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The Property of Predictable Representation of the Sum of Independent Semimartingales

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1. Introduction

Let $X=(X_t)_{t\geqq 0}$ be a semimartingale admitting of the canonical representation:

$$X_t=X_0+\alpha_t+X_t^c+\int_0^t \int_{|x|>1} x\,d\mu+\int_0^t \int_{|x|\leqq 1} x\,d(\mu-v)$$

where (a) $X^c=(X_t^c)_{t\geqq 0}$ is a null-initial-valued continuous local martingale, i.e. the continuous martingale part of semimartingale X; (b) $\alpha=(\alpha_t)_{t\geqq 0}$ is a null-initial-valued predictable process of bounded variation; (c) μ is the measure of jumps of semimartingale X, i.e. the integer random measure

$$\mu(dt,dx)=\sum_s \mathscr{E}_{(s,\Delta x_s)}(dt,dx)\,I_{\Delta x_s\neq 0}$$

with $\mathscr{E}_{(s,\Delta x_s)}(dt,dx)$ as the unit measure concentrating on $(s,\Delta X_s)$; (d) v is the compensator (predictable dual projection) of μ. Set $\beta=\langle X^c,X^c\rangle$, then (α,β,v) are the predictable characteristics of semimartingale X. Semimartingale X is said to have the property of predictable representation, if for each null-initial-valued local martingale $M=(M_t)_{t\geqq 0}$, there exist two predictable processes $H=(H_t)_{t\geqq 0}$ and $W=(W(t,x))_{t\geqq 0,x\in R}$ such that M can be represented as the sum of the two stochastic integrals:

$$M=H\cdot X^c+W\cdot(\mu-v)$$

where $H\cdot X^c$ and $W\cdot(\mu-v)$ are respectively the continuous component M^c and the pure discontinuous component M^d of M.

It is well-known that processes with independent increments and jump processes have the property of predictable representation with respect to their natural σ-fields (see [2]). R.J. Elliot [1] tried to show the property of predictable representation for the following type of semimartingales with respect to their natural σ-fields: $X=B+Y$, where $B=(B_t)_{t\geqq 0}$ is a Brownian motion, $Y=(Y_t)_{t\geqq 0}$ is a jump process, and B and Y are mutually independent. However,

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he did not obtain the property of predictable representation in the form described above, what was given was only an integral representation under the framework of product spaces. In this paper we consider a more general model than [1].

Let $(X^{(i)}, (\underline{\underline{F}}^i_t)_{t \geqq 0})$, $i=1,2$, be two mutually independent semimartingales having the property of predictable representation, and $(\alpha^{(i)}, \beta^{(i)}, v_i)$ be the predictable characteristics of $X^{(i)}$ $i=1,2$. Pose

$$X = X^{(1)} + X^{(2)}, \quad \underline{\underline{F}}_t = \underline{\underline{F}}^1_t \vee \underline{\underline{F}}^2_t, \quad t \geqq 0.$$

Using the general results about the property of predictable representation of semimartingales in [2], we obtain the necessary and sufficient conditions that $\beta^{(i)}$ and v_i, $i=1,2$, must satisfy in order that semimartingale (X, F) has the property of predictable representation under the assumption that predictable supports of the jump times of $X^{(1)}$ and $X^{(2)}$ are disjoint (Theorem 2). Obviously the case discussed in [1] is merely a particular example of ours.

In [2] another predictable representation is also discussed. Let $M=(M_t)_{t \geqq 0}$ be a local martingale. If for each null-initial-valued local martingale $L=(L_t)_{t \geqq 0}$ there exists a predictable process $H=(H_t)_{t \geqq 0}$ such that L can be represented as a stochastic integral, $L = H \cdot M$, then the local martingale is said to have the property of predictable representation. In order to distinguish the two types of the property of predictable representation, we call the latter the strong property of predictable representation of local martingales. Using similar methods, we discuss the same problem for the strong property of predictable representation of local martingales, and obtain similar results.

2. Preliminaries

We shall always discuss problems in a fixed complete probability space $(\Omega, \underline{\underline{F}}, P)$ with the following assumptions: $F^i = (\underline{\underline{F}}^i_t)_{t \geqq 0}$, $i=1,2$, are two filtrations of sub-σ-fields of $\underline{\underline{F}}$ satisfying the usual conditions, $\underline{\underline{F}}^1_\infty = \bigvee_{t \geqq 0} \underline{\underline{F}}^1_t$, and $\underline{\underline{F}}^2_\infty = \bigvee_{t \geqq 0} \underline{\underline{F}}^2_t$ are mutually independent. Pose

$$\underline{\underline{F}}_t = \underline{\underline{F}}^1_t \vee \underline{\underline{F}}^2_t, \quad t \geqq 0.$$

In this section we shall show that $F = (\underline{\underline{F}}_t)_{t \geqq 0}$ satisfies the usual conditions and certain necessary lemmas. The concepts and notations of martingale theory and stochastic integrals which we adopt are the same as that of [2]. For simplicity, we only deal with real processes (taking values on the real line $(R, \mathscr{B})$).

Lemma 1. *Let $\xi \in \underline{\underline{F}}^1_\infty$ and $\eta \in \underline{\underline{F}}^2_\infty$ be integrable random variables. Then for all $t \geqq 0$ we have*

$$E(\xi\eta/\underline{\underline{F}}_t) = E(\xi/\underline{\underline{F}}^1_t)\, E(\eta/\underline{\underline{F}}^2_t). \tag{1}$$

Proof. Let $A \in \underline{\underline{F}}^1_t$ and $B \in \underline{\underline{F}}^2_t$, in view of their independence

$$\begin{aligned}\int_{AB} \xi\eta\, dP &= \int_A \xi\, dP \int_B \eta\, dP = \int_A E(\xi/\underline{\underline{F}}^1_t)\, dP \int_B E(\eta/\underline{\underline{F}}^2_t)\, dP \\ &= \int_{AB} E(\xi/\underline{\underline{F}}^1_t)\, E(\eta/\underline{\underline{F}}^2_t)\, dP.\end{aligned}$$

Obviously, $E(\xi/\underline{\underline{F}}_t^1)\,E(\eta/\underline{\underline{F}}_t^2)$ is $\underline{\underline{F}}_t$-measurable, hence (1) follows.

