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A Stochastic Integral in Storage Theory

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1. Introduction

Throughout the following $\mathbf{R}=[0,+\infty)$, $\overline{\mathbf{R}}=[0,+\infty]$, $\mathbf{N}=\{0,1,2,\ldots\}$, $\mathscr{R}$ and $\overline{\mathscr{R}}$ the Borel subsets of $\mathbf{R}$ and $\overline{\mathbf{R}}$ respectively, and $\mathscr{R}^*$ the universally measurable subsets of $\overline{\mathbf{R}}$. If $(E,\mathscr{E})$ and $(F,\mathscr{F})$ are measurable spaces and $f\colon E\to F$ is measurable relative to $\mathscr{E}$ and $\mathscr{F}$, then we write $f\in\mathscr{E}/\mathscr{F}$; if $(F,\mathscr{F})=(\overline{R},\overline{\mathscr{R}})$ then we simply write $f\in\mathscr{E}$. By $\sigma(\cdot)$ we denote the σ-algebra generated by $(\cdot)$. If N is a transition probability from $(E,\mathscr{E})$ into $(F,\mathscr{F})$, that is, $N(x,\cdot)$ is a probability on $\mathscr{F}$ and $N(\cdot,A)$ is in $\mathscr{E}$, and $f\in\mathscr{F}$, then we write Nf for the function defined as

$$Nf(x)=\int N(x,dy)\,f(y),\qquad x\in E.$$

Our notation and terminology follow that of [1].

Let $(W,\mathscr{I},P)$ be a probability space and let $\{A_t\colon t\geqq 0\}$ be a process on W with stationary independent non-negative increments taking values in $(\mathbf{R},\mathscr{R})$ with $P\{A_0=0\}=1$. It is possible to, and we do, take $t\to A_t(w)$ to be right-continuous non-decreasing for each $w\in W$. Let $\mathscr{I}_t=\sigma(A_s\colon s\leqq t)$. Throughout the first four sections we assume the jump rate to be finite.

Let $\Omega=\overline{\mathbf{R}}\times W$ and, for each probability measure μ on $\overline{\mathscr{R}}$, define P^μ to be the product measure $\mu\times P$ on $\overline{\mathscr{R}}\times\mathscr{I}$. In particular, when μ is the Dirac measure ε_x concentrated at $x\in\overline{\mathbf{R}}$, we write P^x for P^μ. Let $\mathbf{U}=\{P^\mu\colon \mu$ a probability on $\overline{\mathscr{R}}\}$, $\mathscr{F}$ the completion of $\overline{\mathscr{R}}\times\mathscr{I}$ with respect to $\mathbf{U}$, and $\mathscr{F}_t$ the completion of $\overline{\mathscr{R}}\times\mathscr{I}_t$ in $\mathscr{F}$ with respect to $\mathbf{U}$.

For $\omega=(x,w)\in\Omega$, we shall write $A_t(\omega)=A_t(w)$ (there should be no confusion because of this) for all $t\geqq 0$, and put $X_0(\omega)=x$.

Let r be a Lipschitz continuous strictly increasing function from $\overline{\mathbf{R}}$ into $\overline{\mathbf{R}}$ so that $r(0)=0$, and consider the equation

$$X_t(\omega)=X_0(\omega)+A_t(\omega)-\int_0^t r\big(X_u(\omega)\big)\,du,\qquad t\geqq 0 \tag{1.1}$$

for each $\omega\in\Omega$; (note that if $\omega=(+\infty,w)$, then $X_t(\omega)=+\infty$ for all $t\geqq 0$). The reader will observe that certain of our results below go through under less stringent assumptions on r.

Ω is the sample space, $\mathscr{F}_t$ the history until t. For the realization $\omega\in\Omega$, $X_0(\omega)$ is the initial content of the dam, $A_t(\omega)$ the total input during $[0,t]$, $X_t(\omega)$ the content at time t. The above equation states that the rate of output at time u is

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$r(X_u(\omega))$, and

$$Z_t(\omega)=\int_0^t r(X_u(\omega))\,du \tag{1.2}$$

is the total output during $[0,t]$. $A=(\Omega,\mathscr{F},\mathscr{F}_t,A_t,P^x)$ will be called the input process, $X=(\Omega,\mathscr{F},\mathscr{F}_t,X_t,P^x)$ the content process, $Z=(\Omega,\mathscr{F},\mathscr{F}_t,Z_t,P^x)$ the output process.

In the next section we shall show that the Eq. (1.1) has a unique solution for each ω and the resulting process $X=(\Omega,\mathscr{F},\mathscr{F}_t,X_t,P^x)$ is a standard Markov process (a strong Markov process on a locally compact space with a countable base with $\mathscr{F}_t=\mathscr{F}_{t+}=\bar{\mathscr{F}}_t$ and which is right-continuous and quasi-left-continuous on $[0,+\infty)$).

The process $\{Z_t\}$ is a continuous additive functional of X and thus is completely characterized by its potential which we shall compute in Section 3. If we define

$$\tau_t(\omega)=\inf\{s: Z_s(\omega)>t\} \tag{1.3}$$

(then τ_t is the time the cumulative output reaches t) then $\tilde{X}=(\Omega,\mathscr{F},\mathscr{F}_{\tau_t},X_{\tau_t},P^x)$ is also a standard Markov process. We shall compute its resolvent and thus specify its transition function. This also gives the distribution of $\tilde{A}_t=A_{\tau_t}$ since $X_{\tau_t}=X_0+A_{\tau_t}-t$.

