\boldsymbol{\Psi}}^t$ mapping nonnegative functions on $\mathbf{K}$ into nonnegative functions and admitting such a representation. We will refer to such operators $\overline{\boldsymbol{\Psi}}^t$ as *completely concave operators*. To avoid technical complication, we impose an additional condition of boundedness on the operators $\mathbf{P}^t$ by postulating that the associated operators $\overline{\boldsymbol{\Psi}}^t$ map bounded functions into bounded functions.

The main result of this paper is an existence theorem for continuous state branching semigroups which we establish by solving an integral equation (for the associated operators $\overline{\boldsymbol{\Psi}}^t$) which generalizes (0.4). This result is stated in Section 3 and proved in Section 5. The associated underlying process is discussed in Section 4. Some basic facts about completely concave operators are collected in Section 1, and the concept of a continuous state branching semigroup is formally introduced in Section 2.

The special case when $\mathbf{K}$ has only one point seems to have been first treated in any generality by Jirina [3], and quite recently is has been studied extensively in a series of papers by Lamperti [4–6]. The case when $\mathbf{K}$ has two points will be treated in a forthcoming paper by Watanabe [9].

My thanks to W. Feller who from the first urged me to consider processes with continuous creation.

Notations. The *phase space* **K** is a separable compact metric space. The collection of bounded Borel measures on **K** is denoted by $\mathscr{V}(\mathbf{K})$ and is referred to as the *configuration space*. This configuration space is given the usual weak topology so that it is a locally compact Hausdorff space. The collection of bounded Borel functions $\varphi \geqq 0$ on **K** is denoted by $\mathscr{B}(\mathbf{K})$. The total mass of μ in $\mathscr{V}(\mathbf{K})$ is denoted by $|\mu|$ and the action of μ on a function φ in $\mathscr{B}(\mathbf{K})$ is denoted by $\mu(\varphi)$; thus, $\mu(\varphi)=\int \mu(dx)\,\varphi(x)$.

We will use the special notations

$$\mathscr{E}^{\mu}\{\varphi\}=1-\exp\{-\mu(\varphi)\}$$

$$\mathscr{E}_0^l\{x\}=\begin{cases}1-\exp\{-lx\} & \text{for } l>1,\\ 1-\exp\{-lx\}-lx & \text{for } 1\geqq l>0,\end{cases}$$

for these particular functionals on $\mathscr{B}(\mathbf{K})$ and functions on $(0,\infty)$.

The expectation notation $\mathbf{E}.[\ldots]$ and the integral notation $\int p.(d\omega)\ldots$ will be used interchangably depending on whether or not we wish to emphasize the dependence on ω.

Section 1. Completely Concave Operators

When treating probabilities p on the half line $[0,\infty)$ it is often useful to work with the Laplace transform $f(\lambda)$ defined for $\lambda>0$ by

$$f(\lambda)=\int p(dx)\exp\{-x\lambda\}.$$

An analogous tool is available when treating probabilities p on the collection $\mathscr{V}(\mathbf{K})$ of bounded Borel measures on the phase space **K**. The role of $\lambda>0$ is played by functions $\varphi\geqq 0$ on **K** and the function $f(\lambda)$ is replaced by the functional

$$\Phi(\varphi)=\int p(d\mu)\exp\{-\mu(\varphi)\}.$$

We begin this section by generalizing the result of Paul Lévy quoted in the introduction.

1.1. Definition. A nonnegative functional Φ on $\mathscr{B}(\mathbf{K})$ is infinitely divisible if for each $l>0$ there is a Baire probability p^l on $\mathscr{V}(\mathbf{K})$ such that

$$[\Phi(\varphi)]^l=\int p^l(dv)\exp\{-v(\varphi)\}. \qquad \#$$

1.2. Definition. A nonnegative functional $\overline{\Psi}$ on $\mathscr{B}(\mathbf{K})$ is completely concave if it can be represented

$$\overline{\Psi}(\varphi)=\int \overline{\Psi}_*(dv)\,\mathscr{E}^v\{\varphi\}+\overline{\Psi}_0(\varphi)$$

with $\overline{\Psi}_0$ in $\mathscr{V}(\mathbf{K})$ and with $\overline{\Psi}_*$ a Baire measure on $\mathscr{V}(\mathbf{K})-\{0\}$ satisfying

$$\int \overline{\Psi}_*(dv)\frac{|v|}{1+|v|}<\infty. \qquad \# \tag{1.1}$$

Theorem 1. *Let Φ be a nonnegative functional on $\mathscr{B}(\mathbf{K})$. A necessary and sufficient condition for Φ to be infinitely divisible is that $\overline{\Psi}=-\log\Phi$ be completely concave.* #

Theorem 1 is the desired generalization of Paul Lévy's result. The proof depends on the following lemma which is itself a special case of the theorem. The first part of the lemma can be established by generalizing in an obvious way the proof given in [2, Chap. XIII, Sec. 7] for the case $p=1$. The second part can be established by examining the behavior along one dimensional subspaces in λ-space. We omit a formal proof.

Lemma 1.1. *The following two conditions are equivalent for f a nonnegative function defined on the collection $[0, \infty)^p$ of nonnegative p-tuples*

$$\lambda=[\lambda_1, \ldots, \lambda_p].$$

(i) *For every $l>0$ there is a probability p^l on the collection $[0, \infty)^p$ of nonnegative p-tuples $x=\{x_1, \ldots, x_p]$ such that*

$$[f(\lambda)]^l=\int p^l(dx) \exp\{-x\cdot\lambda\}$$

where $x\cdot\lambda=x_1\lambda_1+\cdots+x_p\lambda_p$.

(ii) *The function $-\log f$ admits a representation*

$$-\log f(\lambda)=\int \psi_*(dx)[1-\exp\{-x\cdot\lambda\}]+\psi_0\cdot\lambda$$

with ψ_0 in $[0, \infty)^p$ and with ψ_* a measure on $[0, \infty)^p-\{0\}$ satisfying

$$\int \psi_*(dx)\frac{|x|}{1+|x|}<\infty. \tag{1.2}$$

Here 0 denotes $[0, \ldots, 0]$ in $[0, \infty)^p$ and $|x|$ denotes $x_1+\cdots+x_p$.

