Z. Wahrscheinlichkeitstheorie verw. Geb. 8, 113–116 (1967)

A Dominated Ergodic Type Theorem

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Received December 31, 1966

In the special case of independent, identically distributed (i.i.d.) random variables $\{X_n, n \geqq 1\}$, WIENER's dominated ergodic theorem [5] asserts that for $r > 1$,

(1) $$E|X_n|^r < \infty$$

implies

(2) $$E \sup_{n \geqq 1} n^{-r} \left| \sum_{i=1}^{n} X_i \right|^r < \infty$$

while

(3) $$E|X_n|^r \log^+ |X_n| < \infty$$

implies (2) for $r = 1$. Condition (1) is clearly necessary when $r > 1$ and, as has been demonstrated by BURKHOLDER [1], (3) is likewise necessary for (2) when $r = 1$.

Here, it will be shown for i.i.d. $\{X_n, n \geqq 1\}$ with $EX_n = 0$ that for $r > 2$, the same hypothesis (1) implies $E \sup_{n \geqq 1} c_n |S_n|^r < \infty$ where $S_n = \sum_{i=1}^{n} X_i$ and (for example) $c_n = n^{-r/2} (\log n)^{(-r/2k)-\delta}$, $n > 1$, with $\delta > 0$ and k = greatest integer $\leqq r$. The preceding statement holds for $r = 2$ if (1) is strengthened to (3) but is false for $1 \leqq r < 2$ even under (3).

Such results have implications for stopping rules, namely that for $r \geqq 2$ under the stipulated conditions, $E c_t |S_t|^r < \infty$ for all stopping rules t.

The proof of the theorem below, from which the assertions for $r > 2$ and $r = 2$ follow directly, rests upon the classical result of WIENER cited above.

Theorem. *Let $\{X_n, n \geqq 1\}$ be i.i.d. with $EX_n = 0$ and either $E|X_n|^r < \infty$ or $E|X_n|^r \log^+ |X_n| < \infty$ according as $r > 2$ or $r = 2$. If $\{c_n, n \geqq 1\}$ is a positive, decreasing numerical sequence with $c_n = O(n^{-r/2})$, $\sum_{n=1}^{\infty} n^{k-1} c_n^{2k/r} < \infty$ where k = greatest integer $\leqq r$, then $E \sup_{n \geqq 1} c_n |S_n|^r < \infty$.*

The contra-positive statement for $1 \leqq r < 2$ is substantiated by choosing $\{X_n, n \geqq 1\}$ to be i.i.d. with common symmetric stable distribution of characteristic exponent β, $1 \leqq r < \beta < 2$; then (3) holds but since $E|S_n| = C n^{1/\beta}$ for some C in $(0, \infty)$,

$$E^{1/r} \sup_{n \geqq 1} c_n |S_n|^r \geqq E \sup_{n \geqq 1} c_n^{1/r} |S_n| \geqq C \sup_{n \geqq 1} c_n^{1/r} n^{1/\beta} = \infty$$

for c_n as chosen in paragraph 2.

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* Research under NSF Contract GP-4590.

The proof of the theorem will be facilitated by noting the following lemmas.

Lemma 1. *Let $\{X_n, n \geqq 1\}$ be random variables on the probability space (Ω, A, P) with $E|X_n|^k < \infty$, $n \geqq 1$ for some positive integer k; set $\mathscr{F}_0 = (\varphi, \Omega)$ and let $\mathscr{F}_n$ denote the σ-algebra generated by $X_1, X_2, \ldots, X_n$. If $E\{X_{n+1} | \mathscr{F}_n\} = 0$, $n \geqq 0$ and $U_{k,n} = \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} X_{i_1} X_{i_2} \cdots X_{i_k}$ for $n \geqq k$, then $\{U_{k,n}, \mathscr{F}_n, n \geqq k\}$ is a martingale. Moreover, if $E\{X_{n+1}^2 | \mathscr{F}_n\} = \sigma^2 = \text{const} < \infty$, then $E U_{k,n}^2 = \binom{n}{k} \sigma^{2k}$.*

Proof. The lemma is commonplace for $k = 1$ and otherwise follows readily from $E|U_{k,n}| < \infty$ and the observation $U_{k,n} - U_{k,n-1} = X_n U_{k-1,n-1}$.

The next lemma was first proved by HÁJEK and RÉNYI [4] for the special case where U_n is a sum of n independent random variables with zero means. However, it is required under the more general circumstances that $U_n = U_{k,n}$. Rather than appeal to the submartingale inequality of CHOW [2] from which it follows easily, we give a simple, direct proof.

