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Properties of the Sample Functions of the Completely Asymmetric Stable Process

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1. Introduction

In [3] Chung, Erdös and Sirao proved the following theorem for the Wiener process $\{X(t)\colon 0 \leqq t \leqq 1\}$.

Theorem 1.1. *Let ϕ be a non-negative, continuous and monotone non-increasing function and $\psi(t^{-1}) = \phi(t)$, then*

$$P\,[\{\omega\colon \text{there exists some } \Delta_0(\omega) \text{ such that } |X(t+\Delta,\omega) - X(t,\omega)| \leqq \Delta^{\frac{1}{2}}\,\phi(\Delta) \text{ for all } 0 \leqq t \leqq 1-\Delta \text{ and } 0 < \Delta \leqq \Delta_0(\omega)\}] = 0 \text{ or } 1$$

according as the integral

$$\int^{\infty} \psi^3(t)\, e^{-\frac{1}{2}\psi^2(t)}\, dt \tag{1.1}$$

diverges or converges.

This theorem generalizes a result of P. Lévy, who proved

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{0 \leqq t \leqq 1-\Delta \\ 0<\Delta<\varepsilon}} \frac{|X(t+\Delta) - X(t)|}{(2\Delta \log(\Delta^{-1}))^{\frac{1}{2}}} = 1 \quad \text{a.s.}$$

Recently, Hawkes [4] proved for the completely asymmetric stable process $\{X(t)\colon 0 \leqq t \leqq 1\}$ with characteristic exponent α with $0 < \alpha < 1$ and $\beta = 1$ (stable subordinator):

Theorem 1.2.

$$\lim_{\varepsilon \downarrow 0} \inf_{\substack{0 \leqq t \leqq 1-\Delta \\ 0<\Delta<\varepsilon}} \frac{X(t+\Delta) - X(t)}{\Delta^{1/\alpha}(\log(\Delta^{-1}))^{-\frac{1-\alpha}{\alpha}}} = (2B(\alpha))^{\frac{1-\alpha}{\alpha}} \quad \text{a.s.}$$

The constant $B(\alpha)$ will be defined in Lemma 2.1.

In this paper we extend the last theorem in a similar way as Theorem 1.1 generalizes Lévy's result. We shall give similar theorems for the completely asymmetric stable processes with characteristic exponent α with $1 < \alpha < 2$ and $\alpha = 1$. The proof is analogous to that of Theorem 1 in [3].

2. Preliminaries

The characteristic function (ch. f.) f of a stable distribution is given by

$$\begin{aligned} \log f(t) &= -|t|^\alpha \{1 - i\beta \operatorname{sgn}(t) \tan(\pi\alpha/2)\} && \text{if } \alpha \neq 1 \\ &= -|t| - i\beta(2/\pi) t \log|t| && \text{if } \alpha = 1 \end{aligned} \tag{2.1}$$

where α and β are real numbers with $0<\alpha \leqq 2$ and $|\beta| \leqq 1$. (See for example [2] or [5].) Distributions with $|\beta|=1$ are commonly called completely asymmetric stable distributions. In this paper, however, we restrict this terminology to the case where $\beta=1$. We have chosen the sign of β in (2.1) such that the completely asymmetric stable laws (with $\beta=1$) are the limit distributions of normed sums of positive independent and identically distributed random variables. For $0<\alpha<1$ we call these stable laws one-sided, since their support is $(0, \infty)$. When $\alpha \geqq 1$ the support is $(-\infty, \infty)$. The following estimates for the tails of the distribution function can be deduced from the expansions for the densities given by Skorohod [9]. (See also Polya and Szegö I, part II, problem 208 and Ibragimov and Linnik [5].) Estimate (2.2) is proved directly in [4].

Lemma 2.1. *Let U be the standard normal random variable and X the r.v. with ch. f. (2.1). If $0<\alpha<1$ and $\beta=1$ then*

$$P[X \leqq x] \sim (2/\alpha)^{\frac{1}{2}} P\left[U \geqq (2B(\alpha))^{\frac{1}{2}} x^{-\frac{\alpha}{2(1-\alpha)}}\right] \quad \textit{for } x \downarrow 0 \tag{2.2}$$

where $B(\alpha) = (1-\alpha)\, \alpha^{\frac{\alpha}{1-\alpha}} (\cos(\pi\alpha/2))^{-\frac{1}{1-\alpha}}$.

