Z. Wahrscheinlichkeitstheorie verw. Geb. 13, 141–149 (1969)

# On the Mean Ergodic Theorem for Weighted Averages

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*Summary.* Let $(\Omega, \Sigma, P)$ be a probability space and let $T$ be a measurable and measure preserving point transformation from $\Omega$ into $\Omega$. Let $f$ be a measurable and square integrable function on $(\Omega, \Sigma, P)$, and let $a_{N,k}$ for $N, k=0, 1, \ldots$ be such that $\sum_k a_{N,k}=1$ for all $N$. The authors investigate conditions on the $a_{N,k}$'s such that the sequence $\sum_{k=0}^{\infty} a_{N,k} f(T^k \cdot)$ converges in mean square for all $(\Omega, \Sigma, P, T)$ and $f$ described above. The special cases $T$ weakly mixing and $T$ strongly mixing are also considered.

## 1. Introduction

Let $(\Omega, \Sigma, P)$ be a probability space and let $T$ be a point transformation from $\Omega$ into $\Omega$ which is measurable and measure preserving. Let $\Sigma_0$ be the $\sigma$-field of sets $A \in \Sigma$ such that $T^{-1} A = A$. Let $L$ be the complex Hilbert space of equivalence classes of measurable, complex valued, square integrable functions on $(\Omega, \Sigma, P)$, and let $L_0$ be the subspace of equivalence classes of functions $f$ such that $E\{f \mid \Sigma_0\}=0$ a.e. Let $a_{N,k}$ for $N, k=0, 1, \ldots$ be non-negative real numbers such that

$$\sum_{k=0}^{\infty} a_{N,k}=1 \qquad \text{for all } N.$$

(We need only require $\lim_{N\to\infty} \sum_{k=0}^{\infty} a_{N,k}=1$, but the generalization so obtained is trivial.) Define the linear operator $T$ on $L$ by $(Tf)(\omega)=f(T\omega)$ and define the operators $S_N$ by

$$S_N(f)=\sum_{k=0}^{\infty} a_{N,k}\, T^k(f). \tag{0}$$

In Section 2 results are obtained giving conditions on the $a_{N,k}$'s in order that a mean ergodic theorem hold; the three cases considered are the cases $T$ general, $T$ weakly mixing, and $T$ strongly mixing. Section 3 contains an example showing that our work applies in some cases not covered by the work of Cohen [5]. Section 4 contains an acknowledgement and some concluding remarks.

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* Research partially sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Grant No. AFOSR-68-1394.

** Research done while this author held a National Science Foundation Traineeship at the University of Missouri, Columbia.

## 2. Results

**Theorem 1.** *$\|S_N f - E\{f|\Sigma_0\}\| \to 0$ for every $f$ in $L$ and every measure preserving transformation $T$ on $(\Omega, \Sigma, P)$ if and only if*

(1) $\lim_{N\to\infty} \sum_{k=0}^{\infty} a_{N,k\alpha+j} = \frac{1}{\alpha}$ *for* $0 \leqq j \leqq \alpha$ *and* $\alpha = 2, 3, \ldots$, *and*

(2) $\lim_{N\to\infty} \sum_{\{k|k\gamma \bmod [0,1) \in [a,b)\}} a_{N,k} = b - a$ *for all* $a, b, \gamma$ *in* $[0, 1)$ *such that* $a < b$ *and* $\gamma$ *is irrational.*

**Theorem 2.** *If $T$ is weakly mixing and*

(3) $\lim_{N\to\infty} \sum_{k\in S} a_{N,k} = 0$ *whenever $S$ is a subset of the non-negative integers having density zero,*

*then $\|S_N f - Ef\| \to 0$ for every $f$ in $L$.*

**Theorem 3.** *If $T$ is strongly mixing then $\|S_N f - Ef\| \to 0$ for every $f$ in $L$ if and only if*

(4) $\max_k a_{N,k} \to 0.$

Note that in Theorem 1 it suffices to consider only $f \in L_0$ and to ask whether $\|S_N f\| \to 0$. If $T$ is ergodic (a fortiori if $T$ is weakly or strongly mixing), then $E\{f|\Sigma_0\} = Ef$ a.e. and again it is sufficient to consider only $f \in L_0$.

