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Compact Abelian Group Extensions of Discrete Dynamical Systems

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Summary. This paper introduces the notion of a free G extension of a dynamical system where G is a compact abelian group. The concept is closely allied to that of generalised discrete spectrum (which includes Abramov's quasi-discrete spectrum as a special case). We give necessary and sufficient conditions for a G extension of a minimal (uniquely ergodic) dynamical system to be minimal (uniquely ergodic) and show that in a certain sense a general G extension lifts these properties. Stable G-extensions always lift these properties if the underlying space is connected. This fact is then used to characterise all uniquely ergodic and minimal affine transformations of a certain three dimensional nilmanifold. The rest of the paper is devoted to the exhibition of group invariants for systems with generalised discrete spectrum. In particular it is shown that such systems always have a compact abelian group as underlying space. A lemma which facilitates this result gives necessary and sufficient conditions for a connected G-extension of a compact abelian group to be a compact abelian group.

§ 0. Introduction and Definitions

Furstenberg's structure theorem for distal dynamical systems [1] shows how a distal system may be constructed by a transfinite sequence of isometric extensions starting from the trivial system. At the moment there seems to be a lack of invariants associated with such a construction, although Knapp's work on distal algebras of functions [2] goes some way towards alleviating this defficiency. It would seem worthwhile, therefore, to restrict one's attention to a smaller class of systems. Systems with quasi-discrete spectrum are distal, but these have received sufficient attention in [3–7]. An intermediate class comprises systems with (for want of a better name) generalised discrete spectrum. In § 4 we introduce the notion of discrete spectrum mod D, where D is a conjugate closed, closed, sub-algebra of the algebra of continuous functions. By starting with the algebra C of constants, forming the mod C eigenfunctions, taking the closed linear span D of these functions, forming the mod D eigenfunctions and so on (a possibly transfinite procedure) we may arrive at the entire algebra of continuous functions. If this is the case we say the system has generalised discrete spectrum. (The procedure is similar to, but stronger than, Knapp's of forming functions almost periodic with respect to a sub-algebra [2]: on the other hand it is weaker than Abramov's of forming quasi-eigenfunctions of order $n+1$ from quasi-eigenfunctions of order n. As one might expect, systems with quasi-discrete spectrum have generalised discrete spectrum and these latter are distal.)

Our purpose, then, is to give a structure theorem (a canonical representation theorem) for minimal dynamical systems with generalised discrete spectrum acting on a compact metric space. The invariant we associate with such systems is a sequence of countable discrete abelian groups or, dually, a sequence of compact abelian groups (§ 4 Theorem 8). § 5 is devoted to a lemma which gives a necessary and sufficient condition for a compact abelian group extension of a compact abelian group to be a compact connected abelian group. The condition is very natural and the lemma is of such a fundamental nature that it may not be new, although we have not been able to find it in the literature. Minimal dynamical systems with generalised discrete spectrum acting on a compact connected metric space may be represented as "skew-product" transformations of a compact abelian group. This is a partial generalisation of the fact that a minimal dynamical system with quasi-discrete spectrum acting on a compact metric space can be represented as an affine transformation of a compact abelian group. The difference in the canonical structure of these types of transformation lies in the "non-affinity" of skew-products. For example a typical transformation of quasi-discrete type on a 2-torus $K \times K$ is

$$S(x, y) = (\alpha x, \beta x^n y), \tag{0.1}$$

whereas generalised-discrete spectrum is typified by

$$S(x, y) = (\alpha x, \varphi(x) y) \tag{0.2}$$

where $\varphi\colon K \to K$. Although the latter type in general fails to be affine, roughly speaking, we can say they act as translations on fibres.

Transformations of type (0.2) were studied by Furstenberg in [8], where conditions for the [minimality], unique ergodicity, and ergodicity of such transformations were given. In §§ 1, 2 we note that these conditions generalise to what we call [simple] compact abelian group extensions of dynamical systems. In essence, most of the ideas of these sections can be attributed to Furstenberg [8]. Our indebtedness to [8] is fully acknowledged.

An interesting corollary of the results in §§ 1, 2 shows that "in general" compact abelian group extensions of [minimal], [uniquely ergodic] systems are [minimal], [uniquely ergodic] (cf. [9] for similar results). This suggests the notion of a stable extension. If an extension is stable the qualification "in general" may be omitted. This simple result enables us to characterise all minimal [uniquely ergodic] affine transformations of a certain 3-dimensional nilmanifold, generalising certain results in [10, 11]. The study of such affine transformations appears in § 3.

§ 6 discusses, without proofs, the purely measure theoretic analogues of the rest of the paper.

A *dynamical system* (X, S) (in this paper) is a compact metric space X, together with a homeomorphism S. A dynamical system (Y, T) is a *factor* (or a *homomorphic image*) of (X, S) if there is a continuous map φ of X onto Y such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{S} & X \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow[T]{} & Y \end{array}$$

commutes ($S \xrightarrow{\varphi} T$). If in particular φ is a homeomorphism we say the two systems are *conjugate* or *homeomorphic*. If $SF = F$ (F closed) implies $F = X$ or ϕ we say that S is *minimal*. If m is a normalised S invariant Borel measure and if

$SF=F$ (F Borel) implies $m(F)=0$ or 1 we say that S is *ergodic* (with respect to m) or alternatively that m is an *ergodic measure* for S. Ergodic measures always exist; if there is only one for a given dynamical system we say that the system is *uniquely ergodic*. (In this case there is only one invariant measure and it is ergodic.) A uniquely ergodic system whose invariant measure is positive on non-empty open sets is minimal. A necessary and sufficient condition for (X, S) to be uniquely ergodic is that

$$\frac{1}{n}\sum_{k=0}^{n-1} f(S^k x) \to I(f) \qquad \text{for all } x\in X \text{ and for all } f\in C(X) \tag{0.3}$$

where $I(f)$ is a constant depending only on f and $C(X)$ is the algebra of complex valued continuous functions. (If (X, S) is uniquely ergodic the convergence in (0.3) is necessarily uniform [12].) S is called *distal* if $S^{n_k} x \to z$, $S^{n_k} y \to z$ implies $x=y$.

Let X be a free G-space where G is a compact abelian group i.e. there is given a continuous map φ of $G\times X$ onto X such that

$$\varphi(g, \varphi(h, x)) = \varphi(g h, x)$$

and $\varphi(g, x)=x$ if and only if g is the identity of G. (If the map φ is understood we shall write $g x$ for $\varphi(g, x)$.) If (X, S) is a dynamical system such that S commutes with G (i.e. $S(g x)=g S x$) then S induces a homeomorphism S' on the G orbit space $X'=X/G$ defined by

$$S' G(x) = G(S x)$$

where $G(x)$ is the G orbit of $x\big(G(x)=\{g x\colon g\in G\}\big)$. If (Y, T) is homeomorphic to (X', S') $(T \xrightarrow{\varphi} S')$ we shall say that (X, S) is a *free G-extension* of (Y, T). π will usually denote the map $(S \xrightarrow{\pi} T)$ defined by $\pi(x)=\varphi^{-1} G(x)$.

(X, S) is a *simple* free G-extension of (Y, T) if for every $\gamma\in\hat{G}$ (the character group of continuous homomorphisms of G into the circle group K) there exists $f_\gamma\in C(X, K)$ satisfying

$$f_\gamma(g x) = \gamma(g)\, f_\gamma(x). \tag{0.4}$$

($C(X, G)$ denotes the group of continuous maps of X into G, if G is a group.) Actually simplicity refers only to the spaces X, Y and the group G and the requirement is that the G space induced on the trivial line bundle $X\times C$ by each character $\gamma\in\hat{G}$ $\big(g\colon (x, c)\to(g x, \gamma(g) c)\big)$ possesses an every non-zero cross-section i.e. there exists $s\colon X\to C\big(s(x)\neq 0\big)$ such that the section $(x, s(x))$ is G invariant, for to obtain f_γ one need only divide $s(x)$ by $|s(x)|$.

