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On Spectra of the Schrödinger Operator with a White Gaussian Noise Potential

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§ 1. Introduction

We consider the spectra of the one-dimensional Schrödinger operator $Hu = -u'' + qu$ with the potential q being a white Gaussian noise. More specifically let $\{B(x), x \geqq 0\}$ be the one-dimensional Brownian motion and let $\mathcal{N}(\lambda, a, b)$ be the number of eigenvalues not exceeding λ for the boundary value problem:

$$Hu(x) = \frac{-d\dfrac{d}{dx} + dB}{dx} u(x) = \lambda u(x), \qquad a < x < b, \tag{1.1}$$
$$u(a) = u(b) = 0.$$

We define the *spectral distribution function* (or the *cumulative density of states* in physical term) of H by

$$N(\lambda) = \lim_{l \to \infty} \frac{\mathcal{N}(\lambda, 0, l)}{l}, \qquad -\infty < \lambda < \infty. \tag{1.2}$$

Frisch and Lloyd [3] and Halperin [5] found that the non-random limit function $N(\lambda)$ takes an exact form:

$$N(\lambda) = \left(\sqrt{2\pi} \int_0^\infty u^{-\frac{1}{2}} \exp\{-\tfrac{1}{6}u^3 - 2\lambda u\}\, du \right)^{-1}. \tag{1.3}$$

It seems however that no precise formulation of the eigenvalue problem (1.1) has been given and that the derivation of (1.3) in [3] and [5] involves some heuristic arguments on diffusion approximation.

In §2 of this paper, we show that the problem (1.1) can be formulated by making use of a symmetric form $\mathscr{E}$ on $L^2(a, b)$ defined by

$$\begin{aligned} &\mathscr{D}[\mathscr{E}] = H_0^1(a, b) \\ &\mathscr{E}(u, v) = \int_a^b u'(x)\, v'(x)\, dx - \int_a^b (u'(x)\, v(x) + u(x)\, v'(x))\, B(x)\, dx. \end{aligned} \tag{1.4}$$

We are indeed led from (1.1) to the form (1.4) by the formal integration

$$\int_a^b u(x) H v(x)\, dx = -\int_a^b u(x) v''(x)\, dx + \int_a^b u(x) v(x)\, dB(x)$$

and by the interpretation of the last integral as the Wiener integral.

We then give in § 3 a simple derivation of (1.2) and (1.3) by proving that $N(\lambda)^{-1}$ is just the mean soujorn time on $[0, \pi)$ of the diffusion process $X(t)$ satisfying the following stochastic differential equation:

$$\begin{aligned} dX(t) &= -\sin^2 X(t) \circ dB(t) + (\cos^2 X(t) + \lambda \sin^2 X(t))\, dt \\ X(0) &= 0. \end{aligned} \tag{1.5}$$

Here the symbol $\circ$ denotes the symmetric stochastic differential due to Stratonovich ([6, 9]).

The Equation (1.5) results from the Sturm-Liouville oscillation theorem [2] and a theorem in Kunita [9] concerning the pathwise approximation of the solution of (1.5) by solutions of those ordinary differential equations which are obtained from (1.5) by replacing $B(t)$ with piecewise linear functions.

At the end of § 3 we relate our derivation of (1.3) to that of Frisch-Lloyd and Halperin. In principle our formulation and procedures apply to the case that the Brownian motion $B(x)$ in (1.1) is replaced by other process with stationary independent increments. We further note that the expression (1.3) readily means the asymptotic behaviours

$$\log N(\lambda) \sim -\tfrac{8}{3}|\lambda|^{\frac{3}{2}}, \qquad \lambda \to -\infty, \tag{1.6}$$

$$N(\lambda) \sim \frac{\sqrt{\lambda}}{\pi}, \qquad \lambda \to \infty, \tag{1.7}$$

once we use the Tauberian theorems of exponential type [4] and Hardy-Littlewood type. Halperin [5] obtained a more detailed formula than (1.6). Various asymptotic behaviours of $N(\lambda)$ of the Schrödinger operators with other types of random potentials are studied in [10] and [11].