*Proof of Theorem* 1. The linear operator $T$ is an isometry so $\{T^n L_0 : n = 0, 1, \ldots\}$ is a non-increasing sequence of subspaces of $L_0$. Define $Z = \bigcap_{n=0}^{\infty} T^n L_0$ and for each $n = 1, 2, \ldots$ let $W_n$ be the orthocomplement of $T^n L_0$ in $T^{n-1} L_0$. Note that $Z, W_1, W_2, \ldots$ are pairwise orthogonal; that $T W_n = W_{n+1}$ for $n = 1, 2, \ldots$; that $TZ = Z$; and that if $f$ is in $L_0$ we can write $f = f_Z + f_1 + f_2 + \cdots$ where $f_Z \in Z$ and $f_n \in W_n$ for $n = 1, 2, \ldots$.

If $f \in W_n$ then

$$\begin{aligned}(S_N(f), S_N(f)) &= \sum_{i,j=0}^{\infty} a_{N,i}\, a_{N,j} (T^i f, T^j f)\\ &= \sum_{j=0}^{\infty} a_{N,j}^2 \|f\|^2 \leqq \|f\|^2 \max_j a_{N,j}.\end{aligned}$$

Now hypothesis (1) is sufficient to guarantee that $\max_j a_{N,j} \to 0$ as $N \to \infty$ so $\|S_N f\| \to 0$ as $N \to \infty$.

$T$ is an isometry of $Z$ *onto* $Z$ so it is one-to-one when restricted to $Z$. In addition $(T^* Tf, g) = (Tf, Tg) = (f, g)$ for all $f, g \in L$ so $T^* T = I$. Let $T^{-1}$ be the inverse of $T$ restricted to $Z$. Then, on $Z$, $T^* = T^*(T T^{-1}) = (T^* T) T^{-1} = T^{-1}$ so that $T T^* = I$ on $Z$ as well. Thus $T$ restricted to $Z$ is unitary.

* * * *

Suppose $f \in Z$. By the spectral theory for unitary operators (applied to $T$ restricted to $Z$) we can write

$$\|S_N f\|^2 = \int_C \left| \sum_{k=0}^{\infty} a_{N,k} \lambda^k \right|^2 dm_f(\lambda) \tag{5}$$

where $C$ is the unit circle and $m_f$ is a finite measure on $C$. Since we are dealing only with functions in $L_0$ (i.e. with things of the form $g-E\{g|\Sigma_0\}$) the only invariant functions must be zero almost everywhere so 1 is not in the point spectrum of $T$ and thus $m_f\{1\}=0$. Since $\sum_{k=0}^{\infty} a_{N,k}=1$ the integrand in (5) is bounded by the integrable function 1. We will show that $\sum_{k=0}^{\infty} a_{N,k}\lambda^k \to 0$ for all $\lambda \neq 1$ such that $|\lambda|=1$. Then, since $m_f\{1\}=0$, the bounded convergence theorem will guarantee that $\|S_N f\|^2 \to 0$ for $f\in Z$.

If $\lambda^\alpha=1$ for some smallest positive integer $\alpha \geqq 2$, then

$$\sum_{k=0}^{\infty} a_{N,k}\lambda^k = \frac{1}{\alpha}\sum_{k=0}^{\alpha-1}\lambda^k + \sum_{j=0}^{\alpha-1}\lambda^j\left[\sum_{k=0}^{\infty} a_{N,k\alpha+j}-\frac{1}{\alpha}\right].$$

Now $\sum_{k=0}^{\alpha-1}\lambda^k=0$ and $\sum_{k=0}^{\infty} a_{N,k\alpha+j}\to\frac{1}{\alpha}$ as $N\to\infty$ by hypothesis (1) so $\sum_{k=0}^{\infty} a_{N,k}\lambda^k\to 0$ as $N\to\infty$.