Theorem 1. *$\underline{F}=(\underline{\underline{F}}_t)_{t\geqq 0}$ satisfies the usual conditions.*

Proof. It suffices to verify the right continuity of $\underline{F}$. For an arbitrary $t_0\geqq 0$, let $t\downarrow t_0$ in (1), then by the well-known Lévy's theorem and the right continuity of $\underline{F}^1$ and $\underline{F}^2$ we have

$$E(\xi\,\eta/\underline{\underline{F}}_{t_0+})=E(\xi/\underline{\underline{F}}_{t_0}^1)\,E(\eta/\underline{\underline{F}}_{t_0}^2)=E(\xi\,\eta/\underline{\underline{F}}_{t_0}).$$

Because the random variables having this form and their linear combinations are dense in $L^1(\underline{\underline{F}}_\infty)$, for each $\zeta\in L^1(\underline{\underline{F}}_\infty)$

$$E(\zeta/\underline{\underline{F}}_{t_0+})=E(\zeta/\underline{\underline{F}}_{t_0}).$$

Hence $\underline{\underline{F}}_{t_0+}=\underline{\underline{F}}_{t_0}$.

Lemma 2. *Let $L=(L_t)_{t\geqq 0}$ be an uniformly integrable $\underline{F}^1$-martingale ($\underline{F}^1$-local martingale), $N=(N_t)_{t\geqq 0}$ be an uniformly integrable $\underline{F}^2$-martingale ($\underline{F}^2$-local martingale). Then LN is an uniformly integrable $\underline{F}$-martingale ($\underline{F}$-local martingale).*

If L is a continuous or pure discontinuous $\underline{F}^1$-local martingale, then L is also a continuous or pure discontinuous $\underline{F}$-local martingale.

Proof. Let L and N be the uniformly integrable $\underline{F}^1$- and $\underline{F}^2$-martingale respectively. Immediately from Lemma 1 we get for all $t\geqq 0$

$$L_t N_t=E(L_\infty N_\infty/\underline{\underline{F}}_t).$$

Therefore, LN is an uniformly integrable $\underline{F}$-martingale.

The remained conclusions of the lemma are somewhat apparent (see [2]), and their proofs are omitted.

Suppose that X is an adapted r.c.l.l. process (right continuous and with finite left hand limits), μ is the measure of jumps of X, ν is the compensator of μ. Set

$$\begin{aligned} &D=\{(\omega,t)\colon \Delta X_t\neq 0\},\\ &a_t=\nu([t]\times R),\qquad J=\{(\omega,t)\colon a_t(\omega)>0\}, \end{aligned}\tag{2}$$

ν can be so chosen that $0\leqq a\leqq 1$. For $W\in\tilde{\mathscr{P}}$, we denote

$$\hat{W}_t=\int W(t,x)\,\nu([t],dx),$$

$$\tilde{W}_t=W(t,\Delta X_t)\,I_D(t)-\hat{W}_t$$

and

$$G^1_{\text{loc}}(\mu)=\{W\in\tilde{\mathscr{P}}\colon (\sum_{s\leqq\cdot}\tilde{W}_s^2)^{\frac{1}{2}}\in\mathscr{A}^+_{\text{loc}}\}.$$

For each $W\in G^1_{\text{loc}}(\mu)$ there exists an unique null-initial-valued pure discontinuous local martingale N such that $\Delta N=\tilde{W}$, where N is called the stochastic integral of W with respect to $(\mu-\nu)$ and denoted $W\cdot(\mu-\nu)$.

Lemma 3. *Suppose that local martingale N can be represented as:*

$$N=V\cdot(\mu-\nu),\qquad V\in G^1_{\text{loc}}(\mu).$$

There exists an unique $W \in G^1_{\mathrm{loc}}(\mu)$ *(up to indistinguishability with respect to* $M_\mu = M_\nu$*) such that* $[a=1] \subset [\hat{W}=0]$ *and* $N = W \cdot (\mu - \nu)$. *(*M_μ *is the Doleans measure generated by* μ.*)*

Proof. By the definition of N we have

$$\Delta N = V - \hat{V} \quad \text{a.e. } M_\mu.$$

Set $U = M_\mu(\Delta N / \tilde{\mathscr{P}}) = V - \tilde{V}$, $W = U + \dfrac{\hat{U}}{1-a} I_{a<1}$. By direct computation it can be deduced that

$$\begin{aligned} &W = V - \hat{V} I_{a=1}, \quad \text{a.e. } M_\mu \\ &\hat{W} = \hat{V} I_{a<1}, \qquad \tilde{W} = \tilde{V}, \end{aligned} \tag{3}$$

so we have $[a=1] \subset [\hat{W}=0]$ and $N = W \cdot (\mu - \nu)$. Uniqueness follows from (3).

Lemma 4. *Let* μ *be a* $\underline{F}^1$*-optional random measure. If the Doleans measure* M_μ *generated by* μ *is* σ*-finite on* $\tilde{\mathscr{P}}^1$ *and* ν *is* $\underline{F}^1$*-compensator of* μ. *Then* ν *is also the* $\underline{F}$*-compensator of* μ.