In Section 4 we consider the limiting distribution of the content process. We show that a limiting distribution for $\{X_t\}$ exists (and compute it) if

$$\sup_x r(x)>m$$

where m is the expected rate of input. If $\sup_x r(x)<m$, then $X_t\to+\infty$ as $t\to+\infty$.

In the special case where r is of form $r(x)=cx$ the equation can be solved explicitly in terms of A_t and, then, we are able to give a necessary and sufficient condition for the existence of the limiting distribution. This is put in Section 5.

The dam model considered was proposed by Moran [5]. We employ the methodology of [2] and [3] by considering the process X_t at its points of jumps along with the jump times (the resulting discrete time process is called a Markov renewal process). Section 3 is based on Chapters V and VI of Blumenthal and Getoor [1]. Section 4 ties the results of Çinlar [2], Foguel [4], Orey [6], Pyke and Schaufele [7] to show the existence of a limiting distribution for the imbedded Markov renewal process from which the limiting distribution of X_t is computed directly.

2. Content Process

From the general theory of processes with stationary independent increments we can write

$$A_t(\omega)=a\,t+\tilde{A}_t(\omega), \qquad t\geqq 0 \tag{2.1}$$

where $a\geqq 0$ is a constant and $\tilde{A}_t(\omega)$ is a right-continuous step function. We will denote the jump times by $\tau_1(\omega), \tau_2(\omega), \ldots$, set $\tau_0(\omega)=0$, and let $\alpha_1(\omega), \alpha_2(\omega), \ldots$ be the magnitudes of the corresponding jumps.

It follows from (1.1) and (2.1) that $X_t(\omega)$ satisfies the differential equation

(2.2) $$dx_t = a\,dt - r(x_t)\,dt$$

for $t \in (\tau_n(\omega), \tau_{n+1}(\omega))$ for any $n \in \mathbf{N}$. The following puts together all the relevant facts we need about (2.2). We omit the proof.

(2.3) **Lemma.** *The Eq.* (2.2) *with initial condition* $x_0 = x$ *has a unique solution* $q(x, t)$.

a) *For fixed* $x \in \mathbf{R}$, $t \to q(x,t)$ *is monotone continuous.*

b) *For fixed* $t \in \mathbf{R}$, $x \to q(x,t)$ *is non-decreasing continuous.*

c) *The mapping* $(x,t) \to q(x,t)$ *is in* $\mathscr{R}^2$.

d) $$-\frac{\partial}{\partial t} q(x,t) = [r(x) - a] \frac{\partial}{\partial x} q(x,t). \quad \square$$

(2.4) **Theorem.** *Define, for each* $\omega \in \Omega$,

$$\xi_0(\omega) = X_0(\omega),$$
$$\xi_{n+1}(\omega) = q(\xi_n(\omega), \tau_{n+1}(\omega) - \tau_n(\omega)) + \alpha_{n+1}(\omega), \qquad n \in \mathbf{N}$$

recursively. Then, we have

$$X_t(\omega) = q(\xi_n(\omega), t - \tau_n(\omega)) \qquad \text{if } \tau_n(\omega) \leqq t < \tau_{n+1}(\omega).$$

Proof. It follows from (2.1) that $X_t(\omega)$ satisfies (2.2) in any interval $(\tau_n(\omega), \tau_{n+1}(\omega))$, and at point $\tau_n(\omega)$, $X_t(\omega)$ jumps by $\alpha_n(\omega)$. Thus, if $\xi_n = X_{\tau_n}$, then X_t is as given above for $t \in [\tau_n, \tau_{n+1})$, and $\xi_{n+1} = X_{\tau_{n+1}-0} + \alpha_{n+1}$. $\square$

(2.5) **Theorem.** $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, P^x)$ *is a normal standard Markov process (and since the termination time is* $+\infty$ *a.s., trivially, a Hunt process).*

Proof. a) From the definition of X_0 and the measures P^x, $P^x\{X_0 = x\} = 1$ for any $x \in \mathbf{R}$. Thus we have normality trivially.

b) From the construction, $t \to X_t(\omega)$ is right-continuous and has left-hand limits by Lemma (2.3).

c) *Regularity Conditions.* Both X_t and A_t have the same jump times and the same magnitudes of jumps. Between the jumps, both are deterministic. Thus,

(2.6) $$\mathscr{F}_t^0 \equiv \sigma(X_s\colon s \leqq t) = \sigma(X_0, A_s\colon s \leqq t) = \bar{\mathscr{R}} \times \mathscr{I}_t,$$

and therefore, $\mathscr{F}_t$ is also the completion of $\mathscr{F}_t^0$ in $\mathscr{F}$ relative to $\{P^\mu\colon \mu$ a probability on $\bar{\mathscr{R}}\}$. Thus,

(2.7) $$X_t \in \mathscr{F}_t/\bar{\mathscr{R}}.$$

By right-continuity this implies that X is progressively measurable with respect to $\{\mathscr{F}_t\}$. Further, if T is any $\{\mathscr{F}_t\}$ stopping time,

(2.8) $$X_T \in \mathscr{F}_T/\mathscr{R}^*,$$

since the $\mathscr{F}_t$ are complete with respect to $\mathbf{U} = \{P^\mu\colon \mu$ a probability on $\bar{\mathscr{R}}\}$ and since the completion of $\mathscr{F}_T$ in $\mathscr{F}$ with respect to $\mathbf{U}$ is the same as $\mathscr{F}_T$.