Now suppose $\lambda=e^{2\pi i\gamma}$ with $\gamma$ irrational in (0, 1) and suppose $\varepsilon>0$. Choose a positive integer $M\geqq 2$ such that $|1-e^{2\pi i/M}|<\varepsilon/2$. Define $A_j=\left\{k \mid k \text{ is a non-negative integer and } k\gamma \bmod [0,1) \text{ is in } \left[\frac{j}{M},\frac{j+1}{M}\right)\right\}$, and choose $N_0$ such that $N\geqq N_0$ implies

$$\left|\sum_{k\in A_j} a_{N,k}-\frac{1}{M}\right|<\frac{\varepsilon}{2M} \qquad \text{for } j=0,\dots,M-1.$$

If $N\geqq N_0$ then

$$\begin{aligned}\left|\sum_{k=0}^{\infty} a_{N,k}\lambda^k\right| &\leqq \sum_{j=0}^{M-1}\sum_{k\in A_j} a_{N,k}|\lambda^k-e^{2\pi ij/M}|\\ &\quad+\sum_{j=0}^{M-1}|e^{2\pi ij/M}|\left|\sum_{k\in A_j}a_{N,k}-\frac{1}{M}\right|\\ &\quad+\frac{1}{M}\left|\sum_{j=0}^{M-1}e^{2\pi ij/M}\right|\\ &\leqq \sum_{k=0}^{\infty}a_{N,k}(\varepsilon/2)+\sum_{j=0}^{M-1}\varepsilon/2M+0=\varepsilon.\end{aligned}$$

We have seen that for $f\in L_0$ we can write $f=f_Z+f_1+f_2+\cdots$ with $f_Z\in Z$ and $f_i\in W_i$ for $i=1,2,\dots$, and have shown that $\|S_N f_Z\|\to 0$ and $\|S_N f_i\|\to 0$ for all $i$. This guarantees that $\|S_N f\|\to 0$ completing the proof of the sufficiency of hypotheses (1) and (2).

The necessity of (1) can be shown easily by an example involving a periodic transformation on a space consisting of a finite number of atoms. An example showing that (2) is necessary is somewhat tedious and a little tricky. In that which follows we let $\Omega=[0,1)$, let $\Sigma$ be the collection of Borel subsets of $\Omega$, and let $P$ be Lebesgue measure restricted to $\Sigma$.

Suppose $\gamma\in[0,1)$ is irrational and

$$\lim_{\varepsilon\to 0}\varlimsup_{N\to\infty}\sum_{k\in A_\varepsilon} a_{N,k}\geqq\delta>0 \tag{6}$$

where $A_\varepsilon=\{k\,|\,k\gamma \bmod [0,1) \text{ is in } [\gamma-\varepsilon,\gamma+\varepsilon) \bmod [0,1)\}$.

Let $A=\left[0,\frac{\delta}{2}\right)$, $B=\left[\gamma+\frac{\delta}{6},\gamma+\frac{\delta}{3}\right) \bmod [0,1)$, $f=\frac{2}{\delta}I_A-1$, and $g=\frac{6}{\delta}I_B-1$. Define $T(x)=x-\gamma \bmod [0,1)$. Then

$$\begin{aligned}(S_N(f),g)&=\sum_{k=0}^{\infty}\frac{12\,a_{N,k}}{\delta^2}(I_{T^{-k}A},I_B)-1\\&\geqq\frac{12}{\delta^2}\sum_{k\in A_{\delta/6}}a_{N,k}\,P(T^{-k}A\cap B)-1\\&\geqq\frac{12}{\delta^2}P(B)\sum_{k\in A_{\delta/6}}a_{N,k}-1\\&=\frac{2}{\delta}\sum_{k\in A_{\delta/6}}a_{N,k}-1\end{aligned}$$

so $\varlimsup_{N\to\infty}(S_N(f),g)\geqq 1$ and thus $\|S_N(f)\|$ does not converge to zero as $N\to\infty$.