When ψ_0 and ψ_* exist, they are unique. Moreover each of the probabilities p^l is concentrated on a closed subcone $\mathscr{C}$ of $[0, \infty)^p$ if and only if ψ_0 belongs to $\mathscr{C}$ and ψ_* is concentrated on $\mathscr{C}$. #

Proof of Theorem 1. Suppose that Φ is infinitely divisible and fix $\varphi_1, \ldots, \varphi_p$ in $\mathscr{C}(\mathbf{K})$, the collection of nonnegative continuous functions on $\mathbf{K}$. Let the mapping J of $\mathscr{V}(\mathbf{K})$ into $[0, \infty)^p$ be given by

$$J(v)=[v(\varphi_1), \ldots, v(\varphi_p)].$$

Then

$$[\Phi(\lambda_1\varphi_1+\cdots+\lambda_p\varphi_p)]^l=\int p^l(dv)\exp\{-(Jv)\cdot\lambda\}$$

and so by Lemma 1.1 there are unique ψ_0 and ψ_* respectively belonging to and concentrated on the closed subcone $J[\mathscr{V}(\mathbf{K})]$ and such that

$$-\log\Phi(\lambda_1\varphi_1+\cdots+\lambda_p\varphi_p)=\int\psi_*(dx)[1-\exp\{-x\cdot\lambda\}]+\psi_0\cdot\lambda.$$

Thus there is a unique measure $\overline{\Psi}^0_*$ with domain the sigma algebra on $\mathscr{V}(\mathbf{K})$ generated in the obvious sense by the functions $\varphi_1, \ldots, \varphi_p$ and a nonnegative linear functional $\overline{\Psi}^0_0$ with domain the linear span in $\mathscr{C}(\mathbf{K})$ of the functions $\varphi_1, \ldots, \varphi_p$ such that for any function φ in this linear span,

$$-\log\Phi(\varphi)=\int\overline{\Psi}^0_*(dv)\mathscr{E}^v\{\varphi\}+\overline{\Psi}^0_0(\varphi). \tag{1.3}$$

The system of $\overline{\Psi}^0_*$ and $\overline{\Psi}^0_0$ thus defined as $\varphi_1, \ldots, \varphi_p$ vary satisfies the necessary consistency conditions for there to be a unique measure $\overline{\Psi}_*$ on the Baire algebra

of $\mathscr{V}(\mathbf{K})$ (since $\mathscr{V}(\mathbf{K})$ is locally compact) and a unique bounded measure $\overline{\Psi}_0$ on the Baire algebra of $\mathbf{K}$ such that (1.3) is satisfied for all continuous φ in $\mathscr{B}(\mathbf{K})$. The extension to general φ in $\mathscr{B}(\mathbf{K})$ goes by passage to the limit. Condition (1.1) on $\overline{\Psi}_*$ follows from the corresponding conclusion (1.2) of Lemma 1.1. This establishes necessity, and sufficiency follows upon tracing back the above argument in an obvious manner. #

1.3 Definition. An operator $\overline{\Psi}$ on $\mathscr{B}(\mathbf{K})$ is completely concave if for each x in $\mathbb{K}$ the mapping: $\varphi \to \overline{\Psi}\varphi(x)$ defines a completely concave functional on $\mathscr{B}(\mathbf{K})$. #

Proposition 1.1. *Let $\overline{\Psi}$ be a completely concave operator on $\mathscr{B}(\mathbf{K})$. Then for each measure μ in $\mathscr{V}(\mathbf{K})$ there is a unique probability p^μ on $\mathscr{V}(\mathbf{K})$ such that*

$$\int p^\mu(dv) \exp\{-v(\varphi)\} = \exp\{-\mu(\overline{\Psi}\varphi)\}. \tag{1.4}$$

Moreover $p^{(\mu+\lambda)}$ is the convolution of p^μ and p^λ in the sense that

$$\int p^{(\mu+\lambda)}(dv) F(v) = \int p^\mu(dv_1) \int p^\lambda(dv_2) F(v_1+v_2). \tag{1.5}$$

Proof. For each x in $\mathbf{K}$ we can write

$$\overline{\Psi}\varphi(x) = \int \overline{\Psi}_*(x, dv)\mathscr{E}^v\{\varphi\} + \int \overline{\Psi}_0(x, dy)\varphi(y).$$

The boundedness of $\overline{\Psi}\mathbf{1}$ yields

$$\sup \int \overline{\Psi}_0(x, dy) < \infty \qquad (x \text{ in } \mathbf{K}),$$

$$\sup \int \overline{\Psi}_*(x, dv) \frac{|v|}{1+|v|} < \infty \qquad (x \text{ in } \mathbf{K}),$$

and so the measure $\int \mu(dx)\, \overline{\Psi}_0(x, \cdot)$ on $\mathbf{K}$ is bounded while the measure $\int \mu(dx) \cdot \overline{\Psi}_*(x, \cdot)$ on $\mathscr{V}(\mathbf{K})$ satisfies (1.1). Now the existence of the probability p^μ on $\mathscr{V}(\mathbf{K})$ follows from Theorem 1 and the relation

$$\mu(\overline{\Psi}\varphi) = \int \mu(dx) \int \overline{\Psi}_*(x, dv)\mathscr{E}^v\{\varphi\} + \int \mu(dx) \int \overline{\Psi}_0(x, dy)\varphi(y).$$

The convolution condition (1.5) is immediate when F has the special form $F(v) = \exp\{-v(\varphi)\}$; but such functions generate the full Baire algebra on $\mathscr{V}(\mathbf{K})$. #

An obvious computation establishes

Proposition 1.2. *If $\overline{\Psi}^1$ and $\overline{\Psi}^2$ are completely concave operators, then so is their composite $\overline{\Psi} = \overline{\Psi}^1\,\overline{\Psi}^2$. Moreover*

$$p^\mu(\cdot) = \int p_1^\mu(dv) \int p_2^v(\cdot)$$

where p^μ, p_1^μ, p_2^v are the probabilities determined by (1.4) *for $\overline{\Psi}$, $\overline{\Psi}^1$, and $\overline{\Psi}^2$.* #

Finally, we will need

Proposition 1.3. *Let $\mathscr{B}_0(\mathbf{K})$ be the subcollection of functions φ in $\mathscr{B}(\mathbf{K})$ satisfying $m \leqq \varphi \leqq M$ where $0 < m < M$ are given positive constants. If $\overline{\Psi}^n$ is a sequence of completely concave operators which are uniformly bounded on $\mathscr{B}_0(\mathbf{K})$ and such that $\lim \overline{\Psi}^n\varphi$ exists pointwise for all φ in $\mathscr{B}_0(\mathbf{K})$, then there is a unique completely concave operator $\overline{\Psi}$ such that $\overline{\Psi}\varphi = \lim \overline{\Psi}^n\varphi$ for all φ in $\mathscr{B}(\mathbf{K})$.*