§ 2. Symmetric Form and the Eigenvalue Problem

Consider a real Hilbert space $\mathscr{H}$ with inner product (,). A symmetric bilinear form $\mathscr{E}$ with domain $\mathscr{D}[\mathscr{E}]$ dense in $\mathscr{H}$ is simply called a *symmetric form* on $\mathscr{H}$. It is *lower semibounded* if $\mathscr{E}(u,u)+\gamma(u,u)\geqq 0$, $u\in\mathscr{D}[\mathscr{E}]$, for some constant γ. If, in addition, $\mathscr{D}[\mathscr{E}]$ is complete with norm $\sqrt{\mathscr{E}(u,u)+\gamma'(u,u)}$ for some (equivalently for every) $\gamma'>\gamma$, then $\mathscr{E}$ is said to be *closed*. Suppose furthermore we can extract a strongly $\mathscr{H}$-convergent subsequence from any sequence $u_n\in\mathscr{D}[\mathscr{E}]$ such that $\mathscr{E}(u_n,u_n)+\gamma'(u_n,u_n)<C$ for some constants $\gamma'>\gamma$ and C, then we say that $\mathscr{E}$ satisfies the *complete continuity condition.*

Given a lower semi-bounded closed symmetric form $\mathscr{E}$ on $\mathscr{H}$, there is a unique self-adjoint operator A on $\mathscr{H}$ such that

$$\mathscr{D}(A)\subset\mathscr{D}[\mathscr{E}], \quad \mathscr{E}(u,v)=(Au,v), \qquad u\in\mathscr{D}(A),\ v\in\mathscr{D}[\mathscr{E}]. \tag{2.1}$$

If $\mathscr{E}$ satisfies the additional condition of complete continuity, then the spectrum of A consists of the point spectra of finite multiplicity possessing no accumulation point except for $+\infty$. Therefore we can arrange them as

$$\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \cdots$$

and we may call λ_k the k-th eigenvalue of A.

In this case we have the following *minimax principle* of calculating λ_k: put

$$\lambda(M) = \sup_{u \in M,\, (u,u)=1} \mathscr{E}(u,u)$$

for $M \subset \mathscr{D}[\mathscr{E}]$, then

$$\lambda_k = \inf \lambda(M), \tag{2.2}$$

where M ranges over all k-dimensional linear subspaces of $\mathscr{D}[\mathscr{E}]$. This can be proved in the same way as in [12; 2.5.1] by noting that λ is an eigenvalue of A with an eigenfunction u_0 if and only if $u_0 \in \mathscr{D}[\mathscr{E}]$, $u_0 \neq 0$ and $\mathscr{E}(u_0, v) = \lambda(u_0, v)$ for any $v \in \mathscr{D}[\mathscr{E}]$.

Let us now consider the real L^2-space $L^2(a,b)$. We put

$$\begin{aligned} H_0^1(a,b) = \{&u \in L^2(a,b);\ u \textit{ is absolutely continuous,} \\ &u' \in L^2(a,b) \textit{ and } u(a+) = u(b-) = 0\}, \end{aligned} \tag{2.3}$$

$$|||u|||_1^2 = \int_a^b u'(x)^2\,dx + \int_a^b u(x)^2\,dx, \qquad u \in H_0^1(a,b). \tag{2.4}$$

Denote by $\mathbf{B}(a,b)$ the space of all bounded Borel functions on (a,b). In view of (1.4), we are interested in the symmetric form $\mathscr{E}^h_{(a,b)}$ on $L^2(a,b)$ defined for each $h \in \mathbf{B}(a,b)$ by

$$\begin{aligned} \mathscr{E}^h_{(a,b)}(u,v) &= \int_a^b u'(x)\,v'(x)\,dx \\ &\quad - \int_a^b (u'(x)\,v(x) + u(x)\,v'(x))\,h(x)\,dx \\ \mathscr{D}[\mathscr{E}^h_{(a,b)}] &= H_0^1(a,b). \end{aligned} \tag{2.5}$$

$\mathscr{E}^h_{(a,b)}$ is also denoted by $\mathscr{E}^h$ for simplicity.

Lemma 1. (i) *For each $h \in \mathbf{B}(a,b)$, $\mathscr{E}^h$ is a lower semibounded closed symmetric form on $L^2(a,b)$ satisfying the complete continuity condition.*

(ii) *Suppose $h_n \in \mathbf{B}(a,b)$ converges to $h \in \mathbf{B}(a,b)$ uniformly on (a,b), then*

$$\lim_{n\to\infty} \lambda_k^n = \lambda_k, \qquad k = 1, 2, \ldots, \tag{2.6}$$

where λ_k^n (resp. λ_k) *is the k-th eigenvalue of the self-adjoint operator associated with $\mathscr{E}^{h_n}$* (resp. $\mathscr{E}^h$).