If D is a closed, conjugate closed, sub-algebra of $C(X)$ an *eigen-function* mod D is a function $f\in C(X, K)$ satisfying

$$\frac{f S}{f}\in D.$$

(X, S) is said to have *discrete spectrum* mod D if the linear span of $H(D)$ is dense in $C(X)$ where $H(D)$ denotes the group of eigenfunctions mod D. (If D is the algebra of constant functions discrete spectrum mod D reduces to the classical case.) Generalised discrete spectrum will be defined later.

Throughout this paper (X, S) will denote a dynamical system and G will denote a compact abelian group.

§ 1. Minimal Extensions

Theorem 1. *Let (X, S) be a simple free G-extension of the minimal dynamical system (Y, T). Then (X, S) is minimal if and only if*

$$\frac{f(T\pi x)}{f(\pi x)} = \frac{f_\gamma(Sx)}{f_\gamma(x)} \qquad (S \xrightarrow{\pi} T) \tag{1.1}$$

has no solution $f \in C(Y, K)$, $\gamma \neq 1$.

If (1.1) *has a solution f for f_γ satisfying* (0.4) *then* (1.1) *has a solution f' for each f'_γ satisfying* (0.4).

Proof. Let us clarify the last remark first. If f satisfies (1.1) where $f_\gamma(gx) = \gamma(g) f(x)$ and if $f'_\gamma(gx) = \gamma(g) f'(x)$ then f_γ / f'_γ is G invariant and hence has the form $h \circ \pi$ where $h \in C(Y, K)$. Therefore

$$\frac{f(T\pi x)}{f(\pi x)} = \frac{f'_\gamma(Sx)\, h(\pi S x)}{f'_\gamma(x)\, h(\pi x)} = \frac{f'_\gamma(Sx)\, h(T\pi x)}{f'_\gamma(x)\, h(\pi x)}$$

and $f' = f/h$ satisfies (1.1) with f'_γ in place of f_γ.

Suppose $\gamma \not\equiv 1$, $f \in C(Y, K)$ and

$$\frac{f(\pi S x)}{f(\pi x)} = \frac{f(T\pi x)}{f(\pi x)} = \frac{f_\gamma(Sx)}{f_\gamma(x)},$$

then $f\pi / f_\gamma$ is S invariant and not constant (for this would entail $f(\pi g x) = \gamma(g) f(\pi x) = f(\pi x)$ i.e. $\gamma \equiv 1$) and hence S is not minimal.

On the other hand suppose S is not minimal. Let $F \neq X$ be a closed minimal set for S. Since gF is also a closed minimal set for S, for each $g \in G$ we have $gF = F$ or $gF \cap F = \phi$. Moreover $gF \cap F = \varphi$ for some $g \in G$ for otherwise we would have $\pi^{-1}\pi F = F$ and the minimality of T would imply $F = X$. Hence the closed subgroup $H = \{h\colon hF = F,\ h \in G\}$ of G is proper. Let $\gamma(H) = 1$, $\gamma \neq 1$. If $\pi x = y$ and $gx \in F$ define $f(y) = f_\gamma(gx) = \gamma(g) f_\gamma(x)$. This is well defined for if $hx \in F$ then $\gamma(g) f_\gamma(x) = \gamma(h) f_\gamma(x)$ since $gh^{-1} \in H$. If $y_n \to y$, $\pi x_n = y_n$, $\pi x = y$, $g_n x_n \in F$, $gx \in F$ we must show that $\gamma(g_n) f_\gamma(x_n) \to \gamma(g) f_\gamma(x)$.

By considering a subsequence we may suppose $g_n x_n \to g' x \in F$ (since F is closed and $y_n \to y$) and therefore $f_\gamma(g_n x_n) \to f_\gamma(g' x) = \gamma(g') f_\gamma(x) = \gamma(g) f_\gamma(x)$ since $g' g^{-1} \in H$. Hence $f \in C(Y, K)$ and

$$\frac{f(T\pi x)}{f(\pi x)} = \frac{f(\pi S x)}{f(\pi x)} = \frac{f_\gamma(gSx)}{f_\gamma(hx)} = \frac{\gamma(g) f_\gamma(Sx)}{\gamma(h) f_\gamma(x)}$$

where $gSx = Sgx \in F$ (i.e. $x \in F$) and $hx \in F$.

Hence $gh^{-1} \in H$ and

$$\frac{f(T\pi x)}{f(\pi x)} = \frac{f_\gamma(Sx)}{f_\gamma(x)}.$$

Corollary 1. *Suppose (X, S), (X, S_o) are simple free G extensions of (Y, T). Then $S x = \varphi(\pi x) S_o x$ where $\varphi \in C(Y, G)$ and S is minimal if and only if*

$$\frac{f(T\pi x)}{f(\pi x)} = \gamma(\varphi \pi x) \frac{f_\gamma(S_o x)}{f_\gamma(x)} \tag{1.2}$$

has no solution $f \in C(Y, K)$, $\gamma \neq 1$.

Proof. The proof is immediate once we have shown that $S x = \varphi(\pi x) S_o x$ for some $\varphi \in C(Y, G)$ and this follows from the fact that $\pi S x = T x = S_o x$ since for each x there exists $g = \varphi'(x)$ such that $S x = \varphi'(x) S_o x$. Moreover

$$S g x = g S x = \varphi'(g x) S_o(g x) = g \varphi'(g x) S_o x$$

and we have $S x = \varphi'(g x) S_o x$ i.e. $\varphi'(g x) = \varphi'(x)$ for all $g \in G$ and therefore $\varphi'(x) = \varphi \pi x$ for some $\varphi \in C(Y, G)$. (The continuity of φ is easily proved.)

Corollary 2. *If $X = Y \times G$ $\big(g(y, h) = (y, g h)\big)$ then Eq. (1.2) becomes*

$$\frac{f(T y)}{f(y)} = \gamma \varphi(y) \tag{1.3}$$

on putting $S_o(y, h) = (T y, h)$ and $f_\gamma(y, g) = \gamma(g)$.

Remark 1. In Theorem 1 and its corollaries if X is connected a solution of

$$\frac{f(T\pi x)}{f(\pi x)} = \frac{f_\gamma(S x)}{f_\gamma(x)} \tag{1.4}$$

cannot exist if $\gamma \neq 1$ is of finite order.

Proof. If $\gamma^k = 1$ and Eq. (1.4) has a solution then $f_\gamma(g x) = \gamma(g) f_\gamma(x)$, $f_\gamma^k(g x) = f_\gamma^k(x)$ and for some $h \in C(Y, K)$ $h(\pi x) = f_\gamma^k(x)$. Therefore

$$\frac{f^k(T y)}{f^k(y)} = \frac{h(T y)}{h(y)}$$

i.e. f^k/h is constant and $f^k \pi / f_\gamma^k$ is constant. Since X is connected $f \pi / f_\gamma$ is constant and therefore f_γ is constant on fibres $\pi^{-1} y$ i.e. $\gamma = 1$.