Lemma 2. *If $\{U_n, \mathscr{F}_n, n \geqq 1\}$ is a martingale with $E U_n^2 < \infty$, $n \geqq 1$ and $\{c_n, n \geqq 1\}$ is a positive, decreasing sequence,*

$$P\left\{\max_{1 \leqq j \leqq n} c_j |U_j| \geqq \lambda\right\} \leqq \lambda^{-2} \sum_{j=1}^{n} c_j^2 E(U_j - U_{j-1})^2, \quad \lambda > 0. \tag{4}$$

Proof. Taking $U_0 = 0$ and $\mathscr{F}_0$ to be the trivial σ-algebra, it is readily checked that $W_n = c_n^2 U_n^2 - \sum_{j=1}^{n} c_j^2 E\{(U_j - U_{j-1})^2 | \mathscr{F}_{j-1}\}$, $n \geqq 1$ is a supermartingale with $E W_1 = 0$. Thus, for any stopping time t, setting $t(n) = \min(t, n)$, it follows from a theorem of DOOB [3, p. 302] that

$$E c_{t(n)}^2 U_{t(n)}^2 \leqq E \sum_{1}^{t(n)} c_j^2 E\{(U_j - U_{j-1})^2 | \mathscr{F}_{j-1}\}. \tag{5}$$

Choose $t =$ first index $i \geqq 1$ for which $c_i^2 U_i^2 \geqq \lambda^2$ ($= \infty$, otherwise). The right side of (5) is clearly bounded by the right side of (4) multiplied by λ^2 while the left side of of (5) is at least

$$\int_{[t \leqq n]} c_t^2 U_t^2 \geqq \lambda^2 P\{t \leqq n\}$$

and the lemma follows.

Lemma 3. *If $\{X_n\}$, $\{U_{k,n}\}$ are as in lemma 1, $0 \leqq \alpha < k$ and $\{c_n, n \geqq 1\}$ is a positive, decreasing sequence with*

$$\sum_{n=1}^{\infty} c_n^{2k/(k+\alpha)} n^{k-1} < \infty, \quad \textit{then} \quad E \sup_{n \geqq k} c_n |U_{k,n}|^{1+(\alpha/k)} < \infty.$$

Proof. From lemmas 1 and 2,

$$P\left\{\max_{k \leqq j \leqq n} c_j |U_{k,j}|^{1+(\alpha/k)} \geqq \lambda\right\} = P\left\{\max_{k \leqq j \leqq n} c_j^{k/(k+\alpha)} |U_{k,j}| \geqq \lambda^{k/(k+\alpha)}\right\} \leqq$$

$$\leqq \lambda^{-2k/(k+\alpha)} \sigma^{2k} \sum_{j=k}^{n} c_j^{2k/(k+\alpha)} j^{k-1}.$$

Lemma 3 now follows by integration on λ and monotone convergence. (The conclusion of the lemma also holds when $E\{X_{n+1}^2 | \mathscr{F}_n\} \leqq \sigma^2 = \text{const} < \infty$ but will only be utilized for i.i.d. $\{X_n\}$ with $EX_n^2 = 0$, $EX_n = \sigma^2$.)

Proof of theorem. Consider first the case $2 \leqq r < 3$.

$$E^{2/r} \sup_{n \geqq 2} c_n |S_n|^r = E^{2/r} \sup_{n \geqq 2} c_n \left| \sum_1^n X_i^2 + 2U_{2,n} \right|^{r/2} \leqq$$
$$\leqq E^{2/r} \sup_{n \geqq 2} c_n \left(\sum_1^n X_i^2 \right)^{r/2} + 2E^{2/r} \sup_{n \geqq 2} c_n |U_{2,n}|^{r/2}.$$

The second term on the right is finite by hypothesis and lemma 3. Likewise by hypothesis $c_n \leqq A n^{-r/2}$ for some A in $(0, \infty)$ and so the first term on the right is dominated by $E^{2/r} A \sup_{n \geqq 2} n^{-r/2} \left(\sum_i^n X_i^2 \right)^{r/2}$ which is finite by Wiener's theorem.

Suppose next that $r = k \geqq 3$. Evidently

$$E \sup_{n \geqq k} c_n |S_n|^k \leqq E \sup_{n \geqq k} c_n |S_n^k - k! U_{k,n}| + k! E \sup_{n \geqq k} c_n |U_{k,n}| \tag{6}$$

and by hypothesis and lemma 3, the second term on the right is finite.