Finally, from the construction of P^x, for $A \in \bar{\mathscr{R}}$ and $B \in \mathscr{I}$, $P^x(A \times B) = \varepsilon_x(A) P(B)$ so that $x \to P^x(A \times B)$ is in $\bar{\mathscr{R}}$. By the monotone class theorem then

$$x \to P^x(\Lambda) \quad \text{is in } \bar{\mathscr{R}} \tag{2.9}$$

for any $\Lambda \in \bar{\mathscr{R}} \times \mathscr{I}$.

d) *Strong Markov Property*. Since $(W, \mathscr{I}, \mathscr{I}_t, A_t, P)$ is a process with stationary independent increments, $(W, \mathscr{I}, \mathscr{I}_{t+}, A_t, P)$ is a strong Markov process, and by (2.6), for any $\{\mathscr{F}_{t+}^0\}$ stopping time T,

$$P^x\{A_{T+t} - A_T \in B \mid \mathscr{F}_{T+}^0\} = P^x\{A_t \in B\} \tag{2.10}$$

for all $t \geqq 0$, $B \in \mathscr{R}$ independent of x.

Let T be an $\{\mathscr{F}_{t+}^0\}$ stopping time and define

$$A_t^+ = A_{T+t} - A_T, \qquad X_t^+ = X_{T+t}, \qquad t \geqq 0. \tag{2.11}$$

Then, it follows from (1.1) that we have

$$X_t^+ = X_0^+ + A_t^+ - \int_0^t r(X_u^+)\, du, \qquad t \geqq 0 \tag{2.12}$$

which is the same as (1.1). Since, by (2.10), $\sigma(A_t^+; t \geqq 0)$ is independent of $\mathscr{F}_{T+}^0$, this implies

$$P^x\{X_t^+ \in B \mid \mathscr{F}_{T+}^0\} = P^y\{X_t \in B\} \quad \text{on } \{X_0^+ = y\};$$

that is,

$$P^x\{X_{T+t} \in B \mid \mathscr{F}_{T+}^0\} = P^{X(T)}\{X_t \in B\} \tag{2.13}$$

for all $x \in \mathbf{R}$, $t \geqq 0$, $B \in \mathscr{R}$.

This proves that $(\Omega, \mathscr{F}, \mathscr{F}_{t+}^0, X_t, P^x\}$ is a strong Markov process. By Proposition (8.12) in Chapter I of [1], then

$$\mathscr{F}_t = \mathscr{F}_{t+}, \qquad t \geqq 0. \tag{2.14}$$

Thus, by Theorem (7.3) in Chapter I of [1], if T is any $\{\mathscr{F}_t\}$ stopping time, for each μ there exists a $\{\mathscr{F}_{t+}^0\}$ stopping time T_μ such that $P^\mu\{T \neq T_\mu\} = 0$. Hence, (2.13) implies

$$P^x\{X_{T+t} \in B \mid \mathscr{F}_T\} = P^{X(T)}\{X_t \in B\} \tag{2.15}$$

for all $t \geqq 0$, $x \in \mathbf{R}$, $B \in \mathscr{R}$, and $\{\mathscr{F}_t\}$ stopping times T.

e) If $\bar{\mathscr{F}}_t$ is the completion of $\mathscr{F}_t$ in $\mathscr{F}$ with respect to $\{P^\mu\}$, then obviously $\bar{\mathscr{F}}_t = \mathscr{F}_t$ and together with (2.14) we have

$$\mathscr{F}_t = \mathscr{F}_{t+} = \bar{\mathscr{F}}_t. \tag{2.16}$$

f) *Quasi-Left-Continuity*. Let $\{T_n\}$ be an increasing sequence of $\{\mathscr{F}_t\}$ stopping times with limit T. By the continuity of X_t on intervals (τ_n, τ_{n+1}), $X_{T_n} \to X_T$ everywhere on Ω except on $\Lambda = \bigcup_{k=1}^{\infty} \{T = \tau_k\}$. Thus we need to show that $P^x(\Lambda) = 0$ for all x. Supposing otherwise, if $P^x(\Lambda) > 0$ for some x, then $P^x\{A_{T_n} \to A_T\} =$

$1-P^x(\Lambda)<1$ since the τ_k are discontinuity points for $\{A_t\}$ also. This contradicts the fact that a process with stationary independent increments is quasi-left-continuous. ▯

We shall next derive the transition function

(2.17) $$P_t(x, A)=P^x\{X_t\in A\}, \quad x\in\mathbf{R},\ A\in\mathscr{R}$$

of the process X under the assumption that the rate of jumps is finite. Let, then, $b<\infty$ be the rate of jumps, and let γ be the distribution of the magnitude of a jump.

(2.18) **Lemma.** *For any bounded* $f\in\mathscr{R}^*$ *and* $t\geqq 0$,

(2.19) $$P_t f(x)=e^{-bt} f\circ q(x,t)+\int_0^t\int_0^\infty b e^{-bs}\,ds\,\gamma(dy)\,P_{t-s}f(y+q(x,s)).$$

(2.20) *Remark.* Since X is measurable, the mapping $(x,t)\to P_t(x,A)$ of $\mathbf{R}^2$ into $[0,1]$ for fixed $A\in\mathscr{R}^*$ is $(\mathscr{R}\times\mathscr{R})^{\lambda\times\mu}$-measurable for all finite measures λ and μ on $\mathscr{R}$ (where the σ-algebra in question is the completion of $\mathscr{R}\times\mathscr{R}$ with respect to the product measure $\lambda\times\mu$). Thus, the integral on the right-hand side of (2.19) is well defined. ▯