Theorem 2. *Let (X, S_e) be a simple free G-extension of the minimal dynamical system (Y, T) where X is connected $(S_e \xrightarrow{\pi} T)$. Then $\{g\colon S_g = g S_e$ is minimal$\}$ contains a dense G_δ in G.*

Proof. Let $\mathscr{I}$ be the set of characters of G of infinite order. According to Corollary 1 of Theorem 1 and Remark 1, S_g is not minimal if and only if there exist $\gamma \in \mathscr{I}$, $f_g^\gamma \in C(Y, K)$ satisfying

$$\frac{f_g^\gamma(T\pi x)}{f_g^\gamma(\pi x)} = \gamma(g) \frac{f_\gamma(S_e x)}{f_\gamma(x)}. \tag{1.5}$$

If $h \in C(Y, K)$ also satisfies (1.5) for the same γ, g then f_g^γ/h is constant. In other words, up to a multiplicative constant, there is at most one f_g^γ for each γ, g. Should f_k^γ satisfy (1.5) with g replaced by k we would have

$$\frac{f_g^\gamma(T\pi x)}{f_k^\gamma(T\pi x)}=\gamma(g/k)\frac{f_g^\gamma(\pi x)}{f_k^\gamma(\pi x)}.$$

Hence G_γ the set of $g \in G$ admitting a solution of (1.5) is a coset of some sub-group of G and using the separability of X, $\gamma(G_\gamma)$ is a countable coset of some countable sub-group of K.

$$\begin{aligned}\{g\colon S_g \text{ is not minimal}\} &= \{g\colon \text{for some } \gamma\in\mathscr{I}, g\in G_\gamma\}\\ &= \bigcup_{\gamma\in\mathscr{I}} G_\gamma \subset \bigcup_{\gamma\in\mathscr{I}} \gamma^{-1}\gamma(G_\gamma).\end{aligned}$$

Since $\mathscr{I}$ is countable and $\gamma(G_\gamma)$ is countable the latter set is an F_σ. Moreover $\gamma^{-1}(k)$ is nowhere dense if $k \in K$ since γ is not of finite order. Hence the latter set is a first category F_σ, i.e. $\{g\colon S_g \text{ is minimal}\}$ contains a dense G_δ.

Corollary 1. *If gS_0 is homeomorphic to S_0 for each $g \in G$ then gS_0 is minimal for each $g \in G$.*

Proof. This follows immediately from the existence of at least one g such that gS_0 is minimal.

Corollary 1 motivates the following definition: If (X, S) is a free (not necessarily simple) G extension of (Y, T) we will say that the extension is *stable* if gS is homeomorphic to S for every $g \in G$. In view of Corollary 1, verifying stability is a method of verifying minimality. An analogous result holds for unique ergodicity and we shall apply this method in § 3.

§ 2. Uniquely Ergodic Extensions

In § 1 we needed repeatedly to assume that G-extensions were simple i.e. for each $\gamma \in \hat{G}$ there exists $f_\gamma \in C(X, K)$ satisfying $f_\gamma(gx)=\gamma(g)f_\gamma(x)$. In this section it is enough to assume the existence of such functions in the class $B(X, K)$ of Borel maps of X into K, (*Borel* simple extensions). However, this is no assumption at all as we will now show. The proof imitates Gleason's proof of G invariant local sections for vector bundles.

Lemma 1. *Let X be a free G-extension of the compact metric space Y. Then for each $\gamma \in \hat{G}$ there exists $f_\gamma \in B(X, K)$ satisfying $f_\gamma(gx)=\gamma(g)f_\gamma(x)$.*

Proof. Let $X \xrightarrow{\pi} Y$ and $\gamma \in G$. For each $y \in Y$ we will produce an open neighbourhood $U(y)$ such that on $\pi^{-1}U(y)$ there is a function $F_y \in C(X)$ vanishing nowhere on $\pi^{-1}U(y)$ satisfying $F_y(gx)=\gamma(g)F_y(x)$. Letting $U_1, \ldots, U_k$ be a finite covering of Y selected from $\{U(y)\}$, $U_i = U(y_i)$ and defining $F_i = F_{y_i}/|F_{y_i}|$ on U_i we finally define

$$\begin{aligned} f_\gamma &= F_i \quad \text{on} \quad U_i-(U_1\cup\cdots,\cup U_{i-1}) \qquad i=2,3,\ldots,k\\ &= F_1 \quad \text{on} \quad U_1.\end{aligned}$$

We proceed therefore to produce our neighbourhoods $U(y)$ and functions F_y. Choose $x_o \in \pi^{-1} y$ and define F on $\pi^{-1} y$ by $F(g x_o) = \gamma(g)$. By Tietze's extension theorem extend $F: X \to C, F \in C(X)$. Define $F_y(x) = \int \gamma(h) F(h^{-1} x) dh$ where integration is with respect to normalised Haar measure on G. By the invariance of Haar measure under translation we have

$$F_y(g x) = \gamma(g) F_y(x) \quad \text{and} \quad F_y|_{\pi^{-1} y} = F|_{\pi^{-1} y}$$

and clearly $F_y \in C(X)$. Since $|F_y| = 1$ on $\pi^{-1} y$ $\{x: F_y(x) \neq O\}$ is an open subset of X of the form $\pi^{-1} U(y) \supset \pi^{-1} y$. The lemma is proved.

In the following if m is a normalised Borel measure defined on Y and X is a G-extension of Y, $\tilde{m}$ will denote the normalised Borel measure on X defined by

$$\int f(x) d\tilde{m} = \int \left(\int f(g x) dg \right) dm$$

for Borel functions on X. In this equation the inner integral is with respect to Haar measure producing a function which is constant on fibres $\pi^{-1} y$ making the outer integral meaningful. If m is T invariant and S is a free G-extension of T then $\tilde{m}$ is S invariant.

Theorem 3. *Let (X, S) be a free G-extension of the ergodic dynamical system (Y, T, m). Let $f_\gamma \in B(X, K)$ satisfy $f_\gamma(g x) = \gamma(g) f_\gamma(x)$ where $\gamma \in \hat{G}$. Then $(X, S, \tilde{m})$ is ergodic if and only if*

$$\frac{f(T \pi x)}{f(\pi x)} = \frac{f_\gamma(S x)}{f_\gamma(x)} \qquad a.e.\ (\tilde{m}),\ (S \xrightarrow{\pi} T), \tag{2.1}$$

has no solution $f \in B(Y, K)$, $\gamma \neq 1$.

Proof. Suppose $\gamma \neq 1$ and $f \in B(Y, K)$ satisfies

$$\frac{f(\pi S x)}{f(\pi x)} = \frac{f(T \pi x)}{f(\pi x)} = \frac{f_\gamma(S x)}{f_\gamma(x)}$$

then $f \pi / f_\gamma$ is S invariant and $f \pi / f_\gamma$ is not constant a.e. $(\tilde{m})$ for this would imply $f(\pi g x) = \gamma(g) f(\pi x)$ a.e. i.e. $\gamma \equiv 1$. Hence S is not ergodic.

If S is not ergodic there exists $f \in L^2(X, \tilde{m})$ not constant a.e. satisfying $f(S) = f$. It is not difficult to see that $L^2(X, \tilde{m})$ is spanned by functions of the form $f_\gamma(x) h_\gamma(x)$ where $h_\gamma(x) = h'_\gamma(\pi x)$, $h'_\gamma \in L^2(Y, m)$. Hence

$$f = \sum f_\gamma(x) h_\gamma(x) = \sum f_\gamma(S x) h_\gamma(S x)$$

and

$$\sum_\gamma f_\gamma(x) \overline{f_{\gamma'}(x)} h_\gamma(x) = \sum_\gamma f_\gamma(S x) \overline{f_{\gamma'}(x)} h_\gamma(S x) \tag{2.2}$$

and applying the conditional expectation operator $E(/\pi^{-1} B(Y))$ to both sides of (2.2) we get

$$f_{\gamma'}(x) \overline{f_{\gamma'}(x)} h_{\gamma'}(x) = f_{\gamma'}(S x) \overline{f_{\gamma'}(x)} h_{\gamma'}(S x). \tag{2.3}$$