Proof. Let $(\tilde{A}_n)_{n \geqq 1}$ be a $\tilde{\mathscr{P}}^1$-measurable partition of $\tilde{\Omega} = \Omega \times R_+ \times R$ such that $M_\mu(\tilde{A}_n) < \infty$. For each $B \in \mathscr{B}$, $\int_{[0,t] \times B} I_{\tilde{A}_n} d\nu$ is the $\underline{F}^1$-compensator of $\int_{[0,t] \times B} I_{\tilde{A}_n} d\mu$. Hence by Lemma 2 $\int_{[0,t] \times B} I_{\tilde{A}_n} d\nu$ is also the $\underline{F}$-compensator of $\int_{[0,t] \times B} I_{\tilde{A}_n} d\mu$. Therefore, ν is the $\underline{F}$-compensator of μ.

Lemma 5. *Let* T_i *be a* $\underline{F}^i$*-stopping time,* $i = 1, 2$. *If one of* T_1 *and* T_2 *is totally inaccessible, we have*

$$[\![T_1]\!] \cap [\![T_2]\!] = \emptyset.$$

Proof. Denote by F_i the distribution function of T_i, $i = 1, 2$. Because of the independence of T_1 and T_2, we have

$$P(T_1 = T_2 < \infty) = \int\int I_{x=y} dF_1(x)\, dF_2(y).$$

If $[\![T_1]\!] \cap [\![T_2]\!] \neq \emptyset$, i.e. $P(T_1 = T_2 < \infty) > 0$, there must be a constant c such that $P(T_1 = T_2 = c) > 0$. Hence, none of T_1 and T_2 is totally inaccessible. This is contradictory to our assumption.

Finally, we formulate a simple fact of measure theory.

Lemma 6. *Suppose that* μ_1 *and* μ_2 *are two* σ*-finite measures on the measurable space* $(E, \mathscr{E})$, *and for arbitrary two bounded measurable functions* f_1 *and* f_2, *there exists another measurable function* f *such that*

$$f = f_i, \quad \text{a.e. } \mu_i, \ i = 1, 2.$$

Then μ_1 *and* μ_2 *are mutually singular on* $\mathscr{E}$, *i.e. there exists a set* $A \in \mathscr{E}$ *such that*

$$\mu_1 = I_A \cdot \mu_1, \qquad \mu_2 = I_{A^c} \cdot \mu_2.$$

Proof. Set $\mu=\mu_1+\mu_2$, denote $b_i=\frac{d\mu_i}{d\mu}$, $i=1,2$, we have $b_1+b_2=1$, a.e. μ, and

$$(f-f_i)^2 b_i=0, \qquad \text{a.e. } \mu,\ i=1,2.$$

Hence, $f_1=f=f_2$ on $\{0<b_1<1\}$ $(=\{0<b_2<1\})$. Since f_1 and f_2 are arbitrary, it must be $\mu(0<b_1<1)=0$. Therefore

$$\mu_1=I_{b_1=1}\cdot\mu_1, \qquad \mu_2=I_{b_1=0}\cdot\mu_2.$$

3. The Property of Predictable Representation of the Sum of Independent Semimartingales

In this section we make the following assumptions: $(X^{(i)}, \underline{F}^i)$, $i=1,2$, are two semimartingales having the property of predictable representation, $\underline{\underline{F}}^1_\infty$ and $\underline{\underline{F}}^2_\infty$ are mutually independent. $(\alpha^{(i)}, \beta^{(i)}, \nu_i)$ are the $\underline{F}^i$-predictable characteristics of $X^{(i)}$, and the canonical representation of $X^{(i)}$ is

$$X^{(i)}=X^{(i)}_0+\alpha^{(i)}+M^{(i)}+(xI_{|x|\leqq 1})\cdot(\mu_i-\nu_i)+(xI_{|x|>1})\cdot\mu_i$$

where μ_i is the measure of jumps of $X^{(i)}$, $M^{(i)}$ is the $\underline{F}^i$-continuous martingale part of $X^{(i)}$, and $\beta^{(i)}=\langle M^{(i)}, M^{(i)}\rangle$. We also define D_i, a_i and J_i according to (2).

Lemma 7. *If $J_1J_2=\emptyset$, we have*

$$\begin{aligned}(D_1\cup J_1)\cap(D_2\cup J_2)&=\emptyset,\\ \Delta X^{(1)}\Delta X^{(2)}&=0,\end{aligned} \tag{4}$$

and for all $\underline{F}^i$-local martingales $N^{(i)}$, $i=1,2$,

$$\Delta N^{(1)}\Delta N^{(2)}=0.$$

Proof. Since $D_i\backslash J_i$ is the union of graphs of at most denumerable $\underline{F}^i$-totally inaccessible stopping times, by Lemma 5 we find that

$$(D_i\backslash J_i)\cap D_j=\emptyset, \qquad i=j.$$

Hence (4) can be easily deduced from $J_1J_2=\emptyset$. On the other hand, the following implications hold:

$$[\Delta X^{(i)}\neq 0]\subset D_i, \quad [\Delta N^{(i)}\neq 0]\subset D_i\cup J_i, \qquad i=1,2,$$

so the remained conclusions follow immediately from (4).