Proof. The first jump time $T=\tau_1$ is an $\{\mathscr{F}_{t+}^0\}$, and therefore an $\{\mathscr{F}_t\}$, stopping time. Since X is strong Markov by Theorem (2.5),

$$P^x\{X_t\in B\mid\mathscr{F}_T\}=P_{t-T}(X_T,B) \quad \text{on } \{T\leqq t\}.$$

By Theorem (2.4), $X_T=q(X_0,T)+\alpha_1$, and $X_t=q(X_0,t)$ on $\{T>t\}$. Thus, we have

$$P_t f(x)=E^x[f(X_t)]=E^x[I_{\{T>t\}} f\circ q(x,t)+I_{\{T\leqq t\}}(P_{t-T}f)(\alpha_1+q(x,T))]$$

which yields the lemma. ▯

(2.21) **Theorem.** *Let, for each* $x\in\mathbf{R}$, $B\in\mathscr{R}^2$,

(2.22) $$K(x,B)=\iint b e^{-bs}\,ds\,\gamma(dy)\,I_B(q(x,s)+y,s)$$

and define

(2.23) $$K_0(x,B)=\varepsilon_{(x,0)}(B)$$
$$K_{n+1}(x,B)=\int K(x,d(y,s))\,K_n(y,B-(0,s)).$$

Then, for any bounded $f\in\mathscr{R}$,

(2.24) $$P_t f(x)=\sum_{n=0}^{\infty}\int_{R\times[0,t]} K_n(x,d(y,s))\,e^{-b(t-s)} f\circ q(y,t-s)$$

exists and is the unique solution of (2.19).

Proof. Fix $f\in\mathscr{R}$ bounded and write $f(x,t)=P_t f(x)$ and $g(x,t)=e^{-bt}f\circ q(x,t)$. Then, (2.19) can be rewritten as

(2.25) $$f(x,t)=g(x,t)+\int_{R\times[0,t]} K(x,d(y,s))\,f(y,t-s)$$

with K as defined by (2.22).

Note that $x \to K(x, B)$ is in $\mathscr{R}$ and $B \to K(x, B)$ is a measure on $\mathscr{R}^2$. Thus K is a semi-Markovian kernel and (2.25) is a Markov renewal equation (cf. Çinlar [2] and [3]). Then, that (2.24) exists and is a solution of (2.25) follows from Theorem 8 of [2] since

$$g(x, t) \leqq K(x, \mathbf{R} \times (t, \infty)) = e^{-bt}.$$

That (2.24) is the only solution of (2.25) follows from Theorem (3.13) of [3] since

$$\sup_x K(x, \mathbf{R} \times [0, t]) = 1 - e^{-bt} < 1$$

for some $t > 0$ (because $b < \infty$ by hypothesis). □

(2.26) *Remark.* Let $u(t, x) = e^{bt} P_t f(x)$ for some differentiable function f on $(R, \mathscr{R})$. Then, from Lemma (2.18) we get

$$u(t, x) = f \circ q(x, t) + b \int_0^\infty \gamma(dy) \int_0^t u(s, y + q(x, t-s))\, ds.$$

From this, using (d) of Lemma (2.3), we obtain

$$\left[\frac{\partial}{\partial t} + (r(x) - a)\frac{\partial}{\partial x}\right] u = b \int_0^\infty \gamma(dy)\, u(t, x+y).$$

This is the characteristic equation for X. □

(2.27) *Remark.* Let K be as defined by (2.22) and put, for $\lambda \geqq 0$,

$$K^\lambda(x, A) = \int_0^\infty e^{-\lambda t} K(x, A \times dt).$$

Then K^λ is a sub-Markovian kernel on $(R, \mathscr{R})$. Putting

$$R^\lambda = \sum_n K_n^\lambda$$

we have

$$U^\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\, dt = \int_0^\infty R^\lambda(x, dy) \int_0^\infty e^{-(\lambda+b)t} f \circ q(y, t)\, dt$$

($U^\lambda f$ is called the λ-potential of f). This follows directly from Theorem (2.21) and the relationship between Laplace transforms and convolutions. □

3. Output Process

Consider the process $\{Z_t;\ t \geqq 0\}$ where

(3.1) $$Z_t(\omega) = \int_0^t r(X_s(\omega))\, ds, \quad \omega \in \Omega,\ t \geqq 0.$$

Since X is right-continuous and r continuous, $t \to Z_t(\omega)$ is continuous. It is clear that, for any $s, t \geqq 0$,

(3.2) $$Z_{t+s} - Z_t \in \sigma(X_u\colon t \leqq u \leqq t+s),$$

and that $Z_0 = 0$. Thus, Z is a continuous additive functional of X.

The function

$$f_t(x)=E^x[Z_t] \tag{3.3}$$

is called the characteristic of Z. For fixed $x\in\mathbf{R}$, $f_0(x)=0$, $t\to f_t(x)$ is non-decreasing continuous, and $f_t(x)=E^x[Z_t]\leqq x+E^x[A_t]\leqq x+m\,t$. Conditioning on $\mathscr{F}_t$, we have

$$f_{t+s}(x)=f_t(x)+\int P_t(x,dy)\,f_s(y), \tag{3.4}$$

or directly

$$f_t(x)=\int_0^t\int_0^\infty P_s(x,dy)\,r(y)\,ds. \tag{3.5}$$

The Laplace transform

$$u^\lambda(x)=\lambda\int_0^\infty e^{-\lambda t} f_t(x)\,dt, \qquad \lambda\geqq 0 \tag{3.6}$$

is called the λ-potential of Z. From (3.5) and Remark (2.27) we have, in the notation of (2.27),

$$u^\lambda(x)=\int_0^\infty R^\lambda(x,dy)\int_0^\infty e^{-(\lambda+b)t}\, r\circ q(y,t). \tag{3.7}$$