Proof. By the uniform boundedness of the $\bar{\Psi}^n$ on $\mathscr{B}_0(\mathbf{K})$, there is a constant $C>0$ such that

$$\int \bar{\Psi}^n_*(x, d\mu)\frac{|\mu|}{1+|\mu|}+\int \bar{\Psi}^n_0(x, dy)\leqq C$$

for all n and x. Therefore by the separability of $\mathbf{K}$ and the local compactness in the weak topology of $\mathscr{V}(\mathbf{K})$, there are measures $\bar{\Psi}_*(x, d\mu)$ on $\mathscr{V}(\mathbf{K})$ and $\bar{\Psi}_0(x, dy)$ on $\mathbf{K}$ such that

$$\lim \bar{\Psi}^n \varphi(x)=\int \bar{\Psi}_*(x, d\mu)\mathscr{E}^\mu\{\varphi\}+\int \bar{\Psi}_0(x, dy)\varphi(y) \tag{1.6}$$

for all φ in $\mathscr{B}(\mathbf{K})$ as n runs through an appropriate subsequence depending on x. Thus the proposition will be proved if we show that the limiting measures $\bar{\Psi}_*(x, d\mu)$ and $\bar{\Psi}_0(x, dy)$ are in fact independent of the choice of such a subsequence. For this it suffices to show that the right side of (1.6) is uniquely determined for all φ in $\mathscr{B}(\mathbf{K})$ by its values for φ in $\mathscr{B}_0(\mathbf{K})$. To see this, fix constants m_1, M_1 satisfying $m<m_1<M_1<M$ and consider $\varphi_1, \ldots, \varphi_k$ in $\mathscr{B}_0(\mathbf{K})$ satisfying $m_1\leqq\varphi_1, \ldots, \varphi_k\leqq M_1$. The right side of (1.6) with $\varphi=\lambda_1\varphi_1+\cdots+\lambda_r\varphi_r$ is an analytic function of $\lambda_1, \ldots, \lambda_r>0$ which is uniquely determined as $\lambda_1, \ldots, \lambda_r$ vary over a nonempty open set in complex r-space. Therefore it is uniquely determined for all $\lambda_1, \ldots, \lambda_r>0$ and so the right side of (1.6) is uniquely determined for all φ which can be represented in this way. But this includes all φ in $\mathscr{B}(\mathbf{K})$ that are bounded away from 0. Thus the right side of (1.6) is uniquely determined for all φ in $\mathscr{B}(\mathbf{K})$ that are bounded away from 0, and the last restriction can be removed by a passage to the limit. #

Section 2. Continuous State Branching Semigroups

2.1. Definition. A continuous state branching semigroup is a semigroup of Markovian operators $\mathbf{P}^t$ mapping bounded Baire functions on $\mathscr{V}(\mathbf{K})$ into bounded Baire functions and such that each $\mathbf{P}^t$ can be represented

$$\mathbf{P}^t F(\mu)=\int p^t(\mu, dv)F(v) \tag{2.1}$$

where $p^t(\mu, \cdot)$ is determined by (1.4) for some completely concave operator $\bar{\Psi}^t$ on $\mathscr{B}(\mathbf{K})$. #

We will refer to the operators $\bar{\Psi}^t$ as the logarithmic Laplace transform of the continuous state branching semigroup $\mathbf{P}^t$. The next two propositions are a direct consequence of Propositions 1.1 and 1.2.

Proposition 2.1. *The operators $\bar{\Psi}^t$ defined by* (2.1) *form a semigroup under composition:*

$$\bar{\Psi}^t\bar{\Psi}^s=\bar{\Psi}^{t+s} \qquad s, t>0. \qquad \#$$

Proposition 2.2. *If $\bar{\Psi}^t$ is a semigroup of completely concave operators acting on $\mathscr{B}(\mathbf{K})$, then there is a unique continuous state branching semigroup $\mathbf{P}^t$ with $\bar{\Psi}^t$ the logarithmic Laplace transform.* #

Thus the study of continuous state branching semigroups is equivalent to the study of semigroups of completely concave operators acting on $\mathscr{B}(\mathbf{K})$. Just as in [7], it is possible to introduce a related semigroup of operators which describe the time evolution of the expected mass in various subsets of the phase space $\mathbf{K}$. We define them here for the record, although they play no role in what follows.

2.2. Definition. The mass transition operators $\mathbf{L}_t$ associated with a continuous state branching semigroup $\mathbf{P}^t$ are defined by[1]

$$\mathbf{L}^t\varphi(x)=\int p^t(\delta_x, dv)\, v(\varphi). \qquad \#$$

We state without proof the obvious

Proposition 2.3. *The mass transition operators* $\mathbf{L}^t$ *associated with a continuous state branching semigroup* $\mathbf{P}^t$ *form a semigroup of linear operators mapping nonnegative Baire functions on* $\mathbf{K}$ *into nonnegative Baire functions (not necessarily finite). Moreover if* $\overline{\Psi}^t$ *is the logarithmic Laplace transform for the* $\mathbf{P}^t$, *then*

$$\mathbf{L}^t\varphi(x)=\int \overline{\Psi}^t_*(x, d\mu)\,\mu(\varphi)+\int \overline{\Psi}^t_0(x, dy)\,\varphi(y). \qquad \#$$

Example. Let $\{P_x: x \text{ in } \mathbf{K}\}$ be a single particle Markov process and let $\{\alpha_t\}$ be a multiplicative functional on the underlying trajectory space $\mathbf{W}$. (See paragraphs 3.1 and 3.2 below for definitions.) It is well known (see for example [1, p. 282]) that the operators $\mathbf{R}^t$ defined by

$$\mathbf{R}^t\varphi(x)=\mathbf{E}_x[\varphi(X_t);\alpha_t]$$

form a semigroup of linear operators mapping $\mathscr{B}(\mathbf{K})$ into itself and that each $\mathbf{R}^t$ can be represented

$$\mathbf{R}^t\varphi(x)=\int_K r^t(x, dy)\,\varphi(y)$$

with $r^t(x,\cdot)$ a bounded Borel measure on $\mathbf{K}$. The adjoint semigroup, acting on $\mathscr{V}(\mathbf{K})$, is denoted and defined by

$$\mu\mathbf{R}^t(\cdot)=\int_K \mu(dx)\, r^t(x,\cdot).$$

Obviously the operators $\mathbf{P}^t$ defined on bounded Baire functions F on the configuration space $\mathscr{V}(\mathbf{K})$ by

$$\mathbf{P}^tF(\mu)=F(\mu\mathbf{R}^t)$$

form a continuous state branching semigroup with logarithmic Laplace transform and mass transition operators both given by $\mathbf{R}^t$; that is, $\overline{\Psi}^t\varphi=\mathbf{L}^t\varphi=\mathbf{R}^t\varphi$. The underlying process describes a fluid living in $\mathbf{K}$ which is evolving deterministically with time and which need not be conserved — that is, the total mass can fluctuate with time. Each infinitesimal portion of the fluid follows a particular trajectory ω in $\mathbf{W}$. More precisely, if A is a subset of $\mathbf{W}$ depending only on coordinates X_u with $u\leqq t$ (that is, if A belongs to the past $\mathscr{F}_t$), then we can identify the portion of the fluid at time t which has followed a trajectory lying in A. The mass of this portion is given by $\int \mu(dx)\,\mathbf{E}_x[A;\alpha_t]$ where μ is the initial configuration. Thus the measure $\int \mu(dx)\,\mathbf{E}_x[\cdot;\alpha_t]$ on the past $\mathscr{F}_t$ gives the complete history up to time t of the fluid present at time t. Letting $A=\{\omega:\omega(t)\in A\}$, we have

$$\int \mu(dx)\,\mathbf{E}_x[X_t\in A;\alpha_t]=\mu\mathbf{R}^t(A)$$

and therefore $\mu\mathbf{R}^t$ is the instantaneous configuration at time t.

