Z. Wahrscheinlichkeitstheorie verw. Gebiete
38, 223–236 (1977)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

The Strong Laws of Large Numbers for Quasi-Stationary Sequences

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Let $\{\xi_k\}$ be a sequence of rv's with $E\xi_k=0$, $E\xi_k^2=1$ $(k=1,2,\dots)$. Assume that with some functions $\varphi_1(k)$ and $\varphi_2(k)$ defined on the non-negative integers we have either (1) $\sup_n |E(\xi_n\xi_{n+k})|\leqq\varphi_1(k)$ or (2) $\sup_n E\left[\frac{1}{k}\sum_{i=n+1}^{n+k}\xi_i\right]^2\leqq\varphi_2(k)$ $(k=1, 2,\dots;\ \varphi_i(0)=1)$, and set $w(n)=\sum_{k=0}^{n-1}\varphi_1(k)\,(n=1,2,\dots)$. The main results read as follows:

Theorem 1. *Suppose that* (1) *holds with a* $\varphi_1(k)$ *for which* $w(n)$ *satisfies the condition* $\frac{w(2n)}{w(n)}\geqq q>1$ $(n\geqq n_0)$, *furthermore, a sequence* $\{\lambda_n\}$ *of positive numbers is such that* $\lambda_n^2\,w(n)$ *is non-increasing and* $\sum_{n=1}^{\infty}\lambda_n^2\,w(n)<\infty$. *Then* $\lambda_n\sum_{k=1}^{n}\xi_k\to 0$ *a.e.* $(n\to\infty)$.

Theorem 2. *Let* $\psi(k)$ *be a positive and non-decreasing function defined on the positive integers for which* $\sum_{k=1}^{\infty}\frac{1}{k\psi(k)}<\infty$. *If one of the conditions* (1) *or* (2) *holds with* $\varphi_i(k)=O\left(\frac{1}{\psi(k)}\right)$, *where* $i=1$ *or* 2 *respectively, then* $\frac{1}{n}\sum_{k=1}^{n}\xi_k\to 0$ *a.e.* $(n\to\infty)$.

The rate of convergence in the conclusions of Theorems 1 and 2 as well as the convergence properties of weighted averages are also studied.

§ 1. The Main Results

Let $\{\xi_k\}$ be a sequence of random variables (in abbreviation: rv's) having finite variances. Assume that

$$E\,\xi_k=0,\qquad E\xi_k^2=1\qquad (k=1,\,2,\,\dots).\tag{1.1}$$

The sequence $\{\xi_k\}$ is said to be *quasi-stationary* associated with a function $\varphi(k)$ defined on the non-negative integers if

$$\sup_n |E(\xi_n \xi_{n+k})| \leqq \varphi(k) \quad (k=0, 1, \ldots; \varphi(0)=1). \tag{1.2}$$

Furthermore, $\{\xi_k\}$ is said to be *stationary* (in the weak sense) if

$$E(\xi_n \xi_{n+k}) = R(k) \quad (n=1, 2, \ldots;\ k=0, 1, \ldots),$$

where $R(k)$ is called the *covariance function* of this sequence.

By (1.2) it easily follows that for every sequence $\{a_k\}$ of numbers and for every $b \geqq 0$ and $n \geqq 1$ we have

$$E\left(\sum_{k=b+1}^{b+n} a_k \xi_k\right)^2 \leqq \sum_{k=b+1}^{b+n} a_k^2 + 2\sum_{k=1}^{n-1} \varphi(k) \sum_{j=b+1}^{b+n-k} |a_j a_{j+k}|$$
$$\leqq 2w(n) \sum_{k=b+1}^{b+n} a_k^2, \tag{1.3}$$

where

$$w(n) = \sum_{k=0}^{n-1} \varphi(k) \quad (n=1, 2, \ldots).$$

The main purpose of this note is to provide strong laws of large numbers as consequences of restrictions imposed upon $\varphi(k)$. One possible way to obtain strong laws is that first we prove the a.e. convergence (that is, with probability 1) of the series

$$\sum_{k=1}^{\infty} \lambda_k \xi_k, \tag{1.4}$$

where $\{\lambda_k\}$ is a non-increasing sequence of positive numbers tending to zero, and hence we infer via the well-known Kronecker lemma that

$$\lambda_n s_n \to 0 \quad \text{a.e. } (n \to \infty), \qquad \text{where } s_n = \sum_{k=1}^{n} \xi_k. \tag{1.5}$$

The following theorem of Gapoškin [2, Theorems 1–3] covers a lot of earlier results relating to the a.e. convergence of (1.4).