If $\alpha=1$ and $\beta=1$ then

$$P[X \leqq x] \sim 2^{\frac{1}{2}} P[U \geqq 2(\pi e)^{-\frac{1}{2}} e^{-\pi x/4}] \quad \textit{for } x \to -\infty. \tag{2.3}$$

If $1<\alpha<2$ and $\beta=1$ then

$$P[X \leqq x] \sim (2\alpha)^{\frac{1}{2}} P\left[U \geqq (2B(\alpha))^{\frac{1}{2}} (-x)^{\frac{\alpha}{2(\alpha-1)}}\right] \quad \textit{for } x \to -\infty \tag{2.4}$$

where $B(\alpha) = (\alpha-1)\, \alpha^{-\frac{\alpha}{\alpha-1}} |\cos(\pi\alpha/2)|^{\frac{1}{\alpha-1}}$.

Remark. In the case $\alpha=\frac{1}{2}$ and $\beta=1$ the stable r.v. X has the same distribution as U^{-2}. Then (2.2) can be replaced by

$$P[X \leqq x] = P[|U| \geqq x^{-\frac{1}{2}}] \quad \text{for all } x.$$

From (2.1) we obtain the following property. Let $X_1, \ldots, X_n$ be i.i.d. following a stable law, then the sum $S_n = X_1 + \cdots + X_n$ has the same distribution as $n^{1/\alpha} X_1$ if $\alpha \neq 1$ and as $n X_1 + (2/\pi)\beta n \log n$ if $\alpha=1$. This property and Lemma 2.1 were also used in the papers [7] and [8], where generalized laws of the iterated logarithm are proved for completely asymmetric stable random variables.

One may give a construction of stable processes analogous to the one for the Wiener process in [1]. Let X be a r.v. with ch. f. (2.1) then on $(D, \mathscr{D})$ (defined in [1], Chapter 3) there exists a stable measure $P_{\alpha,\beta}$ characterized by the following three properties.

$$P_{\alpha,\beta}[X(0)=0]=1, \tag{2.5}$$

$$\begin{aligned} P_{\alpha,\beta}[X(t)\leqq x] &= P[t^{1/\alpha} X \leqq x] && \text{if } \alpha \neq 1 \\ &= P[tX+(2/\pi)\beta t \log t \leqq x] && \text{if } \alpha = 1, \end{aligned} \tag{2.6}$$

$$X(t) \text{ has stationary and independent increments.} \tag{2.7}$$

For all stable processes the strong Markov property holds. (See [2].) The sample paths are discontinuous, but there are no fixed discontinuities. The completely asymmetric stable processes have only positive jumps. (If $\beta=-1$ the jumps are negative.) For $0<\alpha<1$ the paths are non-decreasing. (See [2].) For $\alpha=1$ and $1<\alpha<2$ we don't have this monotonicity property. In those cases we apply the following lemmas.

Lemma 2.2. *Let $\{X(t)\colon 0\leqq t\leqq 1\}$ be a stable process with $1<\alpha<2$ and $|\beta|\leqq 1$. There exists a positive constant k_α such that for all $t\in(0,1]$ and all negative x*

a) $P\big[\inf_{0\leqq s\leqq t} X(s)\leqq x\big]\leqq k_\alpha P[X(t)\leqq x]$,

b) $P\big[\inf_{0\leqq r\leqq s\leqq t} \{X(s)-X(r)\}\leqq x\big]\leqq k_\alpha^2 P[X(t)\leqq x]$.

Proof. We follow the method of Kiefer in [6]. An analogous method for partial sums is well-known. Let $y<0$ and let Γ be the event that for some $s\in[0,t]$ we have $X(s)<y$. On Γ we define $S=\inf\{s\colon X(s)<y\}$. The right-continuity of the sample paths implies that $X(S)\leqq y$ on Γ. From the strong Markov property and (2.6) it follows that

$$\begin{aligned} P[X(t)-X(s)\leqq 0 \mid \Gamma \wedge S=s] &= P[X(1)\leqq 0] && \text{for } s<t \\ &= 1 && \text{for } s=t \end{aligned}$$

and hence

$$P[X(t)-X(S)\leqq 0\mid\Gamma]\geqq P[X(1)\leqq 0].$$

Define the constant k_α by $k_\alpha^{-1}=P[X(1)\leqq 0]$. Then

$$\begin{aligned} P\big[\inf_{0\leqq s\leqq t} X(s)<y\big] = P[\Gamma] &\leqq k_\alpha P[X(t)-X(S)\leqq 0\wedge\Gamma] \\ &\leqq k_\alpha P[X(t)\leqq y]. \end{aligned}$$

Taking a decreasing sequence $y_n\downarrow x$, we obtain part a) of the lemma.