1 Here δ_x denotes the unit mass concentrated at x.

It will be convenient in Section 4 to have in mind the following picture for the above process. We imagine an "ideal" fluid evolving deterministically as above with the same single particle Markov process as above but with the multiplicative functional being identically 1; thus the total mass is conserved. The "physical" fluid of the above paragraph is somehow carried by this "ideal" fluid and has a mass density given by the multiplicative functional $\{\alpha_t\}$; that is, $\alpha_t(\omega)$ is the density of the physical fluid at time t in the infinitesimal portion of the ideal fluid which has followed the trajectory ω up to time t. More precisely, α_t is the Radon-Nikodym derivative of the measure on the trajectory space **W** describing the past history of the "physical" fluid present at time t with respect to the corresponding measure for the "ideal" fluid. In the special case when α_t can be represented

$$\alpha_t = \exp\left\{\int_0^t p(X_s)\,ds - \int_0^t n(X_s)\,ds\right\} \qquad p, n \geqq 0$$

the ordinary differential equation

$$\frac{d}{dt}\alpha_t = p(X_t)\alpha_t - n(X_t)\alpha_t$$

is satisfied for almost all t along each trajectory. Thus it is reasonable to think of $p(X_t)$ and $n(X_t)$ as the local rate of creation and the local rate of annihilation of the "physical" fluid. However in general, it is not possible to represent $\log \alpha_t$ as the difference of two increasing additive functionals (see paragraph 3.3 for a definition) and such an interpretation is not available. #

Section 3. Statement of the Existence Theorem

The *trajectory space* **W** is the collection of maps $\omega(\cdot)$ from the half line $[0, \infty)$ into the *phase space* **K** which are right continuous and have left hand limits everywhere. The coordinates $\omega(t)$ are denoted by $X_t(\omega)$ or simply X_t. The sigma algebra on **W** generated by the sets $\{X_s \in A\}$ with $s \leqq t$ and A a Borel subset of **K** will be denoted by $\mathscr{F}_t$ and will be referred to as the *past*. The sigma algebra generated by the set theoretic union of the $\mathscr{F}_t$ will be denoted by $\mathscr{F}$. For each $t > 0$ the *shift transformation* θ_t is defined on ω in **W** by

$$\theta_t\,\omega(s) = \omega(t+s).$$

We are given

3.1. *A single particle Markov process.* This is a collection $\{\mathfrak{P}_x : x \in K\}$ of probabilities on $\mathscr{F}$ which satisfy the following three conditions.

(i) For each x in K,

$$\mathfrak{P}_x[\omega(0) = x] = 1.$$

(ii) For each A in $\mathscr{F}$ the function $\mathfrak{P}_x(A)$ is Borel measurable in x.

(iii) For each set A belonging to the past $\mathscr{F}$ and for each bounded $\mathscr{F}$ measurable function f,

$$\int_A \mathfrak{P}_x(d\omega)\, f(\theta_t\,\omega) = \int_A \mathfrak{P}_x(d\omega) \int \mathfrak{P}_{\omega(t)}(d\omega')\, f(\omega'). \qquad \#$$

3.2. *A multiplicative functional.* This is a collection $\{\alpha^t : t \geqq 0\}$ of strictly positive functionals defined on the trajectory space **W** and satisfying:

(i) For each x in **K** the functionals α^t vary continuously with $t \geqq 0$ and $\alpha^0 = 1$ with $\mathfrak{P}_x$ probability one.

(ii) For each $t > 0$ the functional α^t is measurable with respect to the past $\mathscr{F}_t$, and the expectation $\mathbf{E}_x[\alpha^s]$ is uniformly bounded and uniformly bounded away from 0 for $s \leqq t$ and for x in **K**.

(iii) For each pair s, t with $0 \leqq s \leqq t$ and for each x in **K**,

$$\alpha^{s,t}(\omega) = \alpha^{t-s}(\theta_s \omega)$$

with $\mathfrak{P}_x$ probability one. (Here $\alpha^{s,t}$ denotes the ratio $\alpha^t(\alpha^s)^{-1}$.) #

3.3. *An additive functional.* This is a collection $\{\beta_t : t \geqq 0\}$ of finite, non-negative functionals defined on the trajectory space **W** and satisfying:

(i) For each x in **K** the functionals β_t increase and vary continuously with $t \geqq 0$ with $\mathfrak{P}_x$ probability one.

(ii) For each $t > 0$ the functional β_t is measurable with respect to the past $\mathscr{F}_t$.

(iii) For each pair s, t with $0 \leqq s \leqq t$ and for each x in **K**,

$$\beta_t(\omega) - \beta_s(\omega) = \beta_{t-s}(\theta_s \omega)$$

with $\mathfrak{P}_x$ probability one. #

3.4. *A creation kernel.* This is a collection $\{\pi(x, \cdot) : x \in K\}$ of Baire measures on the configuration space $\mathscr{V}(\mathbf{K})$ such that $\pi(x, A)$ is Borel measurable in x for each Baire subset A of $\mathscr{V}(\mathbf{K})$. #

3.5. *A local creation kernel.* This is a collection $\{\pi_0(x, \cdot) : x \in \mathbf{K}\}$ of Borel measures on the half line $(0, \infty)$ such that $\pi_0(x, A)$ is Borel measurable in x for each Borel subset A of $(0, \infty)$. #

3.6. *A nonnegative Borel function γ on* **K**. #

These quantities are assumed to satisfy the following two technical conditions:

3.7. *Technical Condition.* The quantity

$$\mathbf{E}_x\left[\int_0^t \beta(du)\,\alpha^u \left\{ \int_{|\mu| \leqq |} \pi(X_u, d\mu)\,|\mu| + \int_{|\mu| > |} \pi(X_u, d\mu) + \int_0^{1^+} \pi_0(X_u, dl)\,l^2 + \int_{1^+}^{\infty} \pi_0(X_u, dl) + \gamma(X_u) \right\}\right]$$

decreases to 0 uniformly in x as $t \downarrow 0$. #

3.8. *Technical Condition.* For each $t > 0$ the quantities

$$\mathbf{E}_x\left[\int_0^t \beta(du)\,\alpha^u \int_{0^+}^{(1/k)^+} \pi_0(X_u, dl)\,l^2\right]$$

decrease to 0 uniformly in x as $k \uparrow \infty$. #

We formally state our existence theorem in

Theorem 2. *There is a unique continuous state branching semigroup whose logarithmic Laplace transform $\bar{\Psi}^t$ satisfies*

$$\bar{\Psi}^t\varphi(x)=\mathbf{E}_x[\varphi(X_t);\alpha^t]+\mathbf{E}_x\left[\int_0^t\beta(du)\,\alpha^u\left(\int\pi(X_u,d\mu)\,\mathscr{E}^\mu\{\bar{\Psi}^{t-u}\varphi\}\right.\right.$$
$$\left.\left.+\int_0^\infty\pi_0(X_u,dl)\,\mathscr{E}_0^l\{\bar{\Psi}^{t-u}\varphi(X_u)\}-\gamma(X_u)\{\bar{\Psi}^{t-u}\varphi(X_u)\}^2\right)\right]. \tag{3.1}$$

Section 4. Informal Description of the Underlying Process

We restrict our attention to the special case when all of the creation measures $\pi(x,\cdot)$ are probability measures. The case when the $\pi(x,\cdot)$ are bounded can be reduced to this by redefining the additive functional $\{\beta_t\}$, and the general case can then be understood by truncation and passage to the limit. We think in terms of successive generations of fluid and we invoke the artifice of Section 2, imagining that each generation of "physical" fluid is carried by an auxiliary "ideal" fluid which is governed by the conservative deterministic process corresponding to the given single particle Markov process $\{\mathfrak{P}_x: x\in\mathbf{K}\}$.

The density of the 0th generation of physical fluid along an individual trajectory ω is, except for time homogeneity, a continuous state branching process $\{N_t\}$ with phase space having only one point. (See the introduction for a discussion of the time homogeneous case.) By the logarithmic Laplace transform of the process $\{N_t\}$ we mean the family of functions $\psi^{s,t}$ defined for $0\leqq s<t$ by

$$\psi^{s,t}(\lambda)=-\log\mathbf{E}[\exp\{-\lambda N_t\}\,|\,N_s=1].$$

The law for the density process $\{N_t\}$ (depending on ω) is determined by the condition that the $\psi^{s,t}$ are the unique solution of

$$\psi^{s,t}(\lambda)=\alpha^{s,t}\lambda+\int_s^t\beta(du)\,\alpha^{s,u}\left(\int\pi_0(X_u,dl)\,\mathscr{E}_0^l\{\psi^{u,t}(\lambda)\}-\gamma(X_u)\{\psi^{u,t}(\lambda)\}^2\right).$$

The densities N_t for distinct trajectories are identical for t in the initial time segment $[0,\sigma]$ (if any) on which the trajectories agree; the ratios $N_t N_\sigma^{-1}$ are independent for $t\geqq\sigma$.

Discrete amounts of secondary fluid are created at successive random "creation times" $\tau_1,\tau_2,\ldots$ and are associated with certain random "creating trajectories" $\omega_1,\omega_2,\ldots$. The portion of secondary fluid created at each creation time τ is represented by a random point in the configuration space $\mathscr{V}(\mathbf{K})$, with the usual interpretation. Conditioned on the 0th generation, on the creation times τ_i, and on the creating trajectories ω_i, the portions of secondary fluid created at the various creation times are mutually independent and the portion created at a particular creation time τ has the distribution $\pi(\omega(\tau),\cdot)$ where ω is the associated creating trajectory. Conditioned on the 0th generation (and in particular, on the densities N_t for the 0th generation), the number of creation times in the time interval $[s,t]$, associated with creating trajectories belonging to a particular subset Λ of

the trajectory space **W** in the past F_t, is a Poisson variable with mean

$$\int \mu(dx)\, \mathbf{E}_x\Big[\Lambda; \int_{[s,t]} \beta(du)\, N(u)\Big].$$

(Here μ is the initial configuration for the process.) The numbers of creation times associated with trajectories in a fixed subset Λ in the past $\mathscr{F}_t$ but occurring in disjoint subintervals of $[0, t]$ are mutually independent. The distribution of the numbers of creation times associated with trajectories in distinct subsets of **W** does not admit a simple description in general, but can be understood by finite additivity and a passage to the limit from the following special situation. If $0<t_1<\cdots<t_n$ are given times and if

$$\Lambda_1=\{\omega\colon \omega(0)\in A_0, \omega(t_1)\in A_1, \ldots, \omega(t_n)\in A_n\}$$
$$\Lambda_2=\{\omega\colon \omega(0)\in B_0, \omega(t_1)\in B_1, \ldots, \omega(t_n)\in B_n\}$$

are given subsets of the trajectory space **W** such that $A_0=B_0, A_1=B_1, \ldots, A_{i-1}=B_{i-1}$ but A_i and B_i are disjoint, then the numbers of creation times associated with trajectories in Λ_1 and Λ_2 and occurring in the time interval $[0, t_{i-1}]$ are identical while the numbers occurring in the time interval $[t_i, t_n]$ are independent. The obvious generalization of this statement holds if several sets Λ_i of this special form are considered. [We can describe the distribution of the creation times and associated creating trajectories in a less precise but more suggestive manner as follows. The number of creation times occurring for a fixed trajectory ω in a time interval $[s, t]$ is a differential Poisson variable with infinitesimal mean $\int_{[s,t]} \beta(du)\, N_u$. The numbers occurring in disjoint time intervals are independent. The numbers associated with distinct trajectories are identical for the initial segment of the time axis on which they agree (if any) and are independent for all later time intervals.]

The portion of secondary fluid corresponding to a fixed creation time τ evolves from time τ onward according to the same process which governs the 0^{th} generation. The 1^{st} generation of fluid is the sum of the portions of secondary fluid corresponding to all of the creation times. Succeeding generations are triggered in a similar manner: the creation times and creating trajectories which trigger the $(n+1)^{\text{st}}$ generation depend on the n^{th} generation just as the corresponding quantities which trigger the 1^{st} generation depend on the 0^{th} generation.

Section 5. Proof of the Existence Theorem

We begin by establishing a result on local existence and uniqueness.

Lemma 5.1. *For each pair m, M satisfying $0<m<M$ there is a $\tau>0$ with the property that whenever φ in $\mathscr{B}(\mathbf{K})$ satisfies*

$$m\leqq\varphi\leqq M,$$

there is a unique family of functions $\bar{\Psi}^t\varphi$ in $\mathscr{B}(\mathbf{K})$ defined for $0\leqq t\leqq\tau$ and satisfying (3.1). *Moreover there are positive constants m_1, M_1 such that*

$$m_1\leqq\bar{\Psi}^t\varphi\leqq M_1 \tag{5.1}$$

for $t\leqq\tau$.