Here we have used the facts that $h_\gamma(x)$, $h_\gamma(Sx)$ are $\pi^{-1}B(Y)$ measurable and

$$f_\gamma(gx)\overline{f_{\gamma'}(gx)}=\gamma(g)\overline{\gamma'(g)}f_\gamma(x)\overline{f_{\gamma'}(x)}$$

and hence

$$E(f_\gamma \bar{f}_{\gamma'}/\pi^{-1}B(Y))/f_\gamma \bar{f}_{\gamma'} \qquad \text{is } G \text{ invariant}.$$

Using the ergodicity of T it follows that either $E(f_\gamma \bar{f}_{\gamma'}/\pi^{-1}B(Y))=0$ a.e. $(\gamma \neq \gamma')$ or $f_\gamma \bar{f}_{\gamma'}$ is G invariant $(\gamma\neq\gamma')$ which latter plainly is not so. Hence $E(f_\gamma \bar{f}_{\gamma'}/\pi^{-1}B(Y))=0$ a.e. In the same way $E(f_\gamma(Sx)f_{\gamma'}(x)/\pi^{-1}B(Y))=0$ a.e. if $\gamma\neq\gamma'$. (2.3) is therefore justified, and since $h_{\gamma'}(x)$ vanishes only on an S invariant $\pi^{-1}B(Y)$ set the ergodicity of T implies $h_{\gamma'}=0$ on a set of measure zero or one. However since f is not constant a.e. there must exist $\gamma'\neq 1$ such that $h_{\gamma'}$ is zero only on a set of measure zero. By normalisation it follows from (2.3) that (2.1) has a solution in $B(Y,K)$.

Corollary 1. *Suppose $(X,S,\tilde{m})$, $(X,S_o,\tilde{m})$ are free G extensions of (Y,T,m). Then $Sx=\varphi(\pi x)S_o x$ where $\varphi\in C(Y,G)$ and S is ergodic if and only if*

$$\frac{f(T\pi x)}{f(\pi x)}=\gamma(\varphi(\pi x))\frac{f_\gamma(S_o x)}{f_\gamma(x)} \quad a.e. \tag{2.4}$$

has no solution $f\in B(Y,K)$, $\gamma\neq 1$.

Corollary 2. *If $X=Y\times G$ $(g(y,h)=(y,gh))$ then Eq. (2.4) becomes*

$$\frac{f(Ty)}{f(y)}=\gamma(\varphi(y)) \quad \text{on putting} \quad S_o(y,h)=(Ty,h)$$

and $f_\gamma(y,g)=\gamma(g)$.

Remark 2. In Theorem 3 and its corollaries if X is connected a solution of

$$\frac{f(T\pi x)}{f(\pi x)}=\frac{f_\gamma(Sx)}{f_\gamma(x)} \quad \text{a.e.}$$

cannot exist if $\gamma\neq 1$ is of finite order.

Theorem 4. *Let $(X,S_e,\tilde{m})$ be a free G extension of the ergodic dynamical system (Y,T,m) where X is connected $(S_e \xrightarrow{\pi} T)$. Then $\{g\colon S_g=gS_e \text{ is ergodic}\}$ contains a dense G_δ in G.*

Corollary 1. *If the extension $(X,S_e,\tilde{m})$ is stable then gS_e is ergodic for each $g\in G$.*

The *proofs* of Corollaries 1, 2 of Theorem 3, Remark 2, Theorem 4 and its corollary are direct copies of corresponding statements in § 1.

Theorem 5. *Let (X,S) be a free G extension of the uniquely ergodic dynamical system (Y,T) (with unique invariant normalised measure m). Let $f_\gamma\in B(X,K)$ satisfy $f_\gamma(gx)=\gamma(g)f_\gamma(x)$, $\gamma\in\hat{G}$. Then the following are equivalent:*

(i) *(X,S) is ergodic with respect to $\tilde{m}$.*

(ii) *(X,S) is uniquely ergodic.*

(iii) $\dfrac{f(T\pi x)}{f(\pi x)}=\dfrac{f_\gamma(Sx)}{f_\gamma(x)}$ *a.e.* $(\tilde{m})$

has no solution $f\in B(Y,K)$, $\gamma\neq 1$.

Proof. As we have seen if (2.1) has a solution when $\gamma \not\equiv 1$ then S is not ergodic and therefore not uniquely ergodic i.e. (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). We have also seen that (iii) $\Rightarrow$ (i). We need only show that (i) $\Rightarrow$ (ii).

Suppose S is ergodic with respect to $\tilde{m}$ and suppose $\tilde{p}$ is a normalised S invariant Borel measure with respect to which S is ergodic. With $q = \tilde{m}$ or $\tilde{p}$ let

$$E_q = \left\{x: \frac{1}{n}\sum_{i=0}^{n-1} f(S^i x) \to \int f\,dq \text{ for all } f \in C(X)\right\}.$$

By the ergodic theorem $q(E_q) = 1$. Notice that if $x \in E_{\tilde{m}}$ then $g x \in E_{\tilde{m}}$ for all $g \in G$ since

$$\frac{1}{n}\sum_{i=0}^{n-1} f(S^i g x) = \frac{1}{n}\sum_{i=0}^{n-1} f(g S^i x) \to \int f(g x)\,d\tilde{m} = \int f(x)\,d\tilde{m},$$

for all $f \in C(X)$ i.e. $\pi^{-1}\pi E_{\tilde{m}} = E_{\tilde{m}}$. The measure p defined on Y by $p(B) = \tilde{p}\,\pi^{-1} B$ is clearly T invariant and hence $p = m$. Hence $m(\pi E_{\tilde{p}}) = \tilde{p}\,\pi^{-1}\pi E_{\tilde{p}} \geqq \tilde{p}(E_{\tilde{p}}) = 1$ and $m(\pi E_{\tilde{m}}) = \tilde{m}(\pi^{-1}\pi E_{\tilde{m}}) = \tilde{m}(E_{\tilde{m}}) = 1$ and $\pi E_{\tilde{p}}$, $\pi E_{\tilde{m}}$ have a common point. Since $E_{\tilde{m}}$ consists of whole fibres $\pi^{-1} y$, $E_{\tilde{p}}$, $E_{\tilde{m}}$ have a common point x. Hence

$$\int f\,d\tilde{p} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(S^i x) = \int f\,d\tilde{m} \qquad \text{for all } f \in C(X) \text{ and } \tilde{p} \equiv \tilde{m}.$$

Remark 3. Let $(X, S, \tilde{m})$ be a free G extension of the uniquely ergodic dynamical system (Y, T, m). Then Corollary 1 and 2 of Theorem 3, Remark 2, Theorem 4 and its corollary remain true if "ergodicity" is replaced by unique ergodicity.

In view of the fact that $\tilde{m}$ is a measure which is positive on non-empty open sets (if the same is true of m) the unique ergodicity of $(X, S, \tilde{m})$ implies that (X, S) is minimal.

Remark 4. There is a very close analogy between the situation where (X, S) is a non-minimal simple free G-extension of a minimal system (Y, T) and the situation where (X, S) is a non-uniquely ergodic free G extension of a uniquely ergodic system (Y, T) (even when (X, S) is minimal). In the first case a minimal set is translated to either a disjoint or coincident minimal set by elements of G. The set fixing the minimal set is a closed sub-group. In the second case an ergodic measure is translated to either a mutually singular or coincident measure. The set fixing a given ergodic measure is a closed subgroup. From this fact it is easy to see that if an extension is not uniquely ergodic then as many ergodic measures exist as there are cosets of the corresponding sub-group. In case G is the circle a proper closed sub-group is finite and therefore there is either one ergodic measure or uncountably many. This was first noted in [13].