Proof. Applying Schwarz inequality to the second term of $\mathscr{E}^h(u,u)$, we have

$$\mathscr{E}^h(u,u)+(M^2+1)(u,u)\geqq\frac{1}{2(M^2+1)}|||u|||_1^2 \tag{2.7}$$

for any $u\in H_0^1(a,b)$, where $M=\sup_{a<x<b}|h(x)|$. The first assertion (i) is almost clear from this. Particularly the complete continuity condition follows from (2.7) and the Ascoli-Arzelà selection theorem.
To see the assertion (ii), we note

$$|\mathscr{E}^{h_n}(u,u)-\mathscr{E}^h(u,u)|\leqq C_n|||u|||_1^2, \qquad u\in H_0^1(a,b), \tag{2.8}$$

where $C_n=\sup_{a<x<b}|h_n(x)-h(x)|$.

From (2.7) and (2.8), we get

$$\begin{aligned}&\{1-(2M^2+2)C_n\}[\mathscr{E}^h(u,u)+(M^2+1)(u,u)]\\ &\quad\leqq\mathscr{E}^{h_n}(u,u)+(M^2+1)(u,u)\\ &\quad\leqq\{1+(2M^2+2)C_n\}[\mathscr{E}^h(u,u)+(M^2+1)(u,u)].\end{aligned}$$

Hence, by the minimax principle (2.2)

$$\begin{aligned}\{1-(2M^2+2)C_n\}\{\lambda_k+(M^2+1)\}&\leqq\lambda_k^n+(M^2+1)\\ &\leqq\{1+(2M^2+2)C_n\}\{\lambda_k+(M^2+1)\},\end{aligned}$$

which means (2.6) because $\lim_{n\to\infty}C_n=0$. q.e.d.

The spectrum $\{\lambda_k\}$ associated with $\mathscr{E}^h$ can be identified with the solution of the classical Sturm-Liouville eigenvalue problem when h is smooth. Suppose that h is a piecewise differentiable continuous function on (a,b) and that the derivative $h'(x)$ defined at every interiour point x of the differentiable intervals is uniformly bounded on (a,b). Let A be the self-adjoint operator associated with $\mathscr{E}^h$. Integrating by part, we see the equivalence of the following.

$$Au=\lambda u, \qquad u\in H_0^1(a,b). \tag{2.9}$$

$$\begin{aligned}&-u''(x)+h'(x)u(x)=\lambda u(x), \qquad x\in(a,b),\\ &\qquad\qquad u(a+)=u(b-)=0.\end{aligned} \tag{2.10}$$

§ 3. Spectral Distribution Function and a Related Diffusion

Let $\{B(x,\omega), x\geqq 0\}$ be a standard Brownian motion defined on a probability space $(\Omega,\mathscr{B},P)$ such that $B(0,\omega)=0$ and $B(x,\omega)$ is continuous in $x\geqq 0$ for any $\omega\in\Omega$. Together with $B(x,\omega)$, we consider its piecewise linear approximation $\{B_n(x,\omega), x\geqq 0\}$ defined by

$$B_n(x,\omega)=B\left(\frac{[2^nx]}{2^n},\omega\right)+2^n\left(x-\frac{[2^nx]}{2^n}\right)\left\{B\left(\frac{[2^nx]+1}{2^n},\omega\right)-B\left(\frac{[2^nx]}{2^n},\omega\right)\right\}.$$

Take $l>0$ and fix $\omega\in\Omega$. As was mentioned in § 1, we formulate the eigenvalue Problem (1.1) on the interval $(0,l)$ by means of the symmetric form $\mathscr{E}^h_{(0,l)}$ of (2.5) for $h(x)=B(x,\omega)$, $x\in(0,l)$. Let $\mathscr{N}^\omega(\lambda,l)$ be the number of eigenvalues not exceeding λ of the associated self-adjoint operator on $L^2(0,l)$. We also consider the same quantity $\mathscr{N}_n^\omega(\lambda,l)$ for $h(x)=B_n(x,\omega)$, $x\in(0,l)$. In view of Lemma 1, we then have

$$\lim_{n\to\infty}\mathscr{N}_n^\omega(\lambda,l)=\mathscr{N}^\omega(\lambda,l) \tag{3.1}$$

for every continuity point λ of $\mathscr{N}^\omega(\lambda,l)$.

