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A Probabilistic Example of a Nowhere Analytic C^∞-Function*

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It is well known that there exist functions which are infinitely differentiable but nowhere analytic. In fact, Morgenstern [1] has shown that the nowhere analytic functions are by far in the majority among the infinitely differentiable functions on $[0, 1]$. A number of examples, as well as an extensive bibliography, are given by Salzmann and Zeller [2].

It is the purpose of this note to show that another example is provided by the restriction on $[0, 1]$ of the distribution function F of $X = \sum_1^\infty 2^{-n}\xi_n$, where $\xi_1, \xi_2, \ldots$ are independent random variables, uniformly distributed on $[0, 1]$.

One easily verifies that $F(0) = 0$, $F(1) = 1$ and $0 < F(x) < 1$ for all $x \in (0, 1)$. Moreover, since $2\sum_2^\infty 2^{-n}\xi_n$ is independent of ξ_1, and has the same distribution as X,

$$F(x) = \int_0^1 F(2x - y)\, dy = \begin{cases} \int_0^{2x} F(t)\, dt & \text{for} \quad x \in [0, \frac{1}{2}], \\ \int_{2x-1}^{1} F(t)\, dt + 2x - 1 & \text{for} \quad x \in (\frac{1}{2}, 1], \end{cases} \tag{1}$$

which implies that F is a continuous function. For notational convenience we introduce another function f, which we take to be the unique function on $[0, \infty)$ which coincides with F on $[0, 1]$ and satisfies the relations

$$f(x) = \begin{cases} 1 - f(x - 1) & \text{for} \quad x \in (1, 2] \\ -f(x - 2^n) & \text{for} \quad x \in (2^n, 2^{n+1}], \qquad n = 1, 2, \ldots \end{cases} \tag{2}$$

Clearly f is continuous and vanishes only at the nonnegative even integers. Moreover,

$$f(x) = \int_0^{2x} f(t)\, dt \tag{3}$$

for all $x \in [0, \infty)$. For $x \in [0, \frac{1}{2}]$ this is just a restatement of (1), for $x \in (\frac{1}{2}, 1]$ and for $x \in (1, 2]$ it follows from (1) and (2) by easy computations, and for $x \in (2^n, 2^{n+1}]$, $n = 1, 2, \ldots$ we prove (3) by induction on n: If (3) holds for all $x \in [0, 2^n]$ and some $n \geqq 1$, then we have for $x \in (2^n, 2^{n+1}]$

$$f(x) = -f(x - 2^n) = -\int_0^{2x - 2^{n+1}} f(t)\, dt = \int_{2^{n+1}}^{2x} f(t)\, dt = \int_0^{2x} f(t)\, dt,$$

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since (2) implies that

$$\int_0^{2^{n+1}} f(t)\,dt = 0$$

for all $n \geqq 1$.

It follows from (3) that f, being continuous, is differentiable with

$$f'(x) = 2f(2x)$$

for all $x \in [0, \infty)$ and hence, that f is infinitely differentiable with

$$f^{(n)}(x) = 2^{n(n+1)/2} f(2^n x) \tag{4}$$

for all $x \in [0, \infty)$, $n = 1, 2, \ldots$

In view of the fact that f vanishes only at the nonnegative even integers, (4) implies that at any binary rational of the form $x = (2k-1)2^{-n}$ with k and n positive integers all derivatives except the first n vanish. Consequently the Taylor series expansion of f around such a point is a polynomial of degree n, which cannot possibly coincide with f on any neighborhood, since at every binary irrational point all derivatives of f are nonzero. Thus f is nowhere analytic, being singular at all binary rationals.

References

[1] Morgenstern, D.: Unendlich oft differenzierbare nichtanalytische Funktionen. Math. Nachr. **12**, 74 (1954).

[2] Salzmann, H., und K. Zeller: Singularitäten unendlich oft differenzierbarer Funktionen. Math. Z. **62**, 354—367 (1955).

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