§ 3. Applications to Nilmanifolds

Let N be the nilpotent Lie group consisting of matrices

$$\begin{pmatrix} x & y \\ & z \end{pmatrix} \equiv \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \qquad (x, y, z, \text{ real})$$

and D be the uniform discrete subgroup obtained by restricting x, y, z to be integers. Define $X = N/D$ the space of left cosets of D. X is a compact connected three dimensional nil manifold. Define $S\tilde{x}D=\tilde{\alpha}A(\tilde{x})D$ where $\tilde{x},\tilde{\alpha}\in N$ and where A is an automorphism of N such that $AD=D$. S is called an affine transformation of the nilmanifold. If

$$A:\begin{pmatrix} x & y \\ & z \end{pmatrix}\to\begin{pmatrix} \varphi_1 & \varphi_2 \\ & \varphi_3 \end{pmatrix}$$

then since A is an automorphism we have

$$\varphi_i(x,y,z)+\varphi_i(x',y',z')=\varphi_i(x+x',y+y'+xz',z+z') \qquad i=1,3 \tag{3.1}$$

and

$$\varphi_1(x,y,z)\,\varphi_3(x',y',z')+\varphi_2(x,y,z)+\varphi_2(x',y',z')=\varphi_2(x+x',y+y'+xz',z+z'). \tag{3.2}$$

Moreover $AN'=N'$ where N' is the derived group consisting of elements

$$\tilde{r}=\begin{pmatrix} o & r \\ & o \end{pmatrix}$$

and therefore $\varphi_i(o,r,o)=o$, $i=1,3$. From (3.2) we have $\varphi_2(o,y,o)+\varphi_2(o,y',o)=\varphi_2(o,y+y',o)$ i.e. $\varphi_2(o,y,o)=\varepsilon y$ where $\varepsilon=\pm 1$. $(A(N'\cap D)=N'\cap D.)$ From (3.1) we have

$$\varphi_i(x,y,z)=\varphi_i(x,o,z)+\varphi_i(o,y,o)=\varphi_i(x,o,z) \qquad i=1,3,$$

and from (3.2) we have

$$\varphi_2(x,y,z)=\varphi_1(o,y,o)\,\varphi_3(x,o,z)+\varphi_2(o,y,o)+\varphi_2(x,o,z)=\varepsilon y+\varphi_2(x,o,z). \tag{3.3}$$

The circle group $G=N'\cdot D/D$ acts freely on X according to $\tilde{r}\cdot(\tilde{x}D)=\tilde{x}\tilde{r}D$ where $\tilde{r}\in N'$, and the affine transformation sends cosets of N' to cosets of N'. Hence S induces a homeomorphism on

$$Y=X/G=\frac{N/D}{N'\cdot D/D}\simeq N/N'\cdot D,$$

and since $N'\cdot D$ is the group of matrices

$$\begin{pmatrix} a & r \\ & c \end{pmatrix}$$

where a, c are integers and r is real Y is the two dimensional torus $R\times R/Z\times Z$ and the induced homeomorphism is the affine transformation

$$T(x,z)=(\alpha,\gamma)+(ax+bz,cx+dz) \bmod 1$$

where a, b, c, d are integers and $\Delta=ad-bc=\pm 1$.

In order that T should be minimal (or uniquely ergodic) it is necessary and sufficient that the matrix

$$A'=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

should have entries $a=d=1$ and $bc=o$ and the element (α,γ) together with elements (bx, cz) should generate Y [6].

Again from (3.2) we deduce that $\varphi_2(x,o,z)=\varphi_2(x,o,o)+\varphi_2(o,o,z)+xz(1-\varepsilon)$ and on substitution in (3.3) and resubstitution in (3.2) we get

$$c x x' + b z z' + \varphi_2(x,o,o)+\varphi_2(o,o,z)+\varphi_2(x',o,o)+\varphi_2(o,o,z') \\ = \varphi_2(x+x',o,o)+\varphi_2(o,o,z+z')+x'z(1-\varepsilon).$$

Hence

$$\varphi_2(x,o,o)-\frac{c}{2}x^2$$

is a homomorphism of R to R i.e.

$$\varphi_2(x,o,o)=\frac{c}{2}x^2+\lambda x.$$

Similarly

$$\varphi_2(o,o,z)=\frac{b}{2}z^2+\mu z$$

from which we conclude $\varepsilon=1$, b, c are even (one of which, of course, is zero) and λ, μ are integers.

Therefore A has the form

$$\begin{pmatrix} x & y \\ & z \end{pmatrix} \to \begin{pmatrix} x+bz & \frac{c}{2}x^2+\lambda x+\frac{b}{2}z^2+\mu z+y \\ & cx+z \end{pmatrix} \tag{3.4}$$

(where $bc=0$, b, c even, λ, μ integers) if S (and hence T) is minimal or uniquely ergodic.

Above we concluded that $\varepsilon=1$. This implies that S is a free G extension of T. We will show that S is G stable by showing that for every $g\in G$ there exists φ such that $\varphi^{-1}gS\varphi=S$. In fact we shall only consider φ of the form $x\to\beta x$ and solve $g\alpha A(\beta x)=\beta\alpha A(x) \bmod D$ where $g\in N'$. Since N' is central we have to solve

$$\alpha^{-1}\beta^{-1}\alpha A(\beta)=g^{-1} \bmod D. \tag{3.5}$$

With A satisfying (3.4) it is a purely computational matter to verify that the image of the map $\beta\to\alpha^{-1}\beta^{-1}\alpha A(\beta)$ contains a half line of

$$N'=\left\{\begin{pmatrix} o & r \\ & o \end{pmatrix}\right\}$$

and hence the Eq. (3.5) always has a solution β if $g\in N'$. We have therefore proved:

Theorem 6. *In order that an affine transformation $S\tilde{x}D=\tilde{\alpha}A(\tilde{x})D$ of the nil-manifold $X=N/D$ be minimal (or uniquely ergodic) it is necessary and sufficient that S be a free G extension of a minimal (uniquely ergodic) affine transformation* *of the torus* $Y=X/G$ $(G=N'\cdot D/D)$. *Such transformations S are G stable and have the form* $\tilde{x}D\to\tilde{\alpha}A(x)D$ *where*

$$\tilde{\alpha}=\begin{pmatrix}\alpha & \beta\\ & \gamma\end{pmatrix}$$

A is given by (3.4) *and* (α,γ) *together with* (bx, cz) *generate the two dimensional torus.* (It is an easy matter, moreover, to verify that they are distal.)

The analysis of minimal affine transformations of compact abelian groups can also be carried out using these methods as we can see by the following:

Theorem 7. *Let* $Sx=\alpha Ax$ *be an affine transformation of a compact connected metric abelian group X. Let G be a closed sub-group of X such that*

$$G\subset\ker(A-I)\cap\operatorname{Im}(A-I)$$

then S is a (simple) free stable G extension of T (where $TxG=\alpha A(x)G$*) and is minimal, uniquely ergodic, ergodic according as T is minimal, uniquely ergodic, ergodic.*

§ 4. Generalised Discrete Spectrum

Lemma 2. *Let* H_1 *be a sub-group of* $C(X,K)$ *whose linear span is dense in* $C(X)$, *and let D be a closed (conjugate-closed) sub-algebra of* $C(X)$. *If E is a linear operator of* $C(X)$ *to D satisfying*

(i) $E(fg)=E(f)\cdot g,\ f\in C(X),\ g\in D,$

(ii) $E(f)\geqq O$ *if* $f\geqq O$ *with equality only if* $f\equiv O$,

(iii) $E(1)=1,$

(iv) $\overline{E(f)}=E(\bar{f}),$

(v) $E(f)\equiv O$ *if* $f\in H_1-D$,

then there exist a compact abelian group G acting freely on X such that D consists exactly of functions which are G invariant. Moreover for each $\gamma\in\hat{G}$ *there exists* $f_\gamma\in H_1$ *satisfying* $f_\gamma(gx)=\gamma(g)f_\gamma(x)$. *(*$E(f)$ *is simply* $\int f(gx)\,dg$ *where dg is Haar measure on G.)*

Proof. Let m be a normalised Borel measure on X such that $\int f\,dm>O$ if $f\geqq O$ $(f\not\equiv O)$ and define $\tilde{m}$ by $\int f\,d\tilde{m}=\int E(f)\,dm$. Then $\tilde{m}$ is a normalised measure on X and (ii) ensures that $\int f\,d\tilde{m}>O$ if $f\geqq O$ $(f\not\equiv O)$. We may and shall suppose that $H_1\supset D\cap C(X,K)$ by considering $H_1\cdot D\cap C(X,K)$ instead of H_1 if necessary. (Condition (i) would ensure that (v) remains true.) Let $H_2=D\cap C(X,K)$ so that H_2 is a sub-group of H_1. (v) ensures that members of distinct cosets of H_2 are E orthogonal (and therefore $\tilde{m}$ orthogonal) so that $\Gamma=H_1/H_2$ is a countable group since X is a compact metric space.