Now we define

$$X=X^{(1)}+X^{(2)}, \quad \underline{\underline{F}}_t=\underline{\underline{F}}^1_t\vee\underline{\underline{F}}^2_t, \qquad t\geqq 0.$$

From Lemma 2 it is easily seen that $(X,\underline{F})$ is a semimartingale. If we suppose that $J_1J_2=\emptyset$, then D, the graph of jump times of X, and μ, the measure of jumps of X, are as follows:

$$D=D_1\cup D_2, \qquad \mu=\mu_1+\mu_2$$

and (α, β, ν), the predictable characteristics of X:

$$\alpha = \alpha^{(1)} + \alpha^{(2)}, \quad \beta = \beta^{(1)} + \beta^{(2)}, \quad \nu = \nu_1 + \nu_2.$$

Also

$$J = [a > 0] = J_1 \cup J_2, \quad [a = 1] = [a_1 = 1] \cup [a_2 = 1].$$

X admits of the canonical representation:

$$X = X_0 + \alpha + X^c + (xI_{|x| \leqq 1}) \cdot (\mu - \nu) + (xI_{|x| > 1}) \cdot \mu$$

where $X^c = M^{(1)} + M^{(2)}$ is the $\underline{F}$-continuous martingale part of X.

Theorem 2. *If $J_1 J_2 = \emptyset$, $(X, \underline{F})$ has the property of predictable representation if and only if the following conditions hold:*

(a) *$\beta^{(1)}$ and $\beta^{(2)}$ are mutually singular on $\mathscr{P}$, i.e. there exists a set $A \in \mathscr{P}$ such that*

$$\beta^{(1)} = I_A \cdot \beta^{(1)}, \quad \beta^{(2)} = I_{A^c} \cdot \beta^{(2)}. \tag{5}$$

(b) *ν_1 and ν_2 are mutually singular on $\tilde{\mathscr{P}}$, i.e. there exists a set $K \in \tilde{\mathscr{P}}$ such that*

$$\nu_1 = I_K \cdot \nu_1, \quad \nu_2 = I_{K^c} \cdot \nu_2. \tag{6}$$

Proof. Sufficiency. From the relevant propositions given in [2] it easily follows that the assertion that semimartingale $(X, \underline{F})$ has the property of predictable representation is equivalent to the following proposition:

If Q is a probability measure on $\underline{\underline{F}}$ such that $P \sim Q$, $Q = P|_{\underline{\underline{F}}_0}$, the derivative $\frac{dQ}{dP}$ is bounded, and (α, β, ν) are still the predictable characteristics of X with respect to Q, then $Q = P|_{\underline{\underline{F}}_\infty}$.

In our case we shall show that this proposition is valid, but the proof is somewhat involved, and we divide it into several parts.

First, we denote by $Z_t = E\left(\frac{dQ}{dP} \Big/ \underline{\underline{F}}_t\right)$ the density process of Q with respect to P, then $Z = (Z_t)_{t \geqq 0}$ is a bounded $\underline{F}$-martingale with $Z_0 = 1$. Note that (α, β, ν) are still the predictable characteristics of X with respect to Q, and therefore, from (12.19) of [2]

$$\langle X^c, Z \rangle = 0, \quad M_\mu(\Delta Z / \tilde{\mathscr{P}}) = 0,$$

but $X^c = M^{(1)} + M^{(2)}$, hence for each $H \in \mathscr{P}$ we have

$$0 = H \cdot \langle X^c, Z \rangle = H \cdot \langle M^{(1)}, Z \rangle + H \cdot \langle M^{(2)}, Z \rangle.$$

Replacing H by HI_A and HI_{A^c} in the above equation, from (5) we find that

$$\langle M^{(i)}, Z \rangle = 0, \quad i = 1, 2. \tag{7}$$

For each $W \in \tilde{\mathscr{P}}^+$, since $\mu = \mu_1 + \mu_2$, we have

$$\begin{aligned} 0 = M_\mu(\Delta Z W) &= M_{\mu_1}(W M_{\mu_1}(\Delta Z / \tilde{\mathscr{P}})) + M_{\mu_2}(W M_{\mu_2}(\Delta Z / \tilde{\mathscr{P}})) \\ &= M_{\nu_1}(W M_{\mu_1}(\Delta Z / \tilde{\mathscr{P}})) + M_{\nu_2}(W M_{\mu_2}(\Delta Z / \tilde{\mathscr{P}})). \end{aligned}$$

Replacing W by WI_K and WI_{K^c} in the above equation, we find from (6) that

$$M_{\mu_i}(\Delta Z/\tilde{\mathscr{P}})=0, \quad i=1,2. \tag{8}$$

Secondly, we show $Q=P|_{\underline{F}^i_\infty}$, $i=1,2$. By virtue of (7), $M^{(i)}$ remains a continuous $\underline{F}$-local martingale with respect to Q. Define $T^{(i)}_n=\inf\{t\colon |M^{(i)}_t|\geqq n\}$, $i=1,2$. Since $M^{(i)}$ is $\underline{F}^i$-adapted, then $T^{(i)}_n$ are $\underline{F}^i$-stopping times, and $T^{(i)}_n\uparrow\infty$ a.e. Q as $n\to\infty$, and $|(M^{(i)})^{T^{(i)}_n}|\leqq n$. Thus $M^{(i)}$ is also a continuous $\underline{F}^i$-local martingale with respect to Q.

Let $W\in\tilde{\mathscr{P}}^+$. Denote by $\tilde{M}_\mu$ the Doleans measure generated by μ with respect to Q. Evidently $\tilde{M}_\mu(W)=\tilde{M}_\nu(W)$, i.e.

$$\tilde{M}_{\mu_1}(W)+\tilde{M}_{\mu_2}(W)=\tilde{M}_{\nu_1}(W)+\tilde{M}_{\nu_2}(W) \tag{9}$$

(6) tells us $M_{\mu_1}(WI_{K^c})=M_{\nu_1}(WI_{K^c})=0$, $M_{\mu_2}(WI_K)=M_{\nu_2}(WI_K)=0$. Because of $Q\ll P$ we have $\tilde{M}_{\mu_1}(WI_{K^c})=\tilde{M}_{\mu_2}(WI_K)=0$. Putting $W=VI_K$, $V\in\tilde{\mathscr{P}}^1$, in (9)

$$\tilde{M}_{\mu_1}(V)=\tilde{M}_{\mu_1}(VI_K)=\tilde{M}_{\nu_1}(VI_K)=\tilde{M}_{\nu_1}(V)$$

i.e. ν_1 is still the $\underline{F}^1$-compensator of μ_1 with respect to Q. By the same reason, ν_2 is still the $\underline{F}^2$-compensator of μ_2 with respect to Q.