On the other hand, $\mathscr{N}_n^\omega(\lambda,l)$ is the distribution function of the eigenvalues in the classical eigenvalue problem (2.10) with $h(x)=B_n(x,\omega)$ and $(a,b)=(0,l)$, because $B_n(x,\omega)$ is piecewise linear in $x\geqq 0$. Hence the Sturm-Liouville oscillation theorem [2] leads us to the identity

$$\mathscr{N}_n^\omega(\lambda,l)=\left[\frac{Y_n^\lambda(l,\omega)}{\pi}\right] \tag{3.2}$$

where $Y_n^\lambda(x,\omega)$, $x\geqq 0$, is the solution of the ordinary differential equation:

$$\begin{aligned} d\,Y_n^\lambda(x,\omega)&=-\sin^2 Y_n^\lambda(x,\omega)\,B_n'(x,\omega)\,dx+(\cos^2 Y_n^\lambda(x,\omega)\\ &\qquad+\lambda\sin^2 Y_n^\lambda(x,\omega))\,dx \\ Y_n^\lambda(0,\omega)&=0. \end{aligned} \tag{3.3}$$

Now according to a theorem of H. Kunita [9], the solution of (3.3) converges to the solution $Y^\lambda(x,\omega)$ of the following stochastic differential equation:

$$\begin{aligned} d\,Y^\lambda(x)&=-\sin^2 Y^\lambda(x)\,dB(x)\\ &\qquad+(\cos^2 Y^\lambda(x)+\lambda\sin^2 Y^\lambda(x)+\sin^3 Y^\lambda(x)\cos Y^\lambda(x))\,dx \\ Y^\lambda(0)&=0, \end{aligned} \tag{3.4}$$

which can also be written as (1.5) using the symbol $\circ$. More specifically there exist $\Omega_1\subset\Omega$ with $P(\Omega_1)=1$ and a subsequence $\{n_j\}$ such that, for each $\omega\in\Omega_1$,

$$\lim_{n_j\to\infty} Y_{n_j}^\lambda(x,\omega)=Y^\lambda(x,\omega) \tag{3.5}$$

holds for any $x\geqq 0$ and any rational λ.

Lemma 2. *There exists $\Omega_2\subset\Omega$ with $P(\Omega_2)=1$ such that the inequality*

$$0\leqq\frac{Y^\lambda(l,\omega)}{\pi}-\mathscr{N}^\omega(\lambda;l)\leqq 1$$

holds for any $\omega\in\Omega_2$, rational $l>0$ and rational λ.

Proof. Combining (3.1), (3.2) and (3.5), we arrive at the above inequality for $\omega\in\Omega_1$, real $l>0$ and for any rational λ at which $\mathscr{N}^\omega(\lambda,l)$ is continuous. In view of the right continuity of $\mathscr{N}^\omega(\lambda,l)$ in λ, we can now get Lemma 2 if, for a fixed l, $Y^\lambda(l,\omega)$ is continuous in rational λ for almost all $\omega\in\Omega_1$. But the last statement is a result of the next lemma and [1; Theorem 12.4].

Lemma 3. *Let a, b and c be bounded Lipschitz continuous functions on R^1 and $X^\lambda(t)$ be the solution of the stochastic differential equation*

$$dX^\lambda(t)=a(X^\lambda(t))\,dB(t)+(b(X^\lambda(t))+\lambda c(X^\lambda(t)))\,dt$$
$$X^\lambda(0)=0.$$

Then, for each $T>0$ and $K>0$,

$$E[|X^\lambda(t)-X^{\lambda'}(t)|^2]\leqq C(\lambda-\lambda')^2 e^{CT},$$
$$t\in[0,T],\ |\lambda|,|\lambda'|\leqq K,\ \lambda,\lambda'\text{: rational},$$

for some constant $C>0$.

Proof. This follows from inequality

$$f(t)\leqq C\int_0^t f(s)\,ds+C(\lambda-\lambda')^2,$$

$f(t)$ being the left hand side of the desired inequality. q.e.d.

Theorem. *There exists $\Omega_0\subset\Omega$ with $P(\Omega_0)=1$ such that for each $\omega\in\Omega_0$*

$$\lim_{l\to\infty}\frac{\mathcal{N}^\omega(\lambda,l)}{l}=(E(\tau^\lambda))^{-1}\quad \textit{for every } \lambda, \tag{3.6}$$

where τ^λ is the first hitting time of $\{\pi\}$ of the solution $Y^\lambda(x)$ of (3.4):

$$\tau^\lambda(\omega)=\inf\{x>0;\ Y^\lambda(x,\omega)=\pi\}. \tag{3.7}$$

More explicitly

$$E(\tau^\lambda)=\sqrt{2\pi}\int_0^\infty u^{-\frac{1}{2}}\exp\{-\tfrac{1}{6}u^3-2\lambda u\}\,du. \tag{3.8}$$