We next consider the random time change effected by Z. Let

$$T_t(\omega)=\inf\{s\colon Z_s(\omega)>t\}, \qquad t\geqq 0 \tag{3.8}$$

for each $\omega\in\Omega$. Since Z is continuous non-decreasing, T_t is strictly increasing right-continuous. For each $t\geqq 0$, T_t is an $\{\mathscr{F}_s\}$ stopping time. Define

$$\bar{X}_t=X_{T_t}, \qquad \bar{A}_t=A_{T_t}, \qquad \bar{\mathscr{F}}_t=\mathscr{F}_{T_t}. \tag{3.9}$$

Noting that $Z_{T_t}=t$ by right-continuity of Z, (1.1) implies that

$$\bar{X}_t=X_0+\bar{A}_t-t. \tag{3.10}$$

It is known (cf. [1] p. 212) that, since X is a standard Markov process and Z is continuous and strictly increasing (since r is Lipschitz, $X_t\neq 0$)

$$\bar{X}=(\Omega,\mathscr{F},\bar{\mathscr{F}}_t,\bar{X}_t,P^x)$$

is also standard Markov process.

Further, for $x>0$, $r(x)>0$ and thus $P^x\{Z_t>0\}=1$ for $t>0$ which implies that $P^x\{T_0=0\}=1$ and hence $P^x\{\bar{X}_0=x\}=1$. On the other hand, if $x=0$, $P^x\{T_0=\tau_1\}=1$ and thus $P^0\{\bar{X}_0=\alpha_1>0\}=1$. So, $\bar{X}$ is normal except for $x=0$.

Next we will compute the resolvent of $\bar{X}$ (and thus specify its transition function, and by (3.10), that of $\bar{A}_t$).

(3.11) **Theorem.** *Let, for $\lambda>0$, $f\in\mathscr{R}^*$ bounded,*

$$W^\lambda f(x)=E^x\left[\int_0^\infty e^{-\lambda t} f(\bar{X}_t)\,dt\right]. \tag{3.12}$$

Then,

$$W^\lambda f(x)=e^{-\lambda x}\sum_{n=0}^\infty K_n\, g(x) \tag{3.13}$$

where

$$g(x)=\int_0^\infty \exp[-(b+\lambda a)t+\lambda q(x,t)](f\cdot r)\circ q(x,t)\,dt \tag{3.14}$$

and

$$K(x,A)=\int_0^\infty b\,e^{-(b+\lambda a)t}\,dt \int_{A-q(x,t)} e^{-\lambda z}\gamma(dz), \tag{3.15}$$

for $x\in\mathbf{R}$, $A\in\mathscr{R}$.

Proof. Let $f\in\mathscr{R}^*$ be bounded, $\lambda>0$. It follows from (3.8) and (3.9) that (cf. [1] Lemma (2.2) in Chapter V)

$$\int_0^\infty e^{-\lambda t} f(\overline{X}_t)\,dt=\int_0^\infty \exp(-\lambda Z_s)\,f(X_s)\,dZ_s.$$

Replacing Z_s by $X_0+A_s-X_s$ and dZ_s by $r(X_s)\,ds$ and putting

$$h(x)=e^{-\lambda x} f(x)\,r(x), \tag{3.16}$$

we have

$$\begin{aligned} W^\lambda f(x)&=E^x\left[\exp(-\lambda X_0)\int_0^\infty \exp(-\lambda A_s)\,h(X_s)\,ds\right]\\ &=e^{-\lambda x}\int_0^\infty E^x[\exp(-\lambda A_s)\,h(X_s)]\,ds, \end{aligned} \tag{3.17}$$

where we use normality of X, and Fubini's theorem to change the order of integration.

Noting that $(\Omega,\mathscr{F},\mathscr{F}_t,(X_t,A_t),P^x)$ has the strong Markov property, and that the first jump time $T=\tau_1$ is an $\{\mathscr{F}_t\}$ stopping time, we have

$$\begin{aligned} &E^x[\exp(-\lambda A_t)\,h(X_t)|\mathscr{F}_T]\\ &\quad=\begin{cases} e^{-\lambda a t}\,h\circ q(x,t) & \text{on } \{T>t\},\\ e^{-\lambda A(T)}\,E^{X(T)}[\exp(-\lambda A_{t-T})\,h(X_{t-T})] & \text{on } \{T\leqq t\}. \end{cases} \end{aligned} \tag{3.18}$$

Thus, if we write

$$k(x,t)=E^x[\exp(-\lambda A_t)\,h(X_t)], \qquad k(x)=\int_0^\infty k(x,t)\,dt; \tag{3.19}$$

then we have from (3.18)

$$\begin{aligned} k(x,t)&=\exp(-b\,t-\lambda\,a\,t)\,h\circ q(x,t)\\ &\quad+\iint_{\mathbf{R}\times[0,t]} b\,e^{-bs}\,ds\,\gamma(dy-q(x,s))\exp[-\lambda(a\,s+y-q(x,s))]\,k(y,t-s) \end{aligned}$$

which gives

$$\begin{aligned} k(x)&=\int_0^\infty \exp(-b\,t-\lambda\,a\,t)\,h\circ q(x,t)\,dt\\ &\quad+\int_0^\infty b\,e^{-bs}\,ds\int_0^\infty \gamma(dy-q(x,s))\exp[-\lambda(a\,s+y-q(x,s)]\,k(y). \end{aligned}$$