Now suppose that (6) does not hold for any $\gamma$, $\gamma\in[0,1)$ with $\gamma$ irrational for any $\delta>0$. Suppose, however, that (2) fails to hold for fixed $a,b,\gamma\in[0,1)$ with $a<b$ and $\gamma$ irrational. Assume

$$\varlimsup_{N\to\infty}\sum_{\{k|k\gamma \bmod [0,1)\in[a,b)\}}=b-a+\delta$$

with $\delta>0$. (If not we will use $[0,1)-[a,b)$ and the same argument.) By our assumption about expression (6) there exists an $\varepsilon$ such that $0<\varepsilon<b-a$ and

$$\varlimsup_{N\to\infty}\sum_{k\in B_\varepsilon}a_{N,k}\leqq\frac{\delta}{2}$$

where $B_\varepsilon=\{k\,|\,k\gamma \bmod [0,1) \text{ is in } [b-\varepsilon,b+\varepsilon) \bmod [0,1)\}$. Let $T(x)=x-\gamma \bmod [0,1)$ as before and define

$$f=\frac{1}{\varepsilon}I_{[0,\varepsilon)}-1 \quad\text{and}\quad g=\frac{1}{b-a}I_{[a,b)}-1.$$

Then

$$\begin{aligned}(S_N(f),g)&=\frac{1}{\varepsilon(b-a)}\sum_{k=0}^{\infty}a_{N,k}(I_{T^{-k}[0,\varepsilon)},I_{[a,b)})-1\\&\geqq\frac{1}{\varepsilon(b-a)}\sum_{\{k|k\gamma \bmod [0,1) \text{ is in } [a,b-\varepsilon)\}}P(T^{-k}[0,\varepsilon)\cap[a,b))-1.\end{aligned}$$

Since $T^{-k}[0,\varepsilon)=([0,\varepsilon)+k\gamma)\bmod[0,1)$, if $k\gamma \bmod [0,1)\in[a,b-\varepsilon)$ then $T^{-k}[0,\varepsilon)\subset[a,b)$. Thus

$$(S_N(f),g)\geqq\frac{1}{b-a}\sum_{\{k\mid k\gamma \bmod[0,1) \text{ is in } [a,b-\varepsilon)\}} a_{N,k}-1$$

$$\geqq\frac{1}{b-a}\left[\sum_{\{k\mid k\gamma \bmod[0,1) \text{ is in } [a,b)\}} a_{N,k}-\sum_{k\in B_\varepsilon} a_{N,k}\right]-1$$

so

$$\overline{\lim_{N\to\infty}}\,(S_N(f),g)\geqq\frac{1}{b-a}\left[\overline{\lim_{N\to\infty}}\sum_{\{k\mid k\gamma \bmod[0,1)\in[a,b)\}} a_{N,k}-\overline{\lim_{N\to\infty}}\sum_{k\in B_\varepsilon} a_{N,k}\right]-1$$

$$\geqq\frac{1}{b-a}\left[(b-a+\delta)-\frac{\delta}{2}\right]-1>0.$$

Thus $\|S_N(f)\|$ can not converge to zero as $N\to\infty$.

*Proof of Theorem* 2. The proof is the same as that of Theorem 1 prior to **** except that it uses (3) instead of (1) to guarantee that $\max_j a_{N,j}\to 0$ as $N\to\infty$.

Since $T$ is weakly mixing and since we are dealing only with functions in $L_0$ (i.e. their means are already subtracted out) the point spectrum of $T$ restricted to $Z$ is empty.

Suppose $f\in Z$. At this point we apply the argument of [6, pp. 40 and 41] to get

$$\frac{1}{N}\sum_{k=0}^{N-1}|(T^k f,f)|\to 0 \quad \text{as} \quad N\to\infty,$$

or equivalently to get the existence of a set $S$ of positive integers of density zero such that $(T^k f,f)\to 0$ as $k\to\infty$ provided $k\notin S$. We have

$$\|S_N(f)\|^2=\sum_{j,k} a_{N,j}a_{N,k}(T^j f,T^k f)$$

$$\leqq\sum_k a_{N,k}^2\|f\|^2+2\sum_{j<k} a_{N,j}a_{N,k}|(T^{k-j}f,f)|$$

$$\leqq\|f\|^2\max_k\{a_{N,k}\}+2\sum_{\alpha=1}^{\infty}|(T^\alpha f,f)|\sum_{k=0}^{\infty} a_{N,k}a_{N,k+\alpha}.$$