Theorem A. *Let $\{\xi_k\}$ be a quasi-stationary sequence of rv's associated with some $\varphi(k)$. Then any one of the following conditions ensures that the series* (1.4) *converges a.e.:*

(i) *$\{\lambda_n\}$ is a sequence of numbers for which*

$$\sum_{n=2}^{\infty} \lambda_n^2 w(n)(\log n)^2 < \infty; \tag{1.6}$$

(ii) *$\varphi(k)$ is non-increasing,*

$$1 < p_1 \leqq \frac{\varphi(k)}{\varphi(2k)} \leqq p_2 < 2 \quad (k \geqq k_0),$$

and

$$\sum_{n=4}^{\infty} \lambda_n^2 w(n)(\log\log n)^2 < \infty;$$

(iii) *$\varphi(k)$ is non-increasing and for $\lambda_k = k^{-1}$ we have*

$$\sum_{k=8}^{\infty} \frac{\varphi(k)}{k} \log k\,(\log\log\log k)^2 < \infty.$$

Gapoškin pointed out that these results are best possible even for stationary sequences.

Theorem A immediately implies, among others, the following strong laws:

(i) *If $\{\lambda_n\}$ is a sequence of positive numbers satisfying* (1.6), *then* (1.5) *holds.*

(ii) *If $\varphi(k) = O[k^{-\alpha}(\log k)^{\beta}]$ (which is equivalent to $w(n) = O[n^{1-\alpha}(\log n)^{\beta}]$), where $0 < \alpha < 1$ and β is arbitrary, then for each $\varepsilon > 0$ we have*

$$\frac{s_n}{n^{(2-\alpha)/2}(\log n)^{(1+\beta)/2}(\log\log n)^{(3+\varepsilon)/2}} \to 0 \quad a.e. \quad (n \to \infty). \tag{1.7}$$

(iii) *If for an $\varepsilon > 0$ we have*

$$\varphi(k) = O\left(\frac{1}{(\log k)^2 \log\log k(\log\log\log k)^{3+\varepsilon}}\right), \tag{1.8}$$

then

$$n^{-1}s_n \to 0 \quad a.e. \quad (n \to \infty). \tag{1.9}$$

We note that the assertion under (i) was obtained by Petrov [6, Theorems 2 and 4] in a slightly more special form as follows: *Let $\{\xi_k\}$ be a stationary sequence with the covariance function $R(k)$. Then, for each $\varepsilon > 0$,*

$$\frac{s_n}{\left[n\left(\sum_{k=0}^{n-1} |R(k)|\right)\right]^{1/2} (\log n)^{3/2}(\log\log n)^{(1+\varepsilon)/2}} \to 0 \quad a.e. \quad (n \to \infty).$$

The result under (i) is sharp if for every $\delta > 0$ we have $w(n) = O(n^{\delta})$. Thus, for example, *if $\varphi(k) = O[k^{-1}(\log k)^{\alpha-1}]$, or equivalently $w(n) = O[(\log n)^{\alpha}]$, where α is an arbitrary number, then*

$$\frac{s_n}{n^{1/2}(\log n)^{(\alpha+3)/2}(\log\log n)^{(1+\varepsilon)/2}} \to 0 \quad a.e. \quad (\varepsilon > 0;\, n \to \infty);$$

if $\varphi(k) = O[(k \log k)^{-1}(\log\log k)^{\alpha-1}]$, or equivalently $w(n) = O[(\log\log n)^{\alpha}]$, then

$$\frac{s_n}{n^{1/2}(\log n)^{3/2}(\log\log n)^{(\alpha+1+\varepsilon)/2}} \to 0 \quad a.e. \quad (\varepsilon > 0;\, n \to \infty).$$

On the other hand, the strong laws under (ii) and (iii) are not sharp in general. In fact, we will prove below that a factor $\log\log n$ in the denominator of (1.7), and a factor $\log k(\log\log\log k)^2$ in the denominator of (1.8) are superfluous. Similar observations can be made also in the cases when we start with Theorems 6 and 7 of Gapoškin [2] in order to obtain strong laws for stationary sequences.

We mention one more result. The proving technique of Serfling gives a somewhat sharper result than that in [9, Theorem 2.1]: *If for an* $\varepsilon > 0$ *we have*

$$\varphi(k) = O\left(\frac{1}{(\log k)^2 \log\log k (\log\log\log k)^{1+\varepsilon}}\right), \tag{1.10}$$

then (1.9) *holds.*

Although (1.10) requires less than (1.8), from our Theorem 2 below it follows that here a factor $\log k$ in the denominator can be omitted without weakening the conclusion.