Noting the definitions (3.14), (3.15), (3.16) this becomes

$$k=g+K\,k. \tag{3.20}$$

We have $K(x, \mathbf{R}) = \frac{b}{b+\lambda a} \gamma^\lambda \leqq \gamma^\lambda < 1$ since $\lambda > 0$ where $\gamma^\lambda = \int_0^\infty e^{-\lambda x} \gamma(dx)$. Therefore, $R = \sum_n K_n$ is well defined and $A \to R(x, A)$ is a finite measure. The solution of (3.20) then is

$$k = R g.$$

This together with (3.19) and (3.17) yields (3.13). □

4. Limit Theorems

In this section we shall consider the limiting distribution of X_t as t approaches infinity. First we want to show that, in the decomposition (2.1), we can take $a=0$ without loss of generality.

Suppose $a>0$ in (2.1). If $\sup_x r(x) < a$, then $t \to X_t(\omega)$ is a strictly increasing function and $X_t(\omega) \to +\infty$ as $t \to \infty$ for each $\omega \in \Omega$.

If $a>0$ and $\sup r(x) = a$, then $t \to X_t(\omega)$ is non-decreasing, and there exists a hitting time T such that

$$X_t(\omega) = A_t(\omega) - a\,t - C(\omega)$$

for all $t \geqq T(\omega)$, (and $C(\omega)$ is independent of t). Therefore, then X_t and $A_t - a\,t$ are shifted copies of each other; and $X_t(\omega) \to +\infty$ as $t \to \infty$ for each $\omega \in \Omega$.

Finally suppose $a>0$, $\sup r(x) > a$, let x^* be such that $r(x^*) = a$ (since r is continuous, such a point exists), and set $T(\omega) = \inf\{t: X_t(\omega) > x^*\}$. Then, X_t is strictly increasing for $t < T$, and $X_{T+t} \geqq x^*$, $r(X_{T+t}) \geqq a$ for all $t \geqq 0$. On the other hand, $P^x\{T < \infty\} = 1$ and $E^x(T) < \infty$ for all x. Thus, the set $[0, x^*)$ is distributive and $P^x\{X_t < x^*\} \to 0$ as $t \to \infty$ for $x < x^*$ (of course we have $P^x\{X_t < x^*\} = 0$ for $x \geqq x^*$). Hence, it is sufficient to consider the limiting behavior of X restricted to $[x^*, \infty]$. Then, putting $\tilde{r}(x) = r(x) - a$ we can rewrite (1.1) as $X_t = X_0 + \tilde{A}_t - \int_0^t \tilde{r}(X_u)\,du$.

Hence, from here on, we assume $a=0$. We also assume the jump rate b to be finite, and denote by m the input rate:

$$m = b \int_0^\infty x\,\gamma(dx)$$

(then $E^x(A_t) = m\,t$ for all t).

Below we shall show that $P^x\{X_t \in B\} \to 0$ for all bounded B as $t \to \infty$ if $\sup_x r(x) < m$. If $\sup_x r(x) > m$, then $P^x\{X_t \in B\} \to \nu(B)$ as $t \to \infty$ for some probability measure ν independent of x and we will compute ν. The situation in the case $\sup r(x) = m$ is not clear. Our conjecture is that there exists a σ-finite (but not finite) invariant measure for P_t in that case.

Let η_n be the left-hand limit of X_t at τ_n, that is,

$$\begin{aligned} &\eta_1 = q(X_0, \tau_1) \\ &\eta_{n+1} = q(\eta_n + \alpha_n, \tau_{n+1} - \tau_n), \qquad n = 1, 2, \ldots, \end{aligned} \tag{4.1}$$

and put

$$\mathcal{M}_n = \sigma(X_0, \eta_1, \ldots, \eta_n; \tau_1, \tau_2, \ldots, \tau_n) \qquad n = 1, 2, \ldots. \tag{4.2}$$

The following (whose proof we omit as fairly easy) is central to our development (cf. [2] for the definitions).

(4.3) **Proposition.** $(\Omega, \mathscr{F}, \mathscr{M}_n, (\eta_n, \tau_n), P^x)$ *is a delayed Markov renewal process with semi-Markov kernel Q (that is,*

$$P^x\{\eta_{n+1}\in A, \tau_{n+1}-\tau_n\in B|\mathscr{M}_n\}=Q(\eta_n, A\times B)$$

for all $A, B\in\mathscr{R}$ *and* $n=1, 2, \ldots$*), where*

$$Q(x, \Gamma)=\iint b\,e^{-bs}\,ds\,\gamma(dy)\,I_\Gamma(q(x+y, s), s) \tag{4.4}$$

for all $x\in\mathbf{R}, \Gamma\in\mathscr{R}^2$.

(4.5) **Corollary.** $Y=(\Omega, \mathscr{F}, \mathscr{M}_n, \eta_n, P^x)$ *is a Markov chain with transition kernel*

$$N(x, A)=Q(x, A\times\mathbf{R})=\iint b\,e^{-bs}\,ds\,\gamma(dy)\,I_A\circ q(x+y, s). \tag{4.6}$$

We next will show that the assumptions of Foguel [4] are satisfied by N and further that the σ-finite invariant measure ν he constructs for N is actually finite in our case.

Throughout the following we denote by T_f the operator defined as

$$T_f\,g(x)=f(x)\,g(x)$$

and write T_A instead of T_f when $f=I_A$. The following is of interest on its own.