Now assumption (3) guarantees that $\max_k\{a_{N,k}\}\to 0$ as $N\to\infty$ and

$$\begin{aligned}\sum_{\alpha=1}^{\infty}|(T^\alpha f,f)|\sum_{k=0}^{\infty}a_{N,k}a_{N,k+\alpha}\leqq{}&\|f\|^2\sum_{k=0}^{\infty}a_{N,k}\sum_{\alpha=1}^{M}a_{N,k+\alpha}\\&+\sup_{\substack{\alpha\geqq M\\ \alpha\notin S}}|(T^\alpha f,f)|\sum_{k=0}^{\infty}a_{N,k}\sum_{\alpha=1}^{\infty}a_{N,k+\alpha}\\&+\|f\|^2\sum_{k=0}^{\infty}a_{N,k}\sum_{\alpha\in S}a_{N,k+\alpha}\\ \leqq{}&M\|f\|^2\max_k\{a_{N,k}\}+\sup_{\substack{\alpha>M\\ \alpha\notin S}}|(T^\alpha f,f)|\\&+\|f\|^2\sum_{k=0}^{\infty}a_{N,k}\sum_{\alpha\in S}a_{N,k+\alpha}.\end{aligned}\tag{7}$$

If $\varepsilon>0$, we can make the first two terms in (7) less than $\varepsilon$ by first choosing $M$ large enough that $\sup_{\substack{\alpha\geqq M\\ \alpha\notin S}}|(T^\alpha f, f)|<\varepsilon/2$ and then choosing $N$ large enough that $\max_k\{a_{N,k}\}<\varepsilon/2M\|f\|^2$. It remains to show that

$$\sum_{k=0}^{\infty} a_{N,k}\sum_{\alpha\in S} a_{N,k+\alpha}\to 0 \quad \text{as} \quad N\to\infty.$$

Suppose not, that $\varepsilon>0$, and that

$$\sum_{k=0}^{\infty} a_{N,k}\sum_{\alpha\in S} a_{N,k+\alpha}\geqq 3\varepsilon$$

for infinitely many values of $N$. Now for each positive integer $m$

$$\sum_{k=0}^{\infty} a_{N,k}\sum_{\alpha\in S} a_{N,k+\alpha}\leqq \sum_{k=0}^{m} a_{N,k}+\max_{k>m}\sum_{\alpha\in S} a_{N,k+\alpha}.$$

The sum $\sum_{k=0}^{m} a_{N,k}\to 0$ as $N\to\infty$ so $\max_{k>m}\sum_{\alpha\in S} a_{N,k+\alpha}\geqq 2\varepsilon$ for infinitely many values of $N$. Thus we can find strictly increasing sequences of positive integers $\{N_m\}$ and $\{k_m\}$ such that $\sum_{\alpha\in S} a_{N_m,k_m+\alpha}\geqq 2\varepsilon$ for all $m$. Since $\sup_k\{a_{N,k}\}\to 0$ as $N\to\infty$, we could have chosen $\{N_m\}$ and $\{k_m\}$ so that $\sum_{\substack{\alpha\in S\\ \alpha>m}} a_{N_m,k_m+\alpha}\geqq\varepsilon$ for all $m$. If we define $S_m=\{k_m+\alpha\,|\,\alpha\in S, \alpha>m\}$, then we will find a subsequence $\{m_\nu\}$ such that $S^*=\bigcup_{\nu=1}^{\infty} S_{m_\nu}$ has density zero. (Then, of course,

$$\sum_{\alpha\in S^*} a_{N_{m_\nu},\alpha}\geqq \sum_{\alpha\in S_{m_\nu}} a_{N_{m_\nu},\alpha}\geqq \sum_{\substack{\alpha\in S\\ \alpha>m}} a_{N_{m_\nu},k_{m_\nu}+\alpha}\geqq\varepsilon$$

giving a contradiction.) If $A$ is a subset of the positive integers, define $\#(A)$ to be the cardinality of $A$, and define