Now we prove part b). We first remark that the process $\{\tilde{X}(r)\colon 0\leqq r\leqq t\}$, where

$$\tilde{X}(r)=X(t)-X\big((t-r)-\big),$$

is again a stable process with the same parameters α and β. (We define $X(0-)=0$.) The paths of the processes $\tilde{X}(r)$ and $X(t)$ are in D. Therefore we have for almost all ω

$$\inf_{0\leqq r\leqq t}\{X(t,\omega)-X(t-r,\omega)\}=\inf_{0\leqq r\leqq t}\big\{X(t,\omega)-X\big((t-r)-,\omega\big)\big\}.$$

Let Γ_1 be the event that for some r and s in $[0,t]$ with $r<s$ we have $X(s,\omega)-X(r,\omega)<x$. The r.v. S is defined on Γ_1 by

$$S(\omega)=\inf\{s\colon \exists\, r<s \text{ with } X(s,\omega)-X(r,\omega)<x\}.$$

Let $\varepsilon > 0$. On the event $\{\Gamma_1 \wedge S = s\}$ we define the r.v. R by

$$R(\omega) = \sup\{r\colon X(s,\omega) - X(r,\omega) < x + \varepsilon\}.$$

Then, for all $\omega \in \Gamma_1$, at least one of the inequalities

$$X(S(\omega),\omega) - X(R(\omega),\omega) \leqq x + \varepsilon \quad \text{or} \quad X(S(\omega),\omega) - X(R(\omega)-,\omega) \leqq x + \varepsilon$$

hold. The r.v. S is a stopping time, therefore

$$P[X(t) - X(S) \leqq 0 \mid \Gamma_1] \geqq k_\alpha^{-1}.$$

Then

$$\begin{aligned} P[\Gamma_1] &\leqq k_\alpha P[X(t) - X(S) \leqq 0 \wedge \Gamma_1] \\ &= k_\alpha P[X(t) - X(R-) \leqq X(S) - X(R-) \wedge X(t) - X(R) \leqq X(S) - X(R) \wedge \Gamma_1] \\ &\leqq k_\alpha P[\min(X(t) - X(R-), X(t) - X(R)) \leqq x + \varepsilon] \\ &\leqq k_\alpha P[\inf_{0 \leqq r \leqq t} (X(t) - X(r-)) \leqq x + \varepsilon] \\ &\leqq k_\alpha^2 P[X(t) \leqq x + \varepsilon]. \end{aligned}$$

Part b) follows if we take sequences $x_n \downarrow x$ and $\varepsilon_n \downarrow 0$. □

Lemma 2.3. *Let $\{X(t)\colon 0 \leqq t \leqq 1\}$ be a completely asymmetric stable process with $\alpha = \beta = 1$. Let the function $x(p)$ such that for some constants c_1 and c_2 $c_1 < -x(p) + (2/\pi)\log p + 2/\pi < c_2$ for $p \geqq p_0$. There exists a positive constant k_1 such that for all $t \in (0,1]$ and for sufficiently large p*

a) *for* $0 \leqq (t-s)/t \leqq p^{-1}$

$$\begin{aligned} P\left[\inf_{t - tp^{-1} \leqq s \leqq t} \frac{X(s) - (2/\pi)\, s \log s}{s} \leqq -x(p)\right] &\leqq k_1 P\left[\frac{X(t) - (2/\pi)\, t \log t}{t} \leqq -x(p)\right] \\ &\leqq k_1 P[X(1) \leqq -x(p)]; \end{aligned}$$

b) *for* $0 \leqq r \leqq t p^{-1}$ *and* $0 \leqq (t-s) \leqq t p^{-1}$

$$\begin{aligned} P\left[\inf_{\substack{0 \leqq r \leqq tp^{-1} \\ t - tp^{-1} \leqq s \leqq t}} \frac{X(s) - X(r) - (2/