(4.7) **Theorem.** *Let* $D=\{x: r(x)>c\}$ *be non-empty for some constant* $c>m$. *Then,*

a) $(T_D N T_D)^n\,1(x)\to 0$ *as* $n\to\infty$ *for all* $x\in\mathbf{R}$;

b) $\sum_{n=0}^{\infty}(N T_D)^n\,1(x)<\infty$ *for all* $x\in\mathbf{R}$;

c) $\sum_{n=0}^{\infty}T_C(N T_D)^n\,1$ *is bounded for any bounded set* $C\in\mathscr{R}$.

Proof. Note first that $(NT_D)^n=(NT_D)(T_D N T_D)^{n-1}$ so that b) implies a).

Since r is continuous non-decreasing, the set D is of form $D=(d, \infty)$. Thus, if $q(x, t)\in D$, then $q(x, t)<x-c\,t$ and $x-c\,t\in D$. From (4.1) therefore

$$\{\eta_{k+1}\in D\}\subset\{\eta_{k+1}\in D, \eta_{k+1}<\eta_k+\alpha_k-c(\tau_{k+1}-\tau_k)\in D\}. \tag{4.8}$$

Define

$$S_0=0$$
$$S_n=\sum_{k=1}^{n}[\alpha_k-c(\tau_{k+1}-\tau_k)], \qquad n=1, 2, \ldots. \tag{4.9}$$

Then, from (4.8) we have

$$\{\eta_2\in D, \ldots, \eta_{n+1}\in D\}\subset\{\eta_1+S_1\in D, \ldots, \eta_1+S_n\in D\}=\{U_{\eta_1}>n\} \tag{4.10}$$

where

$$U_x=\inf\{k\geqq 1: x+S_k\notin D\}. \tag{4.11}$$

On the other hand, writting $\tilde{P}^x$ for the conditional probability $P^y\{\cdot|\mathcal{M}_1\}$ on $\{\eta_1 = x\}$, we have

$$(NT_D)^n 1(x) = \tilde{P}^x\{\eta_2 \in D, \ldots, \eta_{n+1} \in D\}; \tag{4.12}$$

thus, from (4.10) and (4.11),

$$\sum_{n=0}^{\infty} (NT_D)^n 1(x) \leqq \sum_{n=0}^{\infty} P^y\{U_x > n\} = E^y[U_x]. \tag{4.13}$$

But $\{S_n\}$ is a random walk and the expected value of one of its steps is $m/b - c/b < 0$ since $c > m$. Thus, $\{S_n\}$ drifts to $-\infty$. From standard results for random walks, since D is of form (d, ∞), $E^y[U_x] < \infty$ for any x. This proves b) via (4.13).

To show c) it is sufficient to take $C = [0, k]$. Then, the result is immediate from b) once we note that

$$\sup_x \sum_{n=0}^{\infty} T_C (NT_D)^n 1(x) = \sup_{x \in C} \sum_{n=0}^{\infty} (NT_D)^n 1(x)$$
$$\leqq \sup_{x \in C} E^y(U_x) \leqq E^y(U_k). \quad \square$$

(4.14) **Lemma.** *If $f \in \mathcal{R}$ is continuous, so is Nf.*

Proof. From Corollary (4.5)

$$Nf(x) = \iint b\, e^{-bs}\, ds\, \gamma(dy)\, f \circ q(x+y, s)$$

and this is continuous since $x \to q(x, t)$ is continuous by Lemma (2.3). $\square$

(4.15) **Theorem.** *If $\sup_x r(x) > m$, then Y has a unique invariant measure ν and $\nu(\mathbf{R}) = 1$.*

Proof. By the hypothesis of the theorem, there exists $c > m$ with $D = \{x: r(x) > c\}$ non-empty. Thus, Theorem (4.7) holds. The statement a) of (4.7) is assumption (2.1) of Foguel [4], and Lemma (4.14) is assumption (3.5) in [4]. Thus all the results of [4] hold for the process Y. In particular, the theorem of that paper shows that there exists a measure ν for Y which is constructed as follows.

Let C be a closed interval containing the complement $\complement D$ of D; and let f be a continuous function with

$$0 \leqq f \leqq 1, \quad f = 0 \text{ on } \complement D, \quad f = 1 \text{ on } \complement C.$$

Then

$$N_\infty = \sum_{n=0}^{\infty} (NT_f)^n NT_{1-f}$$

is a well defined contraction on bounded measurable functions defined on C. Further, by the corollary to Lemma 3 of [4], there exists a probability measure λ, on C, with $\lambda N_\infty = \lambda$. We have

$$\nu = \sum_{n=0}^{\infty} \lambda (NT_f)^n \tag{4.16}$$

as the desired σ-finite measure satisfying $\nu N = \nu$.

Since $f \leqq I_D$, $NT_f \leqq NT_D$ and by iteration we have $(NT_f)^n \leqq (NT_D)^n$. Thus, from (4.16),

$$v\,1 \leqq \lambda \sum_{n=0}^{\infty} (NT_D)^n\, 1 \leqq \sup_{x\in C} \sum_{n=0}^{\infty} (NT_D)^n\, 1(x)$$

since λ is a probability measure concentrated on C. Hence $v\,1<\infty$ by c) of Theorem (4.7). Then we can take $v(\mathbf{R})=1$ by a suitable normalization.