$$R_N(A)=\frac{\#[A\cap\{1,\ldots,N\}]}{N}.$$

Note that $\sup_N R_N(S_m)\leqq \sup_{N\geqq m} R_N(S)$ which converges to zero as $m\to\infty$ since $S$ is of density 0. Choose $\{m_\nu\}$ strictly increasing and such that $\sup_N R_N(S_{m_\nu})\leqq 2^{-\nu}$. Then for every positive integer $\beta$

$$\begin{aligned}
R_N(S^*) &\leqq \sum_{\nu=1}^{\infty} R_N(S_{m_\nu})\\
&\leqq \sum_{\nu=1}^{\beta} R_N(S_{m_\nu})+\sum_{\nu>\beta}\sup_N R_N(S_{m_\nu})\\
&\leqq \beta R_N(S)+2^{-\beta}.
\end{aligned}$$

If $\varepsilon_1 > 0$ we can choose $\beta$ so that $2^{-\beta} < \varepsilon_1/2$, and then choose $N$ so that

$$\sup_{n \geqq N} R_n(S) < \varepsilon_1/2\beta.$$

Then if $n \geqq N$ we have $R_n(S^*) < \varepsilon_1$ and by definition $S^*$ is of density zero. As argued above, this leads to a contradiction. Thus

$$\sum_{k=0}^{\infty} a_{N,k} \sum_{\alpha \in S} a_{N,k+\alpha} \to 0 \quad \text{as} \quad N \to \infty,$$

and for $f$ in $Z$ we have $\|S_N(f)\| \to 0$ as $N \to \infty$.

Since for $f \in L_0$ we can write $f = f_Z + f_1 + f_2 + \cdots$ with $f_Z$ in $Z$ and $f_i$ in $W_i$ for $i = 1, 2, \ldots$; since the $S_N$'s are linear operators on $L$ with $\|S_N\| \leqq 1$ for all $N$; and since $\|S_N f_Z\| \to 0$ and $\|S_N f_i\| \to 0$ for all $i$; it follows that $\|S_N f\| \to 0$ for every $f \in L_0$.

*Proof of Theorem* 3 is straightforward and omitted. The theorem statement is included for completeness and because it is the obvious generalization of [3].

## 3. An Example

Considerable effort has been put into the study of the convergence (in some sense) of sums of the form

$$\sum_{k=0}^{\infty} a_{N,k} f(T^k x) \tag{8}$$

to $\int f dP$. Many of the results obtained have required that $f$ be such that the random variables $\{f(T^k x)\}$ be independent or very strong mixing in some sense; this type of result is not closely related to the results obtained in this paper.

The results in [1], [2], and [4] are much deeper than those obtained here and are of a somewhat different type. They yield almost everywhere convergence but, of course, require rather strong assumptions on the $a_{N,k}$'s.

The only result (known to the authors) with which the result given here might be in competition is the result of Cohen [5] which states:

**Theorem** (Cohen). *If $T$ is a linear operator from a Banach space $B$ to $B$ such that $\|T^n\| \leqq M < \infty$ for $n = 1, 2, \ldots$, if $a_{n,j}$ is a regular matrix such that*

$$\lim_{k \to \infty} \sum_{j=k}^{\infty} |a_{N,j+1} - a_{N,j}| = 0 \tag{9}$$

*uniformly in $N$, if $x \in B$, and if $\left\{L_N x = \sum_{j=1}^{\infty} a_{N,j} T^j x\right\}$ is a weakly compact set, then there is an $x_0$ in $B$ such that $\lim_{N \to \infty} L_N x = x_0 = T x_0$.*

The following is an example of a matrix $a_{N,k}$ of coefficients for which the theorems of this paper are applicable, but for which (9) does not hold. Let $b_1 = 0$

and for each $n \geqq 1$ let $b_{N+1} = b_n + 4n$. Let $I$ be the set of non-negative integers $i$ such that either

a) $i$ is even and there exists a positive integer $n$ such that $b_n \leqq i \leqq b_n + 2n - 2$, or

b) $i$ is odd and there exists a positive integer $n$ such that $b_n + 2n + 1 \leqq i \leqq b_{n+1} - 1$.