Proof. For any $\tau>0$ there are positive constants a, A, B such that

$$a\leqq \mathbf{E}_x[\alpha^t]\leqq A$$

$$\mathbf{E}_x\left[\int_0^t \beta(du)\,\alpha^u\{\gamma(X_u)+\int_{|\mu|\leqq 1}\pi(X_u,d\mu)\,|\mu|+\int_{|\mu|>1}\pi(X_u,d\mu)+\int_{0^+}^{1^+}\pi_0(X_u,dl)\,l^2+\int_{1^+}^{\infty}\pi_0(X_u,dl)\right\}\Bigg]\leqq B$$

for all x in $\mathbf{K}$ and for all $t\leqq\tau$. Also the constant $B\downarrow 0$ as $\tau\downarrow 0$. We define $\bar{\Psi}^t_{(n)}\varphi$ by induction on n as follows whenever it makes sense:

$$\bar{\Psi}^t_{(0)}\varphi(x)=\mathbf{E}_x[\alpha^t;\varphi(X_t)]$$

$$\bar{\Psi}^t_{(n+1)}\varphi(x)=\mathbf{E}_x[\alpha^t;\varphi(X_t)]+\mathbf{E}_x\left[\int_0^t\beta(du)\,\alpha^u\Big(\int\pi(X_u,d\mu)\,\mathscr{E}^\mu\{\bar{\Psi}^{t-u}_{(n)}\varphi\}\right. \tag{5.2}$$

$$\left.+\int\pi_0(X_u,dl)\,\mathscr{E}^l_0\{\bar{\Psi}^{t-u}_{(n)}\varphi(X_u)\}-\gamma(X_u)\{\bar{\Psi}^{t-u}_{(n)}\varphi(X_u)\}^2\Big)\right].$$

Clearly

$$\bar{\Psi}^t_{(n)}\varphi(x)\leqq MA+B$$

and therefore

$$\bar{\Psi}^t_{(n)}\varphi(x)\geqq ma-B(MA+B)^2 \tag{5.3}$$

for all $t\leqq\tau$ and for all x. (In establishing (5.3) we have used the elementary inequality $1-\exp\{-x\}-x\geqq -x^2$.) Since $B(MA+B)^2\leqq ma$ for sufficiently small τ, we conclude that for such τ the definitions (5.2) make sense and that there are constants m_1, M_1 such that

$$m_1\leqq\bar{\Psi}^t_{(n)}\varphi(x)\leqq M_1$$

for all $t\leqq\tau$ and for all n. Next we note the inequalities

$$\begin{aligned}
&|\{\bar{\Psi}^t_{(n+1)}\varphi(x)\}^2-\{\bar{\Psi}^t_{(n)}\varphi(x)\}^2|\leqq 2M_1\,|\bar{\Psi}^t_{(n+1)}\varphi(x)-\bar{\Psi}^t_{(n)}\varphi(x)|\\
&|\mathscr{E}^\mu\{\bar{\Psi}^t_{(n+1)}\varphi\}-\mathscr{E}^\mu\{\bar{\Psi}^t_{(n)}\varphi\}|\leqq C\sup_x|\bar{\Psi}^t_{(n+1)}\varphi(x)-\bar{\Psi}^t_{(n)}\varphi(x)| \qquad (5.4)\\
&|\mathscr{E}^l_0\{\bar{\Psi}^t_{(n+1)}\varphi(x)\}-\mathscr{E}^l_0\{\bar{\Psi}^t_{(n)}\varphi(x)\}|\leqq\begin{cases}C\,|\bar{\Psi}^t_{(n+1)}\varphi(x)-\bar{\Psi}^t_{(n)}\varphi(x)| & \text{for } l>1,\\ M_1\,l^2\,|\bar{\Psi}^t_{(n+1)}\varphi(x)-\bar{\Psi}^t_{(n)}\varphi(x)| & \text{for } l\leqq 1,\end{cases}
\end{aligned}$$

where C, depending on m_1, is defined by

$$C=\sup(l\exp\{-lm_1\})\qquad 1\leqq l<\infty.$$

The first inequality is obvious and the others follow upon applying the Mean Value Theorem to the functions $1-\exp\{x\}-x$ and $1-\exp\{-x\}$. Applying these inequalities to (5.2) and letting

$$a_n=\sup|\bar{\Psi}^t_{(n+1)}\varphi(x)-\bar{\Psi}^t_{(n)}\varphi(x)|\qquad (x \text{ in } \mathbf{K},\ 0\leqq t\leqq\tau),$$

we deduce

$$a_{n+1}\leqq a_n B(2M_1+C).$$

But $B(2M_1+C)<1$ for τ sufficiently small; thus $a_n\downarrow 0$ and the $\bar{\Psi}^t_{(n)}\varphi$ converge to the desired solution $\bar{\Psi}^t\varphi$. Uniqueness of $\bar{\Psi}^t\varphi$ follows by a similar argument. #

Lemma 5.2. *The following statement is true along a set of trajectories ω in* $\mathbf{W}$ *having* $\mathfrak{P}_x$ *probability one for all* x *in* $\mathbf{K}$. *For each positive integer* k *there is a unique family of positive functionals* $\alpha_k^{s,t}$ *defined for* $0 \leqq s < t$ *and satisfying*

$$\alpha_k^{s,t} = \alpha^{s,t} - \int_s^t \beta(du)\, \alpha^{s,u} \left\{ \int_{(1/k)^+}^{1^+} \pi_0(X_u, dl)\, l + k\gamma(X_u) \right\} \alpha_k^{u,t}. \tag{5.5}$$

Moreover

$$\alpha_k^{r,s}\, \alpha_k^{s,t} = \alpha_k^{r,t} \tag{5.6}$$

whenever $0 \leqq r < s < t$.

Proof. It follows easily from the Technical condition 3.7 that for all $k > 0$ the integral

$$\int_0^t \beta(du)\, \alpha^u \left\{ \int_{(1/k)^+}^{1^+} \pi_0(X_u, dl)\, l + k\gamma(X_u) \right\} \tag{5.7}$$

is finite and varies continuously for a set of trajectories in $\mathbf{W}$ having $\mathfrak{P}_x$ probability one for all x in $\mathbf{K}$. Thus it suffices to consider one such trajectory. Since $\alpha^{s,u} = (\alpha^s)^{-1}\, \alpha^u$, also the integrals

$$\int_s^t \beta(du)\, \alpha^{s,u} \left\{ \int_{(1/k)^+}^{1^+} \pi_0(X_u, dl)\, l + k\gamma(X_u) \right\}$$

are bounded. Choose $\tau > 0$ such that (5.7) is $\leqq \frac{1}{2}$ for $t \leqq \tau$. For $0 \leqq s < t \leqq \tau$, let $\alpha_{k(n)}^{s,t}$ be defined inductively on n by the iteration:

$$\alpha_{k(0)}^{s,t} = \alpha^{s,t}$$

$$\alpha_{k(n+1)}^{s,t} = \alpha^{s,t} - \int_s^t \beta(du)\, \alpha^{s,u} \left\{ \int_{(1/k)^+}^{1^+} \pi_0(X_u, dl)\, l + k\gamma(X_u) \right\} \alpha_{k(n)}^{u,t}.$$