To show that v is the only invariant measure we note that, for any $x\in\mathbf{R}$ and $A\in\mathscr{R}$ with positive Lebesgue measure, there exists n such that $N_n(x, A)>0$. This follows from the nature of the exponential distribution since, if $\beta=\sup\{x\colon \gamma(0, x)<1\}$ and $A=[a_0, a_1]\subset[0, x+\beta-\varepsilon]$, then

$$N(x, A) \geqq \int_{\beta-\varepsilon}^{\beta} \gamma(dy) \int_{B_y} b\, e^{-bs}\, ds>0$$

where $B_y=\{t\colon q(x+y, t)\in A\}$. ∎

(4.17) **Theorem.** *Let* $N(t)=\sup\{n\colon \tau_n\leqq t\}$. *Then, if* $\sup_x r(x)>m$,

$$\lambda(A\times B)=\lim_{t\to\infty} P^x\{\eta_{N(t)}\in A,\, t-\tau_{N(t)}\in B\} \tag{4.18}$$

exists for open sets A, B and is given by

$$\lambda(A\times B)=v(A)\int_B b\, e^{-bs}\, ds \tag{4.19}$$

where v is as defined in Theorem (4.15).

Proof. Existence of the limit (4.18) is assured by Theorem 4 of Orey [6] whose conditions we satisfy as follows. Conditions (i) and (ii) of [6] on the state space are satisfied by $(\mathbf{R}, \mathscr{R})$; condition (iii) is satisfied by our Theorem (4.15); and the finiteness of the integral (0.1) of [6] is evident since v is finite and the expected sojourn times involved are $1/b$.

That (4.19) is true follows from Theorem (3.1) of Pyke and Schaufele [7] (though their state space was discrete, their proof goes through for our case).

(4.20) **Theorem.** *If* $\sup r(x)>m$, *then*

$$\mu(A)=\lim_{t\to\infty} P^x\{X_t\in A\}$$

exists and we have

$$\mu(A)=\iiint v(dx)\,\gamma(dy)\, b\, e^{-bs}\, ds\, I_A\circ q(x+y, s).$$

Proof. From Theorem (2.4) and (4.1) we have

$$X_t=q(\eta_n+\alpha_n, t-\tau_n) \qquad \text{on } \{\tau_n\leqq t<\tau_{n+1}\},$$

that is, if $N(t)$ is defined as in Theorem (4.17),

$$X_t=q(\eta_{N(t)}+\alpha_{N(t)}, t-\tau_{N(t)}).$$

Proof follows now from Theorem (4.17) and the independence of $\eta_n, \alpha_n, \tau_{n+1}-\tau_n$ for each n. ∎

(4.21) *Remark.* Suppose $\sup_x r(x) = c < m$. Then, from (1.1) we have

$$X_t \geqq X_0 + A_t - c\,t.$$

Since $E^x[A_t] = m\,t > c\,t$, this implies that $\lim_t X_t = +\infty$ almost surely for any P^x.

5. A Special Case

Consider the case $r(x) = c\,x$ where $c > 0$ is a constant. According to Theorem (4.20), a limiting distribution for X exists if m is finite. We will now show by an explicit calculation that it may still exist when m is infinite. Furthermore, in this case we do not need to assume the rate of jumps to be finite. For the reasons explained in Section 4 we assume, without loss of generality, that $\{A_t\}$ is a pure jump process. Then,

(5.1) $$E^x[\exp(-\lambda A_t)] = \exp(-t\,g(\lambda))$$

where

(5.2) $$g(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\,\nu(dx)$$

for some measure ν on the Borel subsets of $(0, \infty)$ satisfying $\int (x/(1+x))\,\nu(dx) < \infty$.

(5.3) **Theorem.** *Let $r(x) = c\,x$. In order that X_t have a limiting distribution when t approaches $+\infty$, it is necessary and sufficient that*

(5.4) $$\int_1^\infty (\log x)\,\nu(dx) < \infty.$$

Proof. In this case the Eq. (1.1) can be solved explicitly to yield

(5.5) $$X_t(\omega) = X_0(\omega)\,e^{-ct} + \int_0^t e^{-c(t-s)}\,dA_s(\omega)$$

where the integral on the right-hand side is a Riemann-Stieltjes integral. If we write (5.5) as a limit of Riemann-Stieltjes sums, it follows from (5.1) that

$$\begin{aligned} u_t(x, \lambda) &= E^x[e^{-\lambda X_t}] \\ &= \exp(-\lambda\,x\,e^{-ct}) \exp\left[-\int_0^t g(\lambda\,e^{-cs})\,ds\right]. \end{aligned}$$

If we now make the change of variable $y = \lambda\,e^{-cs}$ above, it becomes clear that $\lim_{t\to\infty} u_t(x, \lambda)$ exists simultaneously with

(5.6) $$\lim_{\varepsilon \downarrow 0} \int_\varepsilon^\lambda \frac{1}{y}\,g(y)\,dy.$$

But this last expression is of the form $\int_0^\infty f_\varepsilon(x)\,\nu(dx)$ where

$$f_\varepsilon(x) = \int_\varepsilon^\lambda \frac{1 - e^{-yx}}{y}\,dy.$$

As $\varepsilon\downarrow 0$, $f_\varepsilon(x)$ increases to the limit

$$f(x)=\int_0^\lambda \frac{1-e^{-yx}}{y}\,dy. \tag{5.7}$$

Thus, by the monotone convergence theorem, the limit (5.6) exists simultaneously with the integral

$$\int_0^\infty f(x)\,\nu(dx). \tag{5.8}$$

But f is locally bounded and asymptotic to $\log x$ as $x\to\infty$. Therefore, the conclusion follows from the continuity theorem for Laplace transforms. □

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