Our reasoning is based on a moment inequality for the maximum cumulative sum. Before stating it in an explicit form, set

$$S_{b,n} = \sum_{k=b+1}^{b+n} a_k \xi_k \quad \text{and} \quad M_{b,n} = \max_{1 \leqq k \leqq n} |S_{b,k}|,$$

where $\{a_k\}$ is a sequence of numbers, $b \geqq 0$ and $n \geqq 1$ integers. In the particular case $b = 0$ we set

$$S_n = S_{0,n} = \sum_{k=1}^{n} a_k \xi_k \qquad (n \geqq 1).$$

Furthermore, set $W(1) = w(1)$ and, for $n \geqq 2$,

$$W^{1/2}(n) = W^{1/2}(m-1) + w^{1/2}(m), \tag{1.11}$$

where m denotes the integral part of $\frac{1}{2}(n+2)$. It is obvious that $W(n)$, together with $w(n)$, is positive and non-decreasing for $n = 1, 2, \ldots$. Furthermore, from (1.11) it follows that if $2^k \leqq n < 2^{k+1}$ with some $k \geqq 0$, then

$$W^{1/2}(n) \leqq W^{1/2}(2^{k+1} - 1) = \sum_{j=0}^{k} w^{1/2}(2^j).$$

The results below are obtained by adapting more or less standard arguments [8] to make use of a recent result [4] which gives bounds for the ν-th moment of $M_{b,n}$ in terms of assumed bounds on the ν-th moment of $|S_{b,n}|$, where $\nu \geqq 1$. The following theorem is a special case of Theorem 4 of [4].

Theorem B. *Suppose that* (1.3) *holds for all* $b \geqq 0$ *and* $n \geqq 1$, *and let* $W(n)$ *be defined by* (1.11). *Then we have*

$$E(M_{b,n}^2) \leqq 2 W(n) \sum_{k=b+1}^{b+n} a_k^2 \tag{1.12}$$

for all $b \geqq 0$ *and* $n \geqq 1$.

We note that if $w(n) \equiv 1$, then $W(n) \leqq (\log 2n)^2$, which follows from $1 + \log 2(m-1) \leqq \log 2n$, the latter being true since $n \geqq 2m - 2$. Further, if $w(n) = n^\beta$ with some $\beta > 0$, then $W(n) \leqq (2n)^\beta / (2^\beta - 1)^{1/\beta}$, etc. From this it is seen that in the latter case (1.12) provides a bound for $E(M_{b,n}^2)$ which is asymptotically optimal as $n \to \infty$ in the sense that it is of the same order of magnitude as the bound obtained for $E(S_{b,n}^2)$. We will show that the situation is the same, whenever $w(n)$

increases so "fast" that

$$\frac{w(2n)}{w(n)} \geqq q > 1 \qquad (n \geqq n_0). \tag{1.13}$$

(See Lemma 1 in Section 2.)

Theorem 1. *Suppose that the quasi-stationary sequence* $\{\xi_k\}$ *is such that* $w(n)$ *satisfies* (1.13), *furthermore, the sequence* $\{\lambda_n\}$ *of positive numbers is such that* $\lambda_n^2 w(n)$ *is non-increasing and*

$$\sum_{n=1}^{\infty} \lambda_n^2 w(n) < \infty. \tag{1.14}$$

Then

$$\lambda_n \sum_{k=1}^{n} \xi_k \to 0 \quad a.e. \quad (n \to \infty). \tag{1.15}$$

We note that if (1.14) holds, then $\lambda_n^2 w(n)$ tends to zero as $n \to \infty$, thus the assumption that $\lambda_n^2 w(n)$ is non-increasing does not mean a strong condition.

Let us introduce the following notation. For a positive and non-decreasing $\psi(k)$ defined on the positive integers write $\psi \in \Psi_c$ if

$$\sum_{k=1}^{\infty} \frac{1}{k\psi(k)} < \infty. \tag{1.16}$$

Further, we say that $\psi(k)$ increases "slowly" if

$$\frac{\psi(2k)}{\psi(k)} \leqq r < 2 \qquad (k \geqq k_0). \tag{1.17}$$

We remark that (1.17) does not contain any strong restriction, since the only case interesting for us is $\psi(k) = O[(\log k)^{1+\varepsilon}]$ with an $\varepsilon > 0$ (owing to (1.16)).

Theorem 2. *Suppose that* $\psi \in \Psi_c$, $\psi(k)$ *satisfies* (1.17), *and the quasi-stationary sequence* $\{\xi_k\}$ *is such that*

$$\varphi(k) = O\left(\frac{1}{\psi(k)}\right). \tag{1.18}$$

Then

$$\frac{1}{n} \sum_{k=1}^{n} \xi_k \to 0 \quad a.e. \quad (n \to \infty). \tag{1.19}$$

For stationary sequences a slightly finer result was proved by Gapoškin [1] as follows: *Let* $\{\xi_k\}$ *be a stationary sequence with the covariance function* $R(k)$. *If*

$$\sum_{k=1}^{\infty} \frac{R(k)}{k} < \infty,$$

then (1.19) *holds.* However, his proof cannot be extended to quasi-stationary sequences, since it is based on the spectral representation of the covariance function.

The following consequences of Theorem 1 are interesting in themselves.