According to the multinomial expansion, $S_n^k - k! U_{k,n}$ is expressible as a finite linear combination with coefficients depending on k but not on n, of terms

$$\sum_{i_1, \ldots, i_k \text{ pairwise different}} X_{i_1}^{r_1} X_{i_2}^{r_2} \cdots X_{i_k}^{r_k} \quad \text{where} \quad r_j \geqq 0, \ \sum_{j=1}^k r_j = k, \ r_j \neq 1 .$$

Each of the latter, in turn, is expressible as a finite linear combination, with coefficients again independent of n, of terms

$$\prod_{i=1}^m \left(\sum_{j=1}^n X_j^{h_i} \right)$$

where $1 \leqq m < k$, $1 \leqq h_i \leqq k$, $\sum_{i=1}^m h_i = k$. Thus, $\sup c_n |S_n^k - k! U_{k,n}|$ is bounded by a similar finite linear combination of terms

$$\prod_{i=1}^m \sup c_n^{h_i/k} \left| \sum_{j=1}^n X_j^{h_i} \right|.$$

For $m = 1$, necessarily $h_1 = k$ and so by hypothesis

$$E \sup_n c_n \left| \sum_{j=1}^n X_j^k \right| \leqq A E \sup_n n^{-k/2} \sum_{j=1}^n |X_j|^k \leqq$$
$$\leqq A E \sum_{j=1}^\infty j^{-k/2} |X_j|^k < \infty .$$

If $m \geqq 2$,

$$E \prod_{i=1}^m \sup c_n^{h_i/k} \left| \sum_{j=1}^n X_j^{h_i} \right| \leqq \prod_{i=1}^m E^{h_i/k} \sup c_n \left| \sum_{j=1}^n X_j^{h_i} \right|^{k/h_i} . \tag{7}$$

When $2 \leqq h_i < k$, by hypothesis and Wiener's theorem

$$E \sup c_n \left| \sum_{j=1}^n X_j^{h_i} \right|^{k/h_i} \leqq A E \sup n^{-k/h_i} \left(\sum_{j=1}^n |X_j|^{h_i} \right)^{k/h_i} < \infty . \tag{8}$$

If in (7) exactly s of the h_i are unity (necessarily $0 \leqq s \leqq k-2$), say $h_{m-s+1} = \cdots = h_m = 1$, the bound on the right side of (7) is replaceable by

$$E^{s/k} \sup c_n \left| \sum_{j=1}^{n} X_j \right|^k \prod_{i=1}^{m-s} E^{h_i/k} \sup c_n \left(\sum_{j=1}^{n} |X_j|^{h_i} \right)^{k/h_i} \tag{9}$$

and according to (8), each term under the product sign $\prod_1^{m-s}$ in (9) is finite.

Thus, recalling (6), it follows that for some finite constants $A_0, A_1, \ldots, A_{k-2}$

$$E \sup_{n \geqq k} c_n |S_n|^k \leqq A_0 + \sum_{s=1}^{k-2} A_s E^{s/k} \sup_{n \geqq k} c_n |S_n|^k \tag{10}$$

from which the theorem (for $r = k \geqq 3$) is a simple consequence.

Finally, if $k < r < k+1$, $(k > 2)$, setting $\alpha = r - k$

$$E \sup_{n \geqq k} c_n |S_n|^r \leqq E \sup_{n \geqq k} c_n |S_n|^\alpha |S_n^k - k! \, U_{k,n}| + k! \, E \sup_{n \geqq k} c_n |S_n|^\alpha |U_{k,n}| \,. \tag{11}$$

Clearly,

$$E \sup c_n |S_n|^\alpha |U_{k,n}| \leqq E^{\alpha/r} \sup c_n |S_n|^r E^{k/r} \sup c_n |U_{k,n}|^{1+(\alpha/k)} \tag{12}$$

with the last term of the product in (12) finite by lemma 3. Since previous representations still apply to $S_n^k - k! \, U_{k,n}$, it suffices from (11) to consider such terms as

$$E \sup c_n^{\alpha/r} |S_n|^\alpha \prod_{i=1}^{m} \sup c_n^{h_i/r} \left| \sum_{j=1}^{n} X_j^{h_i} \right|$$

and an argument similar to that in the integral case together with (11) establishes the analogue of (10) and consequently the theorem.

Finally, it may be remarked that the condition $c_n = O(n^{-r/2})$ is necessary, and that it would be of interest to have minimal conditions on $\{X_n\}$ under which comparable results would obtain for $c_n = (n \log \log n)^{-r/2}$. Clearly, such a choice of c_n is the best that might be hoped for.

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