Arguments similar to those in the proof of Lemma 5.1 show that the $\alpha_{k(n)}^{s,t}$ converge boundedly to functions $\alpha_k^{s,t}$ satisfying $\frac{1}{2}\, \alpha^{s,t} \leqq \alpha_k^{s,t} \leqq \alpha^{s,t}$ and the Eq. (5.5) and that the $\alpha_k^{s,t}$ are unique. The semigroup property (5.6) for $0 \leqq s < t \leqq \tau$ follows from

$$\begin{aligned}
\alpha_k^{r,s}\, \alpha_k^{s,t} &= \alpha^{r,s}\, \alpha^{s,t} - \int_r^s \beta(du)\, \alpha^{r,u} \left\{ \int_{(1/k)^+}^{1^+} \pi_0(X_u, dl)\, l + k\gamma(X_u) \right\} \alpha_k^{u,s}\, \alpha_k^{s,t} \\
&\quad - \alpha^{r,s} \int_s^t \beta(du)\, \alpha^{s,u} \left\{ \int_{(1/k)^+}^{1^+} \pi_0(X_u, dl)\, l + k\gamma(X_u) \right\} \alpha_k^{u,t} \\
&= \alpha^{r,t} - \int_r^s \beta(du)\, \alpha^{r,u} \left\{ \int_{(1/k)^+}^{1^+} \pi_0(X_u, dl)\, l + k\gamma(X_u) \right\} \alpha_k^{u,s}\, \alpha_k^{s,t} \\
&\quad - \int_s^t \beta(du)\, \alpha^{r,u} \left\{ \int_{(1/k)^+}^{1^+} \pi_0(X_u, dl)\, l + k\gamma(X_u) \right\} \alpha_k^{u,t}
\end{aligned}$$

and from the uniqueness of solutions to (5.5). The extension of our local solution to a global solution goes in a straightforward manner, using the semigroup property (5.6) to make the extension and the last computation to verify the Eq. (5.5) for the extension. #

For fixed k we define inductively on n:

$$\begin{aligned}
\bar{\Psi}^t_{k(0)}\varphi(x) &= \mathbf{E}_x[\alpha^t_k;\varphi(X_t)]\\
\bar{\Psi}^t_{k(n+1)}\varphi(x) &= \mathbf{E}_x[\alpha^t_k;\varphi(X_t)]+\mathbf{E}_x\Big[\int_0^t \beta(du)\,\alpha^u_k\Big(\int \pi(X_u,d\mu)\,\mathscr{E}^u\{\bar{\Psi}^{t-u}_{k(n)}\varphi\}\\
&\quad+\int_{(1/k)^+}^{\infty}\pi_0(X_u,dl)\,[1-\exp\{-l\,\bar{\Psi}^{t-u}_{k(n)}\varphi(X_u)\}]\\
&\quad+k^2\gamma(X_u)\,[1-\exp\{-l\,\bar{\Psi}^{t-u}_{k(n)}\varphi(X_u)\}]\Big)\Big].
\end{aligned}\tag{5.8}$$

It is clear that the operators $\bar{\Psi}^t_{k(n)}$ increase with n to operators $\bar{\Psi}^t_k$ satisfying (5.8) with the subscripts n deleted. These operators $\bar{\Psi}^t_k$ satisfy in addition the equation

$$\begin{aligned}
\bar{\Psi}^t_k\varphi(x) &= \mathbf{E}_x[\alpha^t;\varphi(X_t)]+\mathbf{E}_x\Big[\int_0^t \beta(du)\,\alpha^u\Big(\int \pi(X_u,d\mu)\,\mathscr{E}^u\{\bar{\Psi}^{t-u}_k\varphi\}\\
&\quad+\int_{(1/k)^+}^{\infty}\pi_0(X_u,dl)\,\mathscr{E}^l_0\{\bar{\Psi}^{t-u}_k\varphi(X_u)\}\\
&\quad-2k^2\gamma(X_u)\,\mathscr{E}^{1/k}_0\{\bar{\Psi}^{t-u}_k\varphi(X_u)\}\Big)\Big]
\end{aligned}\tag{5.9}$$

as can be verified by a straightforward computation after substituting (5.5) into (5.8) (with $\bar{\Psi}^t_{k(n)}\varphi$ replaced by $\bar{\Psi}^t_k\varphi$). Now it follows easily from (5.9) and from the proof of Lemma 5.1 that (5.1) is valid not only for $\bar{\Psi}^t$ but for $\bar{\Psi}^t_k$ which, together with the defining iteration for $\bar{\Psi}^t_k$, implies that $\bar{\Psi}^t_k$ is completely concave. Furthermore, the estimates (5.4) are valid with $\bar{\Psi}^t_{(n+1)}$ replaced by $\bar{\Psi}^t_{k+1}$ and with $\bar{\Psi}^t_{(n)}$ replaced by $\bar{\Psi}^t_k$. Letting

$$b_k=\sup|\bar{\Psi}^t\varphi(x)-\bar{\Psi}^t_k\varphi(x)| \qquad (t\leqq\tau,\ x\in\mathbf{K}),$$

applying these estimates to the difference of the integral Eqs. (3.1) and (5.9) and using the notation in the proof of Lemma 5.1, we get

$$\begin{aligned}
b_k &\leqq b_k(C+M_1)B+\sup_x\mathbf{E}_x\Big[\int_0^t\beta(du)\,\alpha^u\gamma(X_u)\,|\{\bar{\Psi}^{t-u}\varphi(X_u)\}^2\\
&\qquad-2k^2\,\mathscr{E}^{(1/k)}_0\{\bar{\Psi}^{t-u}_k\varphi(X_u)\}|\Big]\\
&\quad+\sup_x\mathbf{E}_x\Big[\int_0^t\beta(du)\,\alpha^u\int_{0^+}^{(1/k)^+}\pi_0(X_u,dl)\,\mathscr{E}^l_0\{\bar{\Psi}^{t-u}_k\varphi(X_u)\}\Big].
\end{aligned}$$

Using the elementary inequalities

$$\begin{aligned}
|x^2-2k^2\,\mathscr{E}^{(1/k)}_0\{x\}| &\leqq 2x^3/k\\
|\mathscr{E}^l_0\{x\}| &\leqq x^2/2 \quad \text{for } l\leqq 1,
\end{aligned}$$

we deduce

$$\begin{aligned}
b_k &\leqq b_k(C+M_1)B+2M_1Bb_k+(2BM_1^3)/k\\
&\quad+(M_1^2/2)\sup\mathbf{E}_x\Big[\int_0^t\beta(du)\,\alpha_u\int_{0^+}^{(1/k)^+}\pi_0(X_u,dl)\,l^2\Big].
\end{aligned}$$