Corollary 1. *If* $w(n)$ *satisfies* (1.13), *then* (1.15) *holds for*

$$\lambda_n = \frac{1}{[n w(n) \psi(n)]^{1/2}} \quad \text{with any } \psi \in \Psi_c.$$

Corollary 2. *If* $\varphi(k) = O[k^{-\alpha}(\log k)^\beta]$ *where* $0 < \alpha < 1$ *and* β *is arbitrary, then* (1.15) *holds for*

$$\lambda_n = \frac{1}{[n^{2-\alpha}(\log n)^\beta \psi(n)]^{1/2}} \quad \text{with any } \psi \in \Psi_c.$$

The case $\alpha = 0$ coincides with the most general case when $\varphi(k) \equiv 1$, i.e., when we require nothing on $\{\xi_k\}$ but (1.1).

Corollary 3. *Let* $\{\xi_k\}$ *be an arbitrary sequence of rv's satisfying only* (1.1). *Then* (1.15) *holds for*

$$\lambda_n = \frac{1}{n[\psi(n)]^{1/2}} \quad \text{with any } \psi \in \Psi_c.$$

Now Theorem 2 improves (1.8) and (1.10), while Corollary 2 improves (1.7).

In the meantime, Gapoškin [3] announced without proof a number of results on the strong laws of large numbers for stationary and quasi-stationary sequences, among others, the above Theorem 2. A version of Theorem 2 is also stated in [3], when the condition (1.2) is replaced by

$$E\left[\frac{1}{n}\sum_{k=b+1}^{b+n} \xi_k\right]^2 \leqq \varphi(n) \quad \text{(for all } b \geqq 0,\ n \geqq 1), \tag{1.20}$$

where $\varphi(n)$ satisfies (1.18) with a $\psi \in \Psi_c$. More precisely, the following holds.

Theorem 2'. *Suppose* $\psi \in \Psi_c$, $\psi(k)$ *satisfies* (1.17), *and the sequence* $\{\xi_k\}$ *of rv's is such that* (1.20) *and* (1.18) *hold. Then* (1.19) *follows.*

Theorem 2' can also easily be proved by making use of Theorem B.

Furthermore, Gapoškin states that Theorems 2 and 2' are best possible in the following sense: if $\psi(k)$ is a non-decreasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \frac{1}{k\psi(k)} = \infty,$$

then there exists a quasi-stationary sequence $\{\xi_k\}$ for which

$$|E(\xi_b \xi_{b+n})| = O\left(\frac{1}{\psi(n)}\right) \quad \text{and} \quad E\left[\frac{1}{n}\sum_{k=b+1}^{b+n} \xi_k\right]^2 = O\left(\frac{1}{\psi(n)}\right)$$

for all $b \geqq 0$ and $n \geqq 1$, but $n^{-1} s_n$ diverges a.e.

§ 2. The Proofs of Theorems 1 and 2

We need some auxiliary results, which seem to be known; however, we could not find any reference and thus we carry out their proofs here.

Lemma 1. *Let $w(n)$ be positive and non-decreasing for $n=1, 2, \dots$ and satisfy* (1.13). *Then we have*

$$\sum_{j=0}^{m} w(2^j)=O[w(2^m)]. \tag{2.1}$$

Proof. We may suppose that $n_0=1$ in (1.13). A repeated use of (1.13) gives that

$$w(2^j)\leqq\left(\frac{1}{q}\right)^{m-j} w(2^m) \qquad (j=0, 1, \dots, m),$$

whence, on account of $q>1$, we obtain that

$$\sum_{j=0}^{m} w(2^j)\leqq w(2^m)\sum_{j=0}^{m}\left(\frac{1}{q}\right)^{m-j}=O[w(2^m)],$$

in accordance with (2.1).

Remark 1. In particular, setting $L_{\beta,\gamma,\delta}(n)=1$ for $1\leqq n<8$ and

$$L_{\beta,\gamma,\delta}(n)=(\log n)^{\beta}(\log\log n)^{\gamma}(\log\log\log n)^{\delta}$$

for $n\geqq 8$, by virtue of Lemma 1 any sequence

$$w(n)=\frac{n^{1-\alpha}}{L_{\beta,\gamma,\delta}(n)}$$

satisfies condition (2.1) for $0\leqq\alpha<1$ and for arbitrary β, γ and δ; and consequently, for the sequence $W(n)$ defined by (1.11) we have $W(n)=O[w(n)]$.

Lemma 2. *Let $\psi(k)$ be positive and non-decreasing for $k=1,2,\dots$ and satisfy* (1.17). *Then we have*

$$\sum_{k=1}^{n}\frac{1}{\psi(k)}=O\left(\frac{n}{\psi(n)}\right). \tag{2.2}$$

Proof. We may suppose again that $k_0=1$ in (1.17). For any $n\geqq 1$ let us define $m\geqq 0$ by $2^m\leqq n<2^{m+1}$. Now using (1.17) repeatedly, we find

$$\frac{2^j}{\psi(2^j)}\leqq\left(\frac{r}{2}\right)^{m-j}\frac{2^m}{\psi(2^m)} \qquad (j=0, 1, \dots, m).$$

Since $\psi(k)$ is non-decreasing, hence we get that

$$\sum_{k=1}^{n}\frac{1}{\psi(k)}\leqq\sum_{j=0}^{m}\frac{2^j}{\psi(2^j)}\leqq\frac{2^m}{\psi(2^m)}\sum_{j=0}^{m}\left(\frac{r}{2}\right)^{m-j}=O\left(\frac{n}{\psi(n)}\right).$$

In the last equality we took into account that

$$\psi(n)\leqq\psi(2^{m+1})\leqq r\psi(2^m).$$

Thus the proof of (2.2) is complete.