Summarizing above discussion, the canonical representations of $(X^{(i)},\underline{F}^i)$ remain unchanged with respect to Q, and therefore, its predictable characteristics are also unchanged. In view of $Q=P|_{\underline{F}^i_0}$ and the property of predictable representation of $(X^{(i)},\underline{F}^i)$, we obtain

$$Q=P|_{\underline{F}^i_\infty}, \quad i=1,2. \tag{10}$$

Thirdly, we show that $\underline{F}^1_\infty$ and $\underline{F}^2_\infty$ are mutually independent with respect to Q just as with respect to P. Let ξ and η be $\underline{F}^1_\infty$-measurable and $\underline{F}^2_\infty$-measurable bounded random variables respectively. Set

$$Y^{(1)}_t=E(\xi/\underline{F}^1_t), \quad Y^{(2)}_t=E(\eta/\underline{F}^2_t), \quad t\geqq 0.$$

The conclusion we wish to prove is $E(Z_\infty\xi\eta)=E(Z_\infty\xi)\,E(Z_\infty\eta)$, i.e.

$$E(Z_\infty Y^{(1)}_\infty Y^{(2)}_\infty)=E(Z_\infty Y^{(1)}_\infty)\,E(Z_\infty Y^{(2)}_\infty). \tag{11}$$

The property of predictable representation of $X^{(i)}$, $i=1,2$, leads to

$$\begin{aligned} &Y^{(i)}=Y^{(i)}_0+L^{(i)}+N^{(i)} \\ &L^{(i)}=H^{(i)}\cdot M^{(i)}, \quad N^{(i)}=W_i\cdot(\mu_i-\nu_i), \end{aligned} \qquad i=1,2,$$

where $H^{(i)}\in\mathscr{P}^i$, $W_i\in\tilde{\mathscr{P}}^i$. (Note that stochastic integrals with respect to $\underline{F}^i$ and $\underline{F}$ are consistent.) By (7)

$$\langle L^{(i)},Z\rangle=H^{(i)}\cdot\langle M^{(i)},Z\rangle=0,$$

hence $ZL^{(i)}$, $i=1,2$, are $\underline{F}$-local martingales.

Since Z is a bounded $\underline{F}$-martingale, $\langle N^{(i)},Z\rangle$, $i=1,2$, exist, and by (8) (see (7.39) of [2])

$$\langle N^{(i)},Z\rangle=(W_i M_{\mu_i}(\Delta Z/\tilde{\mathscr{P}}))\cdot\nu_i=0, \quad i=1,2.$$

Hence $ZN^{(i)}$, $i=1,2$, are also $\underline{F}$-local martingales. Consequently, $ZY^{(i)}$, $i=1,2$, are $\underline{F}$-local martingales, and therefore, bounded $\underline{F}$-martingales. Thus

$$E(Z_\infty Y_\infty^{(i)})=E(Z_0 Y_0^{(i)})=EY_0^{(i)}, \qquad i=1,2. \tag{12}$$

On the other hand, by Ito's formula

$$(L^{(1)}+N^{(1)})(L^{(2)}+N^{(2)})=(L^{(1)}+N_-^{(1)})\cdot(L^{(2)}+N^{(2)})+(L^{(2)}+N_-^{(2)})\cdot(L^{(1)}+N^{(1)}) \\ +[L^{(1)}+N^{(1)}, L^{(2)}+N^{(2)}].$$

By Lemma 2, we have $\langle M^{(1)}, M^{(2)}\rangle=0$, hence $\langle L^{(1)}, L^{(2)}\rangle=0$, and $[L^{(1)}+N^{(1)}, L^{(2)}+N^{(2)}]$ is a pure discontinuous local martingale. By Lemma 7

$$[Z,[L^{(1)}+N^{(1)}, L^{(2)}+N^{(2)}]]=\sum_s \Delta Z_s \Delta N_s^{(1)} \Delta N_s^{(2)}=0.$$

Hence

$$[Z,(L^{(1)}+N^{(1)})(L^{(2)}+N^{(2)})] \\ =(L^{(1)}+N_-^{(1)})\cdot[Z,L^{(2)}+N^{(2)}]+(L^{(2)}+N_-^{(2)})\cdot[Z,L^{(1)}+N^{(1)}]$$

is a $\underline{F}$-local martingale. Consequently, $ZY^{(1)}Y^{(2)}$ is a $\underline{F}$-local martingale, and therefore, a bounded $\underline{F}$-martingale. Thus

$$E(Z_\infty Y_\infty^{(1)} Y_\infty^{(2)})=E(Z_0 Y_0^{(1)} Y_0^{(2)})=E(Y_0^{(1)} Y_0^{(2)})=E(Y_0^{(1)})\,E(Y_0^{(2)}).$$

From the above equation and (12), (11) follows.

Lastly, it is easily seen from (10) and (11) that $P=Q|_{\underline{F}_\infty}$, and therefore, $(X,\underline{F})$ has the property of predictable representation.