Let $G=\hat{\Gamma}$ the group of homomorphism of Γ into K (or equivalently the group of homomorphisms of H_1 into K sending H_2 to 1) equipped with the compact open topology so that G is a compact abelian group. In a well known sense $\hat{G}=\Gamma$. We shall define the action of G on X by describing its action on $C(X)$ as a group of isometric isomorphisms. It will be enough to describe the G action on L (the linear span of H_1) as a group of isometric isomorphisms. Every member of L can be written uniquely as

$$f \equiv f_1 h_1 +, \ldots, f_k h_k, \qquad h_i \in D, \ f_i \in H_1 \tag{4.1}$$

$(f_1, \ldots, f_k)$ belonging to distinct cosets of H_2, for should (4.1) be identically zero we would have (by multiplication by $\bar{f_i}$ and application of E) $h_i \equiv O \ i=1, \ldots, k$.

Define for $g \in G$

$$g \cdot f \equiv \sum_{i=1}^{k} g(\tilde{f_i}) f_i h_i \quad \text{where} \quad \tilde{f_i} = f_i H_2.$$

This action is well defined and G acts as a group of algebraic isomorphisms of the algebra L onto itself. (The action also commutes with conjugation.) We need only show that G acts as a group of isometries i.e. $\|g \cdot f\| = \|f\|$. However this will follow if we can show that $\int |g\, f|^2 \, d\tilde{m} = \int |f|^2 \, d\tilde{m}$ for then we can conclude that

$$\|g\, f\| = \lim_{n \to \infty} \left(\int |g\, f|^{2n} \, dm\right)^{1/2n} = \lim_{n \to \infty} \left(\int |f|^{2n} \, d\tilde{m}\right)^{1/2n} = \|f\|.$$

However

$$\int |g\, f|^2 \, d\tilde{m} = \int \left(\sum_{i=1}^{k} g(\tilde{f_i}) f_i h_i \cdot \sum_{i=1}^{k} \overline{g(\tilde{f_i})} \bar{f_i} \bar{k_i} \right) d\tilde{m} = \int \sum_{i=1}^{k} |h_i|^2 \, d\tilde{m} = \int |f|^2 \, d\tilde{m}.$$

We have proved that G acts on X in such a way that

$$f(g\, x) = g(\tilde{f})\, f(x) \qquad \text{for } f \in H_1. \tag{4.2}$$

Hence $g\, x = x$ implies $g(\tilde{f}) = 1$ for all $f \in H_1$ i.e. $g = 1$. Clearly members of D are exactly the G invariant functions. From (4.2) it follows that for each $\gamma = \tilde{f} \in \hat{G}$ there exists f_γ (namely f) satisfying $f_\gamma(g\, x) = g(\tilde{f})\, f_\gamma(x) = \gamma(g)\, f_\gamma(x)$.

Notice if D consists only of constant functions the conditions of the theorem become: The linear span of H_1 is dense in $C(X)$ and there exists a normalised Borel measure m (positive on non-empty open sets) such that $\int f \, dm = O$ if $f \in H_1 - D$.

The conclusion is that X supports an abelian group structure (compatable with the topology of X) such that m is Haar measure and each member of H_1 is a constant times a character.

Theorem 8. *Let (X, S) be minimal and let $(X, S) \xrightarrow{\varphi} (Y, T)$ where (Y, T) is distal and let $D = \{f\, \varphi : f \in C(Y)\}$. In order that (X, S) should have discrete spectrum* mod D *it is necessary and sufficient that S should be a simple free G extension of T where $\varphi(g\, x) = \varphi(x)$ for all $g \in G$.*

Proof. Suppose S is a simple free G extension of T. We may suppose S induces T on X/G so that D is the subspace of $C(X)$ consisting of G invariant functions. Let $f_\gamma \in C(X, K)$, $\gamma \in G$ satisfy $f_\gamma(g\, x) = \gamma(g)\, f_\gamma(x)$, then the closed linear span of functions $f_\gamma \cdot f\, \varphi$ $(f \in C(Y, K))$ is $C(X)$ and

$$\frac{f_\gamma(S\, x)\, f(\varphi\, S\, x)}{f_\gamma(x)\, f\, \varphi(x)}$$

is G invariant i.e. $f_\gamma \cdot f\, \varphi \in H(D)$.

On the other hand if S has discrete spectrum mod D then there exists a linear operator E sending $C(X)$ to D satisfying the conditions of Lemma 2 and such that $E(fS)=E(f)S$ (cf. Proposition 5.5 of [2]). $H(D)$ plays the role of H_1 in Lemma 2 since if $f\in H(D)-D$ $f(S)=hf$ where $h\in D$ and $E(f)S=hE(f)$.

Hence $E(f)/f$ is S invariant and by the minimality of S, $E(f)$ is a constant multiple of $f\notin D$ i.e. $E(f)\equiv 0$. Hence $G=(\widehat{H_1/H_2})$ acts freely on X as described in Lemma 2 and for each $f\in H(D)$,

$$f(gx)=\gamma(g)f(x), \quad f(Sx)=hf. \tag{4.3}$$

Therefore

$$f(gSx)=\gamma(g)f(Sx)=\gamma(g)hf,$$

$$f(Sgx)=h(gx)f(gx)=h_\gamma(g)f$$

and G commutes with S. That S is a simple G extension of T follows from (4.3).

Evidently a minimal homeomorphism has discrete spectrum (i.e. mod C) if and only if it is a simple compact abelian extension of the trivial homeomorphism of a one point space or, in other words, if and only if it is a compact abelian group translation.

Let H_1 be the group of (mod C) eigenfunctions of S and let D_1 be the closed linear span of H_1. Suppose for some ordinal α, D_β a closed (conjugate closed) sub-algebra of $C(X)$ has been defined for each $\beta<\alpha$. If α is a limit ordinal define

$$D_\alpha=\overline{\bigcup_{\beta<\alpha} D_\beta}, \quad H_\alpha=\bigcup_{\beta<\alpha} H_\beta$$

(so that D_α is the closed linear span of H_α). If α is not a limit ordinal define H_α as the group of mod $D_{\alpha-1}$ eigenfunctions and D_α as the closed linear span of H_α. If for some ordinal α, $D_\alpha=C(X)$ we will say S has (generalised) discrete spectrum of order α (for the least such α). Members of H_β are called (generalised) eigenfunctions of order less than or equal to β. $H_o=K$ (the group of constant functions with absolute value one) are zero order eigenfunctions.