The set $I$ contains the circled integers in the "picture" below.

| $b_1$ | | $b_2$ | | | | $b_3$ | | | | | | $b_4$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 5 | 6 | 7 | 12 | 13 | 14 | 15 | 16 | 17 | 24 | 25 | ... |
| 2 | 3 | 8 | 9 | 10 | 11 | 18 | 19 | 20 | 21 | 22 | 23 | ... | | |

Let $I_N = \{k : k \text{ is one of the first } N \text{ integers in } I\}$, and let

$$a_{N,k} = \begin{cases} \dfrac{1}{N} & k \in I_N \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $k \in I_N$ then either $|a_{N,k} - a_{N,k-1}| = 1/N$ or $|a_{N,k+1} - a_{N,k}| = 1/N$ (or both), and that $\# \{k : k \in I_N, k \geqq N+1\} \geqq [N/2] - 1$. Thus

$$\begin{aligned} \sum_{k=N}^{\infty} |a_{N,k+1} - a_{N,k}| &\geqq \tfrac{1}{2} \sum_{\substack{k \geqq N+1 \\ k \in I_N}} [|a_{N,k} - a_{N,k-1}| + |a_{N,k+1} - a_{N,k}|] \\ &\geqq \frac{1}{2N} \, \# \{k : k \in I_N, k \geqq N+1\} \\ &\geqq \frac{N-3}{4N} \to \frac{1}{4}, \end{aligned}$$

and (9) does not hold. It is "obvious" that (3) holds. Rigorous proofs that (1) and (2) hold are routine but very tedious and will not be given here. To prove that (1) holds one may consider the various cases depending on which of $\alpha$ and $j$ are even and odd. The proof depends on the fact that if $\alpha$ is odd then

$$\frac{\# \{k \mid k \in I, b_n \leqq k \leqq b_n + 2n - 2, \alpha \text{ divides } k\}}{\# \{k \mid k \in I, b_n \leqq k \leqq b_n + 2n - 2\}}$$

and

$$\frac{\# \{k \mid k \in I, b_n + 2n + 1 \leqq k \leqq b_{n+1} - 1, \alpha \text{ divides } k\}}{\# \{k \mid k \in I, b_n + 2n + 1 \leqq k \leqq b_{n+1} - 1\}}$$

both converge to $1/\alpha$ as $n \to \infty$; if $\alpha$ is even, one of these converges to $2/\alpha$ and the other to zero depending on whether $j$ is odd or even. To prove that (2) holds one may note that the set $\{(m+2k)\gamma \bmod [0,1) : 1 \leqq k \leqq n\}$ gets in a sense uniformly spread out in $[0,1)$ as $n \to \infty$, and this "uniform spreading out" occurs uniformly in $m$.

## 4. Acknowledgement and Concluding Remarks

A talk bearing some resemblance to this paper was presented by the first author at an ergodic theory symposium held at Oberwolfach in August 1968. The main result was a version of Theorem 1 when $T$ is ergodic. Assumptions (1), (2), and (3) were shown to be sufficient for the desired convergence, and both (1) and (2) were shown to be necessary. Professors Jacobs, Kakutani, and Neveu were responsible for asking whether (1) and (2) were not in fact necessary *and sufficient* for the desired convergence and for providing the proof of this fact (i.e. for deleting condition (3) in the main theorem). The ideas used in the proof differ surprisingly little from those used in the authors' original proof (which incorporated the proof of Theorem 2); the new proof does involve a stronger use of spectral theory. It is clear that "something" related to sets of density zero must be dealt with if the theorem is to hold for weakly mixing transformations which are not strongly mixing. It also would seem reasonable, even without a knowledge of Theorem 1, that (3) is stronger than necessary for Theorem 2. It is easy to inductively construct sequences of coefficients which satisfy (1) and (2) but not (3), and it came as rather a surprise to the authors to discover that (1) and (2) imply the necessary relations (whatever they are) between the coefficients $a_{N,k}$ and the sets of density zero which one encounters when dealing with weakly mixing transformations. This would seem to imply some sort of regularity for these sets of density zero. Questions which arise are: 1) What properties do these sets have?, and 2) Can such sets be characterized in any useful fashion?

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*(Received October 5, 1968)*