Necessity. Suppose that $H^{(i)}$ belongs to $\{H\in\mathscr{P}^i\colon\ H^2\cdot\beta^{(i)}\in\mathscr{A}_{\mathrm{loc}}^+\}$ and $N^{(i)}=H^{(i)}\cdot M^{(i)}$, $i=1,2$. Then $N=N^{(1)}+N^{(2)}$ is a continuous $\underline{F}$-local square integrable martingale. By virtue of the property of predictable representation of $(X,\underline{F})$, there exists a predictable process $H\in\mathscr{P}$ such that

$$H^{(1)}\cdot M^{(1)}+H^{(2)}\cdot M^{(2)}=N^{(1)}+N^{(2)}=H\cdot X^c=H\cdot(M^{(1)}+M^{(2)}).$$

Hence

$$I_{|H|\leqq n}(H^{(1)}-H)\cdot M^{(1)}=I_{|H|\leqq n}(H-H^{(2)})\cdot M^{(2)}. \tag{13}$$

It follows from that lemma $\langle I_{|H|\leqq n}(H^{(1)}-H)\cdot M^{(1)},\ I_{|H|\leqq n}(H-H^{(2)})\cdot M^{(2)}\rangle=0$. So (13) leads to

$$I_{|H|\leqq n}(H^{(1)}-H)^2\cdot\beta^{(1)}=0, \qquad I_{|H|\leqq n}(H-H^{(2)})^2\cdot\beta^{(2)}=0.$$

Since $H^{(1)}$, $H^{(2)}$ and n are arbitrary, by Lemma 6 $\beta^{(1)}$ and $\beta^{(2)}$ are mutually singular on $\mathscr{P}$.

Suppose that W_i belongs to $G_{\mathrm{loc}}^1(\mu_i)$ and $N^{(i)}=W_i\cdot(\mu_i-\nu_i)$, $i=1,2$. Then $N=N^{(1)}+N^{(2)}$ is a pure discontinuous $\underline{F}$-local martingale. According to Lemma 5, W_i can be so chosen that $[a^{(i)}=1]\subset[\hat{W}_i=0]$. On account of the property of predictive representation of $(X,\underline{F})$, there exists a predictable process $W\in G_{\mathrm{loc}}^1(\mu)$ such that $N=W\cdot(\mu-\nu)$. It follows from Lemma 7 that W also belongs to $G^1_{\text{loc}}(\mu_1)$ and $G^1_{\text{loc}}(\mu_2)$, so

$$W_1 \cdot (\mu_1 - \nu_1) + W_2 \cdot (\mu_2 - \nu_2) = N^{(1)} + N^{(2)} = N$$
$$= W \cdot (\mu - \nu) = W \cdot (\mu_1 - \nu_1) + W \cdot (\mu_2 - \nu_2)$$

and

$$(W_1 - W) \cdot (\mu_1 - \nu_1) = (W - W_2) \cdot (\mu_2 - \nu_2).$$

Again according to Lemma 5, W can be so chosen that

$$[a=1] = [a^{(1)}=1] \cup [a^{(2)}=1] \subset [\hat{W}=0],$$

hence

$$[a^{(1)}=1] \subset [\hat{W}_1=0] \cap [\hat{W}=0] \subset [\widehat{W_1 - W}=0],$$
$$[a^{(2)}=1] \subset [\hat{W}_2=0] \cap [\hat{W}=0] \subset [\widehat{W_2 - W}=0].$$

By Lemma 7, $(W_1 - W) \cdot (\mu_1 - \nu_1)$ and $(W - W_2) \cdot (\mu_2 - \nu_2)$ have no common jumps, hence

$$(W_1 - W) \cdot (\mu_1 - \nu_1) = (W - W_2) \cdot (\mu_2 - \nu_2) = 0.$$

Using Lemma 5 again, we find that

$$W_1 - W = 0, \quad \text{a.e. } M_{\nu_1},$$
$$W_2 - W = 0, \quad \text{a.e. } M_{\nu_2}.$$

Again by Lemma 6, ν_1 and ν_2 are mutually singular on $\tilde{\mathscr{P}}$.

Corollary 1. *If at least one of $X^{(1)}$ and $X^{(2)}$ is quasi-left-continuous, then $(X, \underline{F})$ has the property of predictable representation if and only if the following conditions hold:*

(a) *$\beta^{(1)}$ and $\beta^{(2)}$ are mutually singular on $\mathscr{P}$,*
(b) *ν_1 and ν_2 are mutually singular on $\tilde{\mathscr{P}}$.*

Corollary 2. *Let $X^{(1)}$ be a Brownian motion, and $X^{(2)}$ a jump process. If $X^{(1)}$ and $X^{(2)}$ are mutually independent, then $X = X^{(1)} + X^{(2)}$ has the property of predictable representation with respect to its natural σ-fields.*

Proof. It suffices to take $\underline{F}^i$ as the filtration of natural σ-fields of $X^{(i)}$, $i=1,2$. In this case $\underline{F}$ is just the filtration of natural σ-fields of X.

It is natural to ask whether the conclusion is still valid, if the assumption $J_1 J_2 = \emptyset$ in Theorem 2 is removed. Unfortunately no definite results can be established and we shall indicate by example the case $J_1 J_2 \neq \emptyset$.