Let us note that a homeomorphism with generalised discrete spectrum is distal (I am grateful to J. Auslander who pointed out that this could be proved without reference to a canonical representation). For suppose $S^{n_i}x\to z\leftarrow S^{n_i}y$. We shall show that $x=y$ by showing that for each $\beta\leqq\alpha$, $f(S^n x)=f(S^n y)$ when $f\in H_\beta$. This is trivially true for $\beta=0$. Suppose it is true for $\beta<\beta_o$. If β_o is a limit ordinal then it is true for β_o since

$$H_{\beta_o}=\bigcup_{\beta<\beta_o} H_\beta.$$

If β_o is not a limit ordinal then it is true for β_0-1. Let $f\in H_{\beta_o}$, then $f(S)=hf$ where $h\in D_{\beta_o-1}$. Moreover since h can be approximated by linear combinations of elements of H_{β_0-1} we have $h(S^n x)=h(S^n y)$. Hence

$$f(S^{n_i}x)=h(S^{n_i-1}x), \dots, h(x)f(x)$$

$$f(S^{n_i}y)=h(S^{n_i-1}y), \dots, h(y)f(y)$$

and therefore $f(x)=f(y)$ and in the same way $f(S^n x)=f(S^n y)$. The proof is complete by transfinite induction.

Theorem 9.

Let (X, S) be a minimal dynamical system (X connected). Then (X, S) has (generalised) discrete spectrum of order α if and only if it can be represented as (Y, T) where Y is a compact abelian group and T is a homeomorphism of Y for which there is a sequence $\{G_\beta\}$ of closed sub-groups $(0 \leqq \beta \leqq \alpha)$ $Y=G_0 \supset G_{\beta_1} \supsetneqq G_{\beta_2} \supset G_\alpha=\{e\}$ if $\beta_1<\beta_2$ satisfying

(i) *The coset partition of G_β is T invariant for each β i.e. the sub-algebra C_β of $C(Y)$ consisting of functions invariant under G_β is T invariant. (Hence $T(g\,y)=\rho_\beta(g, y)\,Ty$ for $g \in G_\beta$ where $\rho_\beta \in C(G_\beta \times Y, G_\beta)$ and therefore induces a homeomorphism T_β on Y/G_β.)*

(ii) *If β is not a limit ordinal then*

$$G_{\beta-1}=\{g \in Y\colon (g\,G_\beta)\,T_\beta=T_\beta(g\,G_\beta)\}$$
$$=\left\{g \in Y\colon \frac{T(g\,y)}{g\,Ty} \in G_\beta\right\}.$$

(iii) *If β is a limit ordinal $G_\beta=\bigcap_{\beta_1<\beta} G_{\beta_1}$.*

In other words (Y, T) is the inverse limit of the systems $(Y/G_\beta, T_\beta)$ with respect to the natural maps $(Y/G_{\beta+1}) \to (Y/G_\beta)$, $T_{\beta+1} \to T_\beta$ where G_β is defined in (ii) *and* (iii).

Proof. If Y is a compact abelian group and T has the form described in the theorem it is easy to show that T has discrete spectrum of order α. The (generalised) eigenfunctions are easily derived from the characters.

We shall show that if (X, S) has discrete spectrum of order α (and X is connected) it can be given the form described in the theorem.

We may suppose that $D_\beta=\{h\,\varphi\colon h \in C(Y_\beta)\}$ where $\varphi\colon (X, S) \to (Y_\beta, T_\beta)$ in which case, since $H_{\beta+1}$ is the group of mod D_β eigenfunctions, by Theorem 8, $(Y_{\beta+1}, T_{\beta+1})$ is a simple free compact abelian group extension of (Y_β, T_β).

We shall show (cf. § 5, Lemma 3) that the assumption that Y_β is a compact abelian group leads to the conclusion that $Y_{\beta+1}$ is a compact abelian group. The theorem is completed by defining (Y, T) as the limit of the systems (Y_β, T_β).

Remark 5. If X is connected and (X, S) is minimal with generalised discrete spectrum of order α and if for each non-limit ordinal $\beta \leqq \alpha$ the group $D_{\beta-1} \cap C(X, K)$ is a direct factor of H_β then $Y/G \simeq Y/G_{\beta-1} \times G_{\beta-1}/G_\beta$ and T_β under the same natural isomorphism commutes with $G_{\beta-1}/G_\beta$ and induces $T_{\beta-1}$ on $Y/G_{\beta-1}$. Hence

$$T_\beta(y\,G_{\beta-1}, g\,G_\beta)=\left(T_{\beta-1}\,y\,G_{\beta-1}, \varphi_{\beta-1}(y\,G_{\beta-1})\,g\,G_\beta\right)$$

where $\varphi_{\beta-1} \in C(Y/G_{\beta-1}, G_{\beta-1}/G_\beta)$. Denoting the groups $G_{\beta-1}/G_\beta$ by G'_β, X may be represented as $\prod_{\beta \in [1,\alpha]'} G'_\beta$ and S may be represented as

$$S\,g'=S\{g'_\beta\}=\{\varphi_{\beta-1} \prod_{\beta-1} g' \cdot g'_\beta\} \qquad \text{where } \varphi_{\beta-1} \in C\Big(\prod_{[1,\beta-1]} G'_\gamma, G'_\beta\Big),$$

$\prod_{\beta-1}$ is the projection of $\prod_{1,\alpha]'} G'_\beta$ to $\prod_{[1,\beta-1]} G'_\gamma$ and these products are over non-limit ordinals. In particular if $\alpha=\omega$ (the first infinite ordinal) then S has the skew-product form $S(g'_1, g'_2, \ldots)=(a\,g'_1, \varphi_1(g_1)\,g'_2, \varphi_2(g'_1, g'_2)\,g'_3, \ldots)$.

§ 5. Lemma on Group Extensions

Lemma 3. *If X is a connected free simple G extension of Y where Y, G are compact abelian groups, then G is a closed sub-group of a compact abelian group X' homeomorphic to X in such a way that the diagram*

$$\begin{array}{ccc} X & \overset{\varphi}{\longleftrightarrow} & X' \\ {\scriptstyle g}\updownarrow & & \updownarrow{\scriptstyle g} \\ X & \underset{\varphi}{\longleftrightarrow} & X' \end{array}$$

is commutative i.e. the G spaces (X, G), (X', G) (natural action) are equivariant. Moreover the induced map sending X/G to X'/G is a group isomorphism.

Proof. We may suppose that $Y=X/G$. Let Γ be the character group of G and let Γ_n be the sub-group of Γ generated by $1, \gamma_1, \ldots, \gamma_n$ where $\Gamma=(1, \gamma_1, \gamma_2, \ldots)$. Let X_n be the space obtained by identifying points $g\,x$ such that $\gamma_i(g)=1$ $i=1, \ldots, n$. (In particular $X_0=Y=X/G$, $X_\infty=X$.) $X_n=X/G_n$ where $G_n=a\,n\,n\,\Gamma_n$. Clearly G/G_n acts freely on X/G_n ($g\,G_n$: $G_n\,x \to G_n\,g\,x$) and X/G_n is a simple G/G_n extension of Y/G. In fact X/G_{n+1} is acted on freely by G_n/G_{n+1} ($g\,G_{n+1}$: $G_{n+1}\,x \to G_{n+1}\,g\,x$) and is a simple G_n/G_{n+1} extension of the orbit space X/G_n.

The character group of G_n/G_{n+1} is Γ_{n+1}/Γ_n where $\gamma_{n+1}\,\Gamma_n(g_n\,G_{n+1})=\gamma_{n+1}(g_n)$ i.e. the character group of G_n/G_{n+1} is one generated (cyclic) unless $G_n=G_{n+1}$, $\Gamma_{n+1}=\Gamma_n$ (i.e. $\gamma_{n+1}\in\Gamma_n$). Assuming X/G_n is a compact abelian group (which is true for $n=0$) we will show that X/G_{n+1} can be given a compact abelian group structure in such a way that G_n/G_{n+1} is a closed sub-group. By induction and the taking of limits the theorem will follow. We have therefore reduced the theorem to the special case where the character group of G is cyclic i.e. G is a circle or G is a finite cyclic group.