Remark 2. By Lemma 2 it follows immediately that

$$\sum_{k=1}^{n}\frac{1}{k^{\alpha}L_{\beta,\gamma,\delta}(k)}=O\left(\frac{n^{1-\alpha}}{L_{\beta,\gamma,\delta}(n)}\right),$$

provided that $0 \leqq \alpha < 1$ and β, γ, δ are arbitrary, but the first non-zero member of α, β, γ and δ is positive.

Lemma 3. *Let $\psi \in \Psi_c$, $\psi(k)$ satisfy* (1.17), *and set*

$$w(n) = 1 + \sum_{k=1}^{n-1} \frac{1}{\psi(k)} \quad (n = 1, 2, \ldots).$$

Then $n^{-2} w(n)$ is non-increasing, $w(n)$ satisfies (1.13) *and*

$$\sum_{n=1}^{\infty} \frac{w(n)}{n^2} < \infty.$$

Proof. The fact that $n^{-2} w(n)$ is non-increasing easily follows from $n^{-1} \psi(n) \leqq w(n)$, which is obvious.

By Lemma 2 we have

$$w(n) \leqq \frac{Cn}{\psi(n)} \quad (n = 1, 2, \ldots), \tag{2.3}$$

where C is a positive constant. Hence

$$\frac{w(2n)}{w(n)} = 1 + \frac{1}{w(n)} \sum_{k=n}^{2n-1} \frac{1}{\psi(k)} \geqq 1 + \frac{1}{rC},$$

i.e., (1.13) holds for $q = 1 + (rC)^{-1}$.

Finally, (2.3) and (1.16) yield

$$\sum_{n=1}^{\infty} \frac{w(n)}{n^2} \leqq \sum_{n=1}^{\infty} \frac{C}{n\psi(n)} < \infty.$$

After these preliminaries we turn to the

Proof of Theorem 1. Set $s_n = \sum_{k=1}^{n} \xi_k$ and let $\varepsilon > 0$. By (1.3) and by Chebyshev's inequality,

$$P[\lambda_n |s_n| \geqq \varepsilon] \leqq \frac{\lambda_n^2}{\varepsilon^2} E(s_n^2) = O(n w(n) \lambda_n^2).$$

Hence

$$\sum_{i=1}^{\infty} P[\lambda_{2^i} |s_{2^i}| \geqq \varepsilon] = O(1) \sum_{i=1}^{\infty} 2^i w(2^i) \lambda_{2^i}^2.$$

Since $\lambda_n^2 w(n)$ is non-increasing, by a well-known theorem of Cauchy the series

$$\sum_{n=1}^{\infty} \lambda_n^2 w(n) \quad \text{and} \quad \sum_{i=1}^{\infty} 2^i w(2^i) \lambda_{2^i}^2$$

simultaneously converge or diverge. In virtue of (1.14) and the Borel-Cantelli lemma, with probability 1,

$$\lambda_{2^i} |s_{2^i}| < \varepsilon \quad \text{for all } i \text{ large enough.} \tag{2.4}$$

Now we want to apply Theorem B. (1.13) implies via Lemma 1 that $W^{1/2}(n) = O[w^{1/2}(n)]$, where $W(n)$ is defined by (1.11). Thus (1.12) provides

$$P[\lambda_{2^i} M_{2^i,2^i} \geqq \varepsilon] \leqq \frac{1}{\varepsilon^2} \lambda_{2^i}^2 E(M_{2^i,2^i}^2) = O(2^i w(2^i) \lambda_{2^i}^2).$$

Using again the Borel-Cantelli lemma, we can see that, with probability 1,

$$\lambda_{2^i} M_{2^i,2^i} < \varepsilon \quad \text{for all } i \text{ large enough.} \tag{2.5}$$

Taking into account the relations (2.4), (2.5) and that

$$|s_n| \leqq |s_{2^i}| + M_{2^i,2^i} \quad \text{if } 2^i \leqq n \leqq 2^{i+1},$$

we have, with probability 1,

$$\lambda_n |s_n| < 2\varepsilon \quad \text{for all } n \text{ large enough.}$$

Therefore, (1.15) holds.