Set

$$X^{(1)} = \xi I_{[\![1,\infty[\![}, \quad X^{(2)} = \eta I_{[\![1,\infty[\![}$$
$$X = X^{(1)} + X^{(2)} = (\xi + \eta) I_{[\![1,\infty[\![}.$$

$\underline{F}^i$ is taken as the filtration of natural σ-fields of $X^{(i)}$, $i=1,2$, i.e.

$$\underline{\underline{F}}^1_t = \begin{cases} \sigma(\xi), & t \geqq 1, \\ \underline{F}_0, & 0 \leqq t < 1, \end{cases} \quad \underline{\underline{F}}^2_t = \begin{cases} \sigma(\eta), & t \geqq 1, \\ \underline{F}_0, & 0 \leqq t < 1, \end{cases}$$

where $\underline{\underline{F}}_0$ is the σ-field generated by all sets of zero probability. Since $X^{(1)}$ and $X^{(2)}$ are jump processes, each of them has the property of predictable representation.

$$\underline{\underline{F}}_t=\underline{\underline{F}}_t^1 \vee \underline{\underline{F}}_t^2=\begin{cases}\sigma(\xi,\eta), & t\geqq 1,\\ \underline{\underline{F}}_0, & 0\leqq t<1.\end{cases}$$

Let G_1, G_2 and G be distribution functions of ξ,η and $\xi+\eta$ respectively, then

$$\mu_1(dt,dx)=\mathscr{E}_{\{1\}}(dt)\,\mathscr{E}_{\{\xi\}}(dx),\qquad \nu_1(dt,dx)=\mathscr{E}_{\{1\}}(dt)\,G_1(dx),$$
$$\mu_2(dt,dx)=\mathscr{E}_{\{1\}}(dt)\,\mathscr{E}_{\{\eta\}}(dx),\qquad \nu_2(dt,dx)=\mathscr{E}_{\{1\}}(dt)\,G_2(dx).$$

But the measure of jumps μ of X and its compensator ν are

$$\mu(dt,dx)=\mathscr{E}_{\{1\}}(dt)\,\mathscr{E}_{\{\xi+\eta\}}(dx),\qquad \nu(dt,dx)=\mathscr{E}_{\{1\}}(dt)\,G(dx).$$

Each uniformly integrable $\underline{F}$-martingale can be represented as $h(\xi,\eta)\,I_{[\![1,\infty[\![}$, where h is a two dimensional Borel function satisfying $E|h(\xi,\eta)|<\infty$ and $E\,h(\xi,\eta)=0$. But

$$W\cdot(\mu-\nu)=(W(1,\xi+\eta)-EW(1,\xi+\eta))\,I_{[\![1,\infty[\![}.$$

Now the question whether $(X,\underline{F})$ has the property of predictable representation is reduced to the question whether $h(\xi,\eta)$ can be represented as

$$h(\xi,\eta)=f(\xi+\eta).$$

Suppose that the distributions of ξ and η are

$$P(\xi=0)=P(\xi=1)=\tfrac{1}{2},\qquad P(\eta=0)=P(\eta=2)=\tfrac{1}{2}.$$

We denote by $[x]$ and $\{x\}$ the integral part and the fractional part of x respectively. For an arbitrary h

$$h(\xi,\eta)=h\left(2\left\{\frac{\xi+\eta}{2}\right\},\,2\left[\frac{\xi+\eta}{2}\right]\right)=f(\xi+\eta),$$
$$h(\xi,\eta)\,I_{[\![1,\infty[\![}=f(x)\cdot(\mu-\nu).$$

In this case $(X,\underline{F})$ has the property of predictable representation, but ν_1 and ν_2 are not mutually singular on $\tilde{\mathscr{P}}$.

If the distributions of ξ and η are

$$P(\xi=0)=P(\xi=1)=\tfrac{1}{2},\qquad P(\eta=2)=P(\eta=3)=\tfrac{1}{2},$$

then $h(\xi,\eta)=\xi-\frac{1}{2}$ cannot be represented as $f(\xi+\eta)$. In this case $(X,\underline{F})$ cannot have the property of predictable representation, although ν_1 and ν_2 are mutually singular on $\tilde{\mathscr{P}}$.

4. The Strong Property of Predictable Representation of the Sum of Independent Local Martingales

In this section we make the following assumptions: $(M^{(i)},\underline{F}^i)$, $i=1,2$, are two local martingales having the strong property of predictable representation, again $\underline{\underline{F}}_\infty^1$ and $\underline{\underline{F}}_\infty^2$ are mutually independent. Pose

$$M = M^{(1)} + M^{(2)}, \qquad \underline{\underline{F}}_t = \underline{\underline{F}}_t^1 \vee \underline{\underline{F}}_t^2, \qquad t \geqq 0.$$

Then $(M, \underline{\underline{F}})$ is a local martingale. According to (4.63) of [2] the compensator ν_i of the measure of jumps μ_i of $M^{(i)}$ has the following decomposition:

$$\nu_i(dt, dx) = G^{(i)}_{\omega,t}(dx)\, dB^{(i)}_t, \qquad i = 1, 2,$$

where $B^{(i)}$ is a $\underline{\underline{F}}^i$-predictable increasing process, $G^{(i)}_{\omega,t}(dx)$ is a transition probability from $(\Omega \times R_+, \mathscr{P})$ to $(R, \mathscr{B})$, and they satisfy the following conditions:

$$\begin{aligned} G^{(i)}_{\omega,t}(\{0\}) &= 0, \\ I_{G^{(i)}_{\omega,t}(R)=0} \cdot B^{(i)} &= 0, \end{aligned} \qquad i = 1, 2.$$

Theorem 3. *$(M, \underline{\underline{F}})$ has the strong property of predictable representation if and only if $\beta^{(1)} + B^{(1)}$ and $\beta^{(2)} + B^{(2)}$ are mutually singular on $\mathscr{P}$.*

Proof. Sufficiently. First, since $B^{(1)}$ and $B^{(2)}$ are mutually singular on $\mathscr{P}$, so are ν_1 and ν_2, and therefore, $J_1 J_2 = \emptyset$. By Lemma 7

$$\Delta M^{(1)} \Delta M^{(2)} = 0.$$

For each bounded $\underline{\underline{F}}$-martingale Z, denote $V^{(i)} = M_{\mu_i}(\Delta Z / \tilde{\mathscr{P}})$, then

$$\langle M^{(i)}, Z\rangle = \langle M^{(i)}, Z^c\rangle + (x V^{(i)}) \cdot \nu_i \ll \beta^{(i)} + B^{(i)}, \qquad i = 1, 2.$$

Hence $\langle M^{(1)}, Z\rangle$ and $\langle M^{(2)}, Z\rangle$ are mutually singular on $\mathscr{P}$.