(i) Assume G is a circle. Let m be normalised Haar measure on X/G, and let dg be normalised Haar measure on G, and let $\tilde{m}$ be the measure on X defined by

$$\int f\,d\tilde{m}=\int\left(\int f(g\,x)\,dg\right)dm.$$

Let γ be a generator of $\hat{G}$ and let $f_\gamma(g\,x)=\gamma(g)\,f_\gamma(x)$, $f_\gamma\in C(X,K)$. Let $H=\{f_\gamma^n\,h\,\pi\colon h\in\widehat{X/G}\}$ where π is the natural map of X to X/G. H is a sub-group of $C(X,K)$ whose linear span is dense in $C(X)$. Moreover $\int f_\gamma^n\,h\,\pi\,d\tilde{m}\neq O$ only if $h\equiv 1$, $m=O$ since

$$\int f_\gamma^n(g\,x)\,h\,\pi\,dg=h\,\pi\int\gamma^n(g)\,f(x)\,dg=O \qquad \text{if } n\neq 0 \text{ and if } n=O,\ h\not\equiv 1$$
$$\int\left(\int f_\gamma^0(g\,x)\,h\,\pi\,dg\right)dm=\int h\,dm=O.$$

The conditions of Lemma 2 (special case) apply and case (i) is concluded.

(ii) Suppose X is a free simple G extension of Y where G is a finite cyclic group of order k i.e. suppose X is a k-fold connected covering of the compact connected abelian group $Y (X \xrightarrow{\pi} Y)$.

$\pi^{-1}(y)$ consists of k distinct points. Let δ be a translation invariant metric on Y and let d be a G invariant metric on X. Let

$$m(x) = \inf_{g \neq 1} d(g\,x, x) = m(h\,x) \qquad \text{for all } h \in G.$$

$m(x)$ can be considered as a function on Y and is continuous, hence $m(x) \geqq m$ for some $m > 0$. For each x let

$$U(x) = \left\{ x' : d(x, x') < \frac{m}{2} \right\}.$$

Then $g\,U(x) = U(g\,x)$ and the open balls $U(g\,x)$ are disjoint for fixed x and varying $g \in G$. Sets $\pi\,U(x) = V(\pi\,x)$ form a covering of Y and it is easy to see that for large enough N each set $\pi^{-1}\,y\,Y_N$ will be contained in some

$$\bigcup_{g \in G} U(g\,x) \equiv W(\pi\,x)$$

where Y_n is a decreasing sequence of closed sub-groups $(\bigcap Y_n = e)$ such that Y/Y_n is a torus (cf. [14]). For each coset $y = \tilde{y}\,Y_N$ let $W(\tilde{y})$ be a choice from the finite sub-covering $W_1, \ldots, W_k$ of $\{W(\pi\,x)\}$ such that $\pi^{-1}\,Y_N \subset W(\tilde{y})$. Clearly $\pi^{-1}\,y\,Y_N$ decomposes into closed sets

$$\{\pi^{-1}\,y\,Y_N \cap U(g\,x) : g \in G\} \quad \text{where} \quad W(\tilde{y}) = \bigcup_{g \in G} U(g\,x).$$

We write $u \sim v$ if u, v belongs to one of these sets. $\big(W(\tilde{y})$ consists of k components and we say $u \sim v$ if they belong to the same component and project into the same coset of Y_N.$\big)$ $\sim$ is an equivalence relation and $X' = X/\sim$ is a k-fold covering (in fact a simple G extension) of $Y' = Y/Y_N$. In other words the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ Y & \xrightarrow[\varphi']{} & Y' \end{array}$$

commutes where φ is the projection of X to X' defined by the equivalence relation $\sim$, φ' is the natural map of Y to $Y' = Y/Y_N$ and π' is the projection of X' onto its orbit space Y'. Moreover φ commutes with the action of G. Suppose we have established (ii) for X', Y' in place of X, Y so that X' is a torus and G is a finite cyclic sub-group. The proof would be completed as follows: take γ to be a character on X' extending a non-trivial character of order k on G, so that $\gamma(g\,x') = \gamma(g)\,\gamma(x')$ and define $f_\gamma \in C(X, K)$ as $f_\gamma(x) = \gamma(\varphi\,x)$ so that $f_\gamma(g\,x) = \gamma(\varphi\,g\,x) = \gamma(g)\,\gamma\,\varphi\,x = \gamma(g)\,f_\gamma(x)$. Note that $f_\gamma^k(g\,x) = f_\gamma^k(x) = \gamma^k(\varphi\,x)$ and since γ^k and f_γ^k are G invariant we have $\gamma^k(x') = \gamma'(\pi'\,x')$, $f^k(x) = f'_\gamma(\pi\,x)$ where γ' is a character on Y' and $f' \in C(Y, K)$ and $f'(\pi\,x) = \gamma'(\pi'\,\varphi\,x) = \gamma'(\varphi'\,\pi\,x)$. Since φ' is a group homomorphism and γ' is a character it follows that f' is a character on Y. We can conclude therefore that

$\{f_y^i h\pi\colon i=0, 1, \ldots, k-1,\ h\in \hat{Y}\}$ is a group all of whose elements are mutually orthogonal and whose linear span is dense in $C(X)$. Case (ii) is now easily concluded by applying Lemma 2 again.

It remains only to deal with case (ii) when G is finite cyclic of order k and Y is a torus. X is a finite covering of Y. Let Z be the universal covering group of $Y(Z \xrightarrow{\varphi} Y$, a covering homomorphism) so that [15] if $z\in\varphi^{-1}y$, $x\in\pi^{-1}y$ there exists a unique map $\psi=\psi_{z,x}$ such that

$$\begin{array}{ccc} Z & \xrightarrow{\psi} & X \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\pi} \\ & & Y \end{array}$$

commutes and $\psi(z)=x$. Choose $e'\in\pi^{-1}e$ and let ψ be the map which sends the identity of Z to e'. Note that $\psi(az)=\psi(bz)$ if $\psi(a)=\psi(b)$. Indeed it is sufficient to note that $\psi(ab^{-1}z)=\psi(z)$ since both functions map b to $\psi(a)=\psi(b)$ and $\pi\psi(z)=\varphi(z)$ and

$$\pi\psi(ab^{-1}z)=\varphi(ab^{-1}z)=\varphi(a)\,\varphi(b)^{-1}\varphi(z)=\pi\psi(a)\big(\pi\psi(b)\big)^{-1}\varphi(z)=\varphi(z).$$

Hence if $\psi z_i=x_i$ and $\psi z_i'=x_i$ $(i=1,2)$ then $\psi(z_i z_2)=\psi(z_1' z_2')$ and the definition $x_1\times x_2=\psi(z_1 z_2)$ is unambiguous. It is easy to show that this multiplication makes X into a compact abelian group such that π is a homomorphism onto Y. From this it follows that G is a sub-group of X. Case (ii) is concluded and the proof of the lemma is complete.

§ 6. Measure Theoretic Analogues

Let (X,B,m) be a normalised measure space and let S be a one-one measure preserving transformation of the space onto itself. Most of the considerations of the previous sections have analogues for purely measure theoretic dynamical systems (X,S).

The proofs of these analogues will not be given. Suffice it to say that ergodicity plays the role of minimality, $L^2(X)$ plays the role of $C(X)$ and $M(X,K)$ the group of measurable maps of X to K plays the role of $C(X,K)$. Where connectedness was essential in §§ 1, 2 ergodicity should be strengthened and replaced by total ergodicity (S^n ergodic for $n\neq 0$). Moreover theorems can be given a simpler form due to the fact that a free G extension of Y can be represented as $Y\times G$. Consequently the canonical form of ergodic system with generalised discrete spectrum is a skew-product on a "transfinite" direct product of groups. We shall not stop to give the precise form of these results.

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