Proof of Theorem 2. Without loss of generality, we may assume that $\varphi(k) = \frac{1}{\psi(k)}$ for $k = 1, 2, \ldots$. Lemma 3 shows that the present $w(n)$ satisfies all conditions of Theorem 1 in the special case $\lambda_n = n^{-1}$. The application of Theorem 1 provides (1.19), which was to be proved.

Proof of Theorem 2' is almost immediate after the proofs of Theorems 1 and 2. In fact, by (1.20) we have

$$E\left[\sum_{k=b+1}^{b+n} \xi_k\right]^2 \leqq n^2 \varphi(n) = O\left(\frac{n^2}{\psi(n)}\right) \qquad \text{(for all } b \geqq 0,\ n \geqq 1). \tag{2.6}$$

Hence, for any $\varepsilon > 0$,

$$P[n^{-1} s_n \geqq \varepsilon] \leqq \frac{1}{n^2 \varepsilon^2} E(s_n^2) = O\left(\frac{1}{\psi(n)}\right),$$

and consequently,

$$\sum_{i=1}^{\infty} P[2^{-i} |s_{2^i}| \geqq \varepsilon] = O(1) \sum_{i=1}^{\infty} \frac{1}{\varphi(2^i)} < \infty.$$

Thus $2^{-i} s_{2^i} \to 0$ a.e. $(i \to \infty)$.

After this we apply Theorem B starting with (2.6), where now $w(n) = \frac{n}{\psi(n)}$ and $a_k \equiv 1$. Here $w(n)$ satisfies (1.13) owing to (1.17), and a fortiori, by Lemma 1 we have $W(n) = O[w(n)]$. Thus

$$\sum_{i=1}^{\infty} P[2^{-i} M_{2^i,2^i} \geqq \varepsilon] = O(1) \sum_{i=1}^{\infty} \frac{1}{\psi(2^i)} < \infty,$$

whence $2^{-i} M_{2^i,2^i} \to 0$ a.e. $(i \to \infty)$, which makes the proof complete.

§ 3. The Rate of Convergence

The method used in proving Theorems 1 and 2 is suitable to provide information on the rate of convergence in (1.15) and (1.19). For simplicity, we concern ourselves with the estimation of convergence rates in (1.19).

We begin with

Lemma 4. *Let* $w(n)$ *be positive and non-decreasing for* $n=1,2,\dots$ *and satisfy*

$$\frac{w(2n)}{w(n)} \leqq r < 2 \qquad (n \geqq n_0). \tag{3.1}$$

Then

$$\sum_{j=m}^{\infty} \frac{w(2^j)}{2^j} = O\left(\frac{w(2^m)}{2^m}\right). \tag{3.2}$$

Proof. (3.2) immediately follows from the following elementary consequence of (3.1):

$$w(2^j) \leqq r^{j-m} w(2^m) \qquad (j=m,\, m+1, \dots).$$

Theorem 3. *Suppose that the quasi-stationary sequence* $\{\xi_k\}$ *is such that* $w(n)$ *satisfies* (1.13) *and* (3.1). *Then, for each* $\varepsilon > 0$,

$$P_n = P\left[\sup_{k \geqq n} k^{-1} |s_k| \geqq \varepsilon\right] = O\left(\frac{w(n)}{n}\right). \tag{3.3}$$

Proof. Obviously we have

$$P_n \leqq \sum_{j=m}^{\infty} P\left[\max_{2^j \leqq k < 2^{j+1}} k^{-1} |s_k| \geqq \varepsilon\right] \leqq \sum_{j=m}^{\infty} P[M_{0,2^{j+1}} \geqq 2^j \varepsilon],$$

where $m \geqq 0$ is defined by $2^m \leqq n < 2^{m+1}$. On the one hand, (1.13) ensures that Theorem B can be applied. By (1.12) it follows that

$$P_n = O(1) \sum_{j=m}^{\infty} \frac{w(2^{j+1})}{2^{j+1}}. \tag{3.4}$$

On the other hand, by (3.1) we can use Lemma 4, according to which

$$P_n = O\left(\frac{w(2^{m+1})}{2^{m+1}}\right) = O\left(\frac{w(n)}{n}\right).$$

This completes the proof of (3.3).

Thus we have

Corollary 4. *If* $\varphi(k) = O([n^\alpha L_{\beta,\gamma,\delta}(n)]^{-1})$ *or equivalently*

$$w(n) = O\left(\frac{n^{1-\alpha}}{L_{\beta,\gamma,\delta}(n)}\right),$$

where $0 < \alpha < 1$ *and* β, γ, δ *are arbitrary, then for each* $\varepsilon > 0$ *we have*

$$P\left[\sup_{k \geqq n} k^{-1} |s_k| \geqq \varepsilon\right] = O\left(\frac{1}{n^\alpha L_{\beta,\gamma,\delta}(n)}\right).$$

The case $\alpha=0$ is of special interest. As in this case (3.1) is not satisfied, we have to treat it separately.