Choosing τ and therefore B small enough so that $(3M_1+C)B<1$ and applying the Technical condition 3.8 we see that $b_k \downarrow 0$ as $k \uparrow \infty$. Now it follows from Proposition 1.3 that the operators $\bar{\boldsymbol{\Psi}}^t$ defined in Lemma 5.1 for $t \leqq \tau$ and φ satisfying $m \leqq \varphi \leqq M$ extend uniquely to completely concave operators (also denoted by $\bar{\boldsymbol{\Psi}}^t$) defined for all φ in $\mathscr{B}(\mathbf{K})$. Letting $\varphi=\lambda_1 \varphi_1+\cdots+\lambda_r \varphi_r$ in (3.1), we easily see (using Technical condition 3.7) that both sides are well defined and analytic for $\lambda_1, \ldots, \lambda_r>0$; thus the reasoning of Proposition 1.3 shows that (3.1) is valid for all φ in $\mathscr{B}(\mathbf{K})$. This establishes the existence and uniqueness of a family of completely concave operators $\bar{\boldsymbol{\Psi}}^t$ defined for $t \leqq \tau$ and satisfying (3.1). An argument exactly analagous to that in the proof of Lemma 5.2 establishes the semigroup property for $0 \leqq s<t \leqq \tau$ and extends our local solution to a global solution satisfying both the semigroup property and the Eq. (3.1). The proof of Theorem 2 is complete. #

Remark. The informal discussion of the underlying process in Section 4 suggests that an alternative proof of Theorem 2 might be available at least when there is no nonlocal creation, that is, when $\pi(x, \cdot)$ for all x. This proof splits naturally into two steps:

(i) Define the logarithmic Laplace transform $\psi_\omega^{s,t}$ for the density processes $\{N_t\}$ along individual trajectories ω by solving the integral equation

$$\psi_\omega^{s,t}(\lambda)=\alpha^{s,t}\lambda+\int_s^t \beta(du)\,\alpha^{s,u}\left(\int_0^\infty \pi_0(X_u, dl)\,\mathscr{E}_0^l\{\psi^{u,t}\lambda\}-\gamma(X_u)\{\psi^{u,t}\lambda\}^2\right). \quad (5.10)$$

(ii) Construct the logarithmic Laplace transform $\bar{\boldsymbol{\Psi}}^t$ required in Theorem 2 by "piecing together" the $\psi_\omega^{0,t}$ in a manner consistent with the informal description in Section 4.

The first step easily be carried out by generalizing the argument of [8, Sect. 4]. To carry out the second step, we proceed as follows. For each partition $P=\{0, t_1, \ldots, t_n\}$ with $0<t_1<\cdots<t_n=t$ we define a completely concave operator $\bar{\boldsymbol{\Psi}}_P^t$ by

$$\bar{\boldsymbol{\Psi}}_P^t\varphi(x)=\int \mathfrak{P}_x(d\omega_1)\,\psi_{\omega_1}^{0,t_1}\mathfrak{P}_{\omega_1(t_1)}(d\omega_2)$$
$$\cdot\,\psi_{\omega_2}^{0,t_2-t_1}\cdots\int \mathfrak{P}_{\omega_{n-1}(t_{n-1}-t_{n-2})}(d\omega_n)\,\psi_{\omega_n}^{0,t-t_{n-1}}(\varphi[X_{t-t_{n-1}}(\omega_n)]).$$

The operator $\bar{\boldsymbol{\Psi}}_P^t$ corresponds to the following way of "pasting" together the density processes $\{N_t\}$:

The densities $\{N_s\}$ corresponding to distinct trajectories are identical for $0 \leqq s \leqq t_i$ if the trajectories are identical on the time interval $[0, t_i]$. The ratios $\{N_s(N_{t_{i-1}})^{-1}\}$ are independent for $t_{i-1} \leqq s \leqq t$ unless the trajectories are identical for $0 \leqq s \leqq t_i$.

A straightforward application of Jensen's inequality for concave functions. shows that if P_1 contains P_2 in the obvious sense, then $\bar{\boldsymbol{\Psi}}_{P_1}^t\varphi \geqq \bar{\boldsymbol{\Psi}}_{P_2}^t\varphi$ for all φ in $\mathscr{B}(\mathbf{K})$. The upper envelope of the $\bar{\boldsymbol{\Psi}}_P^t$ as P varies over all possible partitions of $[0, t]$ is the natural candidate for the desired operators $\bar{\boldsymbol{\Psi}}^t$. The semigroup property for this upper envelope can be established without any difficulty. However the verifications of complete concavity and of the Eq. (3.1) raise certain technical problems which we cannot resolve at present without imposing conditions which are much too restrictive. A simple computation using (5.10) and the simple Markov property

establishes the following equation for the operator $\bar{\Psi}_P^t$:

$$\begin{aligned}\bar{\Psi}_P^t \varphi(x) = \int \mathfrak{P}_x(d\omega) \Big[& \varphi(X_t)\alpha^t + \int_0^{t_1} \beta(du)\alpha^u \Big(\int \pi_0(X_u, dl)\, \mathscr{E}_l^0 \{\psi_\omega^{0,t_1-u} \bar{\Psi}_{P_1}^{t-t_1} \varphi(X_{t_1})\} \\ & - \gamma(X_u)\{\psi_\omega^{0,t_1-u} \bar{\Psi}_{P_1}^{t-t_1} \varphi(X_{t_1})\}^2 \Big) \Big] \\ & + \cdots \\ & + \int_{t_{n-1}}^{t} \beta(du)\alpha^u \Big(\int \pi_0(X_u, dl)\, \mathscr{E}_0^l \{\psi_\omega^{0,t-u}[\varphi(X_t)]\} - \gamma(X_u)\{\psi_\omega^{t-u}[\varphi(X_t)]\}^2 \Big) \Big]\end{aligned} \tag{5.11}$$

where $P_1 = [0, t_2 - t_1, \ldots, t_n - t_1]$ etc. This suggests that we should consider first the restriction envelope of the $\bar{\Psi}_P^t$ as P runs through a suitable sequence of partitions and then pass to the limit in (5.11) to verify that the restricted upper envelopes satisfy the Eq. (3.1). The uniqueness of solutions to the Eq. (3.1) would then guarantee that the restricted upper envelope is in fact the full upper envelope. This would prove that the full upper envelope is completely concave and that it satisfies (3.1). Our problem is that we cannot justify the passage to the limit in (5.11). We can establish convergence along individual trajectories in **W** but not convergence of the expectations. #

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