Proof. We take a closer look at the solution $Y^\lambda(x,\omega)$ of (3.4). $\{Y^\lambda(x), x\geqq 0\}$ is a diffusion process (starting from the origin at time 0) possessing the infinitesimal generator

$$\tfrac{1}{2}\sin^4 z\,\frac{d^2}{dz^2}+(\cos^2 z+\lambda\sin^2 z+\sin^3 z\cos z)\frac{d}{dz} \tag{3.9}$$

with coefficients being periodic with period π. The operator (3.9) takes the canonical form $\frac{d}{dm}\frac{d}{ds}$ on each periodic regular interval with

$$ds(z)=\exp(\tfrac{2}{3}\cot^3 z+2\lambda\cot z)\frac{dz}{\sin^2 z},$$

$$dm(z)=2\exp(-\tfrac{2}{3}\cot^3 z-2\lambda\cot z)\frac{dz}{\sin^2 z}.$$

According to the Feller classification of the boundary [7], $(n-1)\pi$ (resp. $n\pi$) is an entrance (resp. exit) boundary of the interval $((n-1)\pi, n\pi)$, $n=1,2,\ldots$.

Furthermore

$$E(\tau^\lambda)=\int_0^\pi (s(\pi)-s(z))\,dm(z)$$

$$=2\int_{-\infty}^{\infty} e^{\frac{2}{3}s^3+2\lambda s}\int_s^{\infty} e^{-\frac{2}{3}t^3-2\lambda t}\,dt\,ds$$

$$=2\int_0^{\infty} e^{-\frac{2}{3}u^3-2\lambda u}\int_{-\infty}^{\infty} e^{-2u^2 s-2us^2}\,ds\,du,$$

which is equal to the right hand side of (3.8).

Above observation readily means that

$$\lim_{l\to\infty}\frac{Y^\lambda(l,\omega)}{\pi l}=(E(\tau^\lambda))^{-1}\qquad \text{a.s.} \tag{3.10}$$

To see this, we put $\tau_n^\lambda(\omega)=\inf\{x>0;\ Y^\lambda(x,\omega)=n\pi\}$, $n=1,2,\ldots$. Then $\tau_n^\lambda=\tau^\lambda+(\tau_2^\lambda-\tau^\lambda)+\cdots+(\tau_n^\lambda-\tau_{n-1}^\lambda)$ is a sum of independent identically distributed random variables. By the law of large numbers, we have $\lim_{n\to\infty}\frac{\tau_n^\lambda(\omega)}{n}=E(\tau^\lambda)$ a.s., which leads us to (3.10) since the following statement holds for almost all $w\in\Omega$: for each real $l>0$, there exists n such that $\tau_n^\lambda(\omega)\leqq l<\tau_{n+1}^\lambda(\omega)$ and hence $\frac{n}{\tau_{n+1}^\lambda(\omega)}\leqq\frac{Y^\lambda(l,\omega)}{\pi l}\leqq\frac{n+1}{\tau_n^\lambda(\omega)}$.

Our conclusion (3.6) is an immediate consequence of (3.10) and Lemma 2 if we observe that $\mathcal{N}^\omega(\lambda,l)$ is monotone not only in λ but also in l in view of the minimax principle (2.2). q.e.d.

Remark. The diffusion $\{Y^\lambda(x),\ x\geqq 0\}$ may be considered by the transformation $w=-\cot z$ as a diffusion process on $[-\infty,+\infty)$ with infinitisimal generator

$$\mathcal{G}u(w)=\tfrac{1}{2}\frac{d^2u}{dw^2}+(w^2+\lambda)\frac{du}{dw} \tag{3.11}$$

and the boundary condition $u(-\infty)=u(+\infty)$.

Let $p(w)$ be the smooth density function of the invariant probability measure of this process. Then $p(w)$ is the unique bounded solution of

$$\tfrac{1}{2}p'(w)-(w^2+\lambda)\,p(w)=-N(\lambda). \tag{3.12}$$

This can be easily obtained either from formula (3.6) or from its more explicit version (1.3). (3.12) implies the following *Rice's formula:*

$$N(\lambda)=\lim_{w\to\pm\infty} w^2 p(w). \tag{3.13}$$

Frisch and Lloyd [3] derived this formula in the case that the process $B(x)$ in (1.1) is the Poisson process instead of the Brownian motion and Halperin [5] started with this formula to get (1.3).

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