Theorem 4. *Suppose that* $\psi\in\Psi_c$, $\psi(k)$ *satisfies* (1.17), *and the quasi-stationary sequence* $\{\xi_k\}$ *is such that* (1.18) *holds. Then, for each* $\varepsilon>0$,

$$P_n=P\big[\sup_{k\geqq n} k^{-1}|s_k|\geqq\varepsilon\big]=O(1)\sum_{k=n}^{\infty}\frac{1}{k\psi(k)}. \tag{3.5}$$

Proof. Lemma 3 shows that $w(n)$ satisfies (1.13). Thus, as in the proof of Theorem 3, we come to (3.4). Then making use of (2.3), we complete our reasoning as follows:

$$P_n=O(1)\sum_{j=m}^{\infty}\frac{w(2^{j+1})}{2^{j+1}}=O(1)\sum_{j=m}^{\infty}\frac{1}{\psi(2^{j+1})}=O(1)\sum_{k=n}^{\infty}\frac{1}{k\psi(k)},$$

where $2^m\leqq n<2^{m+1}$. The last equality in this chain is obtained by the Cauchy theorem mentioned above ($\psi(k)$ is non-decreasing). The proof of (3.5) is finished.

The implications of Theorem 4 are of some interest.

Corollary 5. *For any* $\varepsilon>0$,

$$\varphi(k)=O\left(\frac{1}{kL_{1,1,1+\varepsilon}(k)}\right) \quad \textit{implies} \quad P_n=O\left(\frac{1}{(\log\log\log n)^{\varepsilon}}\right),$$

$$\varphi(k)=O\left(\frac{1}{k\log k(\log\log k)^{1+\varepsilon}}\right) \quad \textit{implies} \quad P_n=O\left(\frac{1}{(\log\log n)^{\varepsilon}}\right),$$

$$\varphi(k)=O\left(\frac{1}{k(\log k)^{1+\varepsilon}}\right) \quad \textit{implies} \quad P_n=O\left(\frac{1}{(\log n)^{\varepsilon}}\right).$$

§ 4. Weighted Averages

Let $\{a_k\}$ be a sequence of numbers, $a_1\neq 0$. We are interested in the convergence properties of

$$S_n=\sum_{k=1}^{n} a_k\xi_k \qquad (n=1,2,\ldots),$$

where we assume that

$$A_n=\sum_{k=1}^{n} a_k^2\to\infty \qquad (n\to\infty). \tag{4.1}$$

Starting with Theorem A of Gapoškin, (1.6) implies

Corollary 6. *If* (4.1) *holds, then for any* $\psi\in\Psi_c$

$$\frac{S_n}{[w(n)A_n\psi(A_n)]^{1/2}\log n}\to 0 \quad a.e. \quad (n\to\infty). \tag{4.2}$$

In fact, this immediately follows via the Kronecker lemma from the fact that the series (1.4) converges a.e. for

$$\lambda_k = [w(k) A_k \psi(A_k)]^{-1/2} (\log k)^{-1} \quad (k \geqq 2).$$

The latter assertion is a simple consequence of (1.6) and the following lemma, applied widely in the theory of numerical series (see, for example, [5, Lemma 1]): *Let* $d_k \geqq 0$ *be the terms of a divergent series with partial sums* D_n $(d_1 > 0)$. *Then the series*

$$\sum_{n=1}^{\infty} \frac{d_n}{D_n \psi(D_n)}$$

converges for any $\psi \in \Psi_c$.

We remark that for independent rv's (when $w(n) \equiv 1$) a stronger result was proved by Petrov [5, Theorem 1]: *If* $\{\xi_k\}$ *is a sequence of independent rv's satisfying* (1.1), *and if* $\{a_k\}$ *is a sequence of numbers satisfying* (4.1), *then, for any* $\psi \in \Psi_c$,

$$\frac{S_n}{[A_n \psi(A_n)]^{1/2}} \to 0 \quad a.e. \quad (n \to \infty).$$

The result (4.2) is sharp in the case $w(n) = O(n^\delta)$ for every $\delta > 0$ (cf. Gapoškin [2, Theorem 8]), in particular, in the case of orthogonal rv's (when again $w(n) \equiv 1$) (see Tandori [10] and Petrov [7]). Nevertheless, it is very probable that if $w(n)$ increases "fast" in the sense of (1.13), then in the denominator of (4.2) the factor $\log n$ is unnecessary. More precisely, we set up the following

Conjecture. Suppose that the quasi-stationary sequence $\{\xi_k\}$ is such that $w(n)$ satisfies (1.13), furthermore, the sequence $\{a_k\}$ of numbers is such that (4.1) and perhaps some more requirements on the "regular" behaviour of A_n are satisfied. Then, for any $\psi \in \Psi_c$,

$$\frac{S_n}{[w(n) A_n \psi(A_n)]^{1/2}} \to 0 \quad \text{a.e.} \quad (n \to \infty). \tag{4.3}$$

In case $a_k \equiv 1$ (i.e., when $A_n = n$) the conjecture coincides with Corollary 1. For the sake of simplicity, we present here a possible generalization of Theorem 2 and Corollaries 2 and 3, respectively.