The proof below is similar to that of Theorem 2. It suffices to verify the following condition which is equivalent to the strong property of predictable representation:

If Q is a probability measure on $\underline{\underline{F}}$ such that $Q \sim P$, $Q = P|_{\underline{\underline{F}}_0}$, the derivative $\frac{dQ}{dP}$ is bounded, and M is still a $\underline{\underline{F}}$-local martingale with respect to Q, then $Q = P|_{\underline{\underline{F}}_\infty}$.

Let $Z_t = E\left(\frac{dQ}{dP} \Big/ \underline{\underline{F}}_t\right)$ be the density process of Q with respect to P, then $Z = (Z_t)_{t \geqq 0}$ is a bounded $\underline{\underline{F}}$-martingale. By Girsanov's theorem, $\langle M, Z\rangle = 0$. (Note that $[M, Z]$ is locally integrable.) Since

$$0 = \langle M, Z\rangle = \langle M^{(1)}, Z\rangle + \langle M^{(2)}, Z\rangle$$

and $\langle M^{(1)}, Z\rangle$ and $\langle M^{(2)}, Z\rangle$ are mutually singular, so

$$\langle M^{(i)}, Z\rangle = 0, \qquad i = 1, 2.$$

Again by Girsanov's theorem, $M^{(i)}$ is still a $\underline{\underline{F}}$-local martingale with respect to Q. But we want to show that $M^{(i)}$ is still a $\underline{\underline{F}}^i$-local martingale with respect to Q. In fact, let T be a $\underline{\underline{F}}^i$-stopping time such that $(M^{(i)})^T$ is an uniformly integrable $\underline{\underline{F}}^i$-martingale. Since $(M^{(i)})^T$ is L^1-bounded and Z_∞ is bounded, $(M^{(i)})^T$ is also bounded with respect to Q, and therefore, $(M^{(i)})^T$ is also a $\underline{\underline{F}}^i$-local martingale. Thus $M^{(i)}$ is a $\underline{\underline{F}}^i$-local martingale with respect to Q. Now by means of the strong property of predictable representation of $(M^{(i)}, \underline{\underline{F}}^i)$ we have

$$Q = P|_{\underline{\underline{F}}^i_\infty}, \qquad i = 1, 2.$$

On the other hand, since $\beta^{(1)}+B^{(1)}$ and $\beta^{(2)}+B^{(2)}$ are mutually singular, $J_1 J_2=\emptyset$ and (5), (6) hold, according to the proof of sufficiency of Theorem 2 we find that $\underline{\underline{F}}_\infty^1$ and $\underline{\underline{F}}_\infty^2$ are also mutually independent with respect to Q. Hence

$$Q=P|_{\underline{F}_\infty},$$

that is $(M, \underline{F})$ has the strong property of predictable representation.

Necessity. Let $N^{(i)}$ be the null-initial-valued $\underline{F}^i$-local martingale:

$$N^{(i)}=H^{(i)}\cdot M^{(i)}, \qquad i=1,2,$$

where $H^{(i)}\in\mathscr{P}^i$ (it may be supposed that $H_0^{(i)}=0$). In this case there exists a predictable process $H\in\mathscr{P}$ (also $H_0=0$) such that

$$N^{(1)}+N^{(2)}=H\cdot M.$$

Put $A_n=I_{|H|\leqq n}$. For each n

$$(H^{(1)}I_{A_n})\cdot M^{(1)}+(H^{(2)}I_{A_n})\cdot M^{(2)}=(HI_{A_n})\cdot M^{(1)}+(HI_{A_n})\cdot M^{(2)}$$
$$[(H^{(1)}-H)I_{A_n}]\cdot M^{(1)}=[(H-H^{(2)})I_{A_n}]\cdot M^{(2)}.$$

By Lemma 2 $[M^{(1)}, M^{(2)}]$ is a $\underline{F}$-local martingale, and therefore,

$$((H-H^{(1)})^2 I_{A_n})\cdot[M^{(1)}, M^{(1)}]=((H^{(2)}-H)(H-H^{(1)})I_{A_n})\cdot[M^{(1)}, M^{(2)}]$$

is a $\underline{F}$-local martingale, since it is also a null-initial-valued increasing process, it must be $((H-H^{(1)})^2 I_{A_n})\cdot[M^{(1)}, M^{(1)}]=0$. Let $n\to\infty$ we obtain

$$0=(H-H^{(1)})^2\cdot[M^{(1)}, M^{(1)}]=(H-H^{(1)})^2\cdot\beta^{(1)}+(x(H-H^{(1)}))^2\cdot\mu_1,$$
$$0=(x^2(H-H^{(1)})^2)\cdot\nu_1=((H-H^{(1)})^2\int x^2 G_{\omega,t}^{(1)}(dx))\cdot B^{(1)}.$$

Since

$$\int x^2 G_{\omega,t}^{(1)}(dx)\neq 0, \qquad \text{a.e. } dB^{(1)}$$

so

$$(H-H^{(1)})^2\cdot B^{(1)}=0,$$
$$(H-H^{(1)})^2\cdot(\beta^{(1)}+B^{(1)})=0.$$

By the same reason we have

$$(H^{(2)}-H)^2\cdot(\beta^{(2)}+B^{(2)})=0.$$

Again by Lemma 6 we find that $\beta^{(1)}+B^{(1)}$ and $\beta^{(2)}+B^{(2)}$ are mutually singular on $\mathscr{P}$.

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