Theorem 5. *Suppose that the quasi-stationary sequence* $\{\xi_k\}$ *is such that* $w(n)$ *satisfies* (1.13), *furthermore, the sequence* $\{a_k\}$ *of numbers is such that* (4.1) *and*

$$w(n) = O\left(\frac{A_n}{\psi(A_n)}\right) \tag{4.4}$$

are satisfied, where $\psi \in \Psi_c$. *Then*

$$A_n^{-1} S_n \to 0 \quad a.e. \quad (n \to \infty). \tag{4.5}$$

Theorem 6. *Suppose that* $\{\xi_k\}$ *is such that* $w(n)$ *satisfies* (1.13), *furthermore,* $\{a_k\}$ *is such that* (4.1) *and*

$$w(n) = O[A_n^{1-\alpha} (\log A_n)^\beta] \tag{4.6}$$

are satisfied, where either $0<\alpha<1$ *and* β *arbitrary or* $\alpha=\beta=0$. *Then, for any* $\psi\in\Psi_c$,

$$\frac{S_n}{[A_n^{2-\alpha}(\log A_n)^\beta\,\psi(A_n)]^{1/2}}\to 0 \quad a.e. \quad (n\to\infty). \tag{4.7}$$

It is obvious that (4.4) and (4.5), further, (4.6) and (4.7) are special cases of (4.3).

Proof. As the proofs of Theorems 5 and 6 run along the same lines as that of Theorem 1, we concern ourselves only with the proof of Theorem 5.

To this effect, define a sequence of integers $1\leqq n_1\leqq n_2\leqq\ldots$ in such a way that

$$A_{n_i-1}\leqq 2^i<A_{n_i} \qquad (i=1,2,\ldots;\ A_0=0). \tag{4.8}$$

This choice is possible by (4.1), and obviously $n_i\to\infty$ as $i\to\infty$.

By (1.3) and Chebyshev's inequality,

$$P[|S_n|\geqq\varepsilon A_n]\leqq\frac{1}{\varepsilon^2A_n^2}E(S_n^2)\leqq\frac{2w(n)}{\varepsilon^2A_n}.$$

Hence, by (4.4)

$$\sum_{i=1}^\infty P[|S_{n_i}|\geqq\varepsilon A_{n_i}]\leqq\frac{2}{\varepsilon^2}\sum_{i=1}^\infty\frac{w(n_i)}{A_{n_i}}=O(1)\sum_{i=1}^\infty\frac{1}{\psi(A_{n_i})}.$$

The series on the right-hand side is convergent since by (4.8)

$$\sum_{i=1}^\infty\frac{1}{\psi(A_{n_i})}\leqq\sum_{i=1}^\infty\frac{1}{\psi(2^i)},$$

and the latter series converges if and only if $\psi\in\Psi_c$ (by the Cauchy theorem). Hence the Borel-Cantelli lemma implies, with probability 1, that

$$A_{n_i}^{-1}|S_{n_i}|<\varepsilon \qquad \text{for all } i \text{ large enough.} \tag{4.9}$$

Now we are going to apply Theorem B. By (1.13) Lemma 1 implies that $W(n)=O[w(n)]$. Therefore, with $v_i=n_{i+1}-n_i-1$,

$$P[M_{n_i,v_i}\geqq\varepsilon A_{n_i}]\leqq\frac{1}{\varepsilon^2A_{n_i}^2}E(M_{n_i,v_i}^2)=O\left(\frac{w(v_i)(A_{n_{i+1}-1}-A_{n_i})}{A_{n_i}^2}\right),$$

provided that i is such that $n_i<n_{i+1}-1$. Observing that

$$\frac{A_{n_{i+1}-1}-A_{n_i}}{A_{n_i}^2}\leqq\frac{4}{A_{n_{i+1}-1}},$$

we get that

$$P[M_{n_i,v_i}\geqq\varepsilon A_{n_i}]=O\left(\frac{w(n_{i+1}-1)}{A_{n_{i+1}-1}}\right).$$

Hence (4.4) implies that

$$\sum_{i=1}^{\infty}{}' P[M_{n_i,v_i}\geqq\varepsilon A_{n_i}]=O(1)\sum_{i=1}^{\infty}{}'\frac{1}{\psi(A_{n_{i+1}-1})}<\infty,$$

where $\sum'$ means that the summation is extended to those i's for which $n_i < n_{i+1} - 1$. By the Borel-Cantelli lemma.

$$A_{n_i}^{-1} M_{n_i, v_i} < \varepsilon \quad \text{for all } i \text{ large enough,}$$

which, together with (4.9), gives the wanted (4.5).

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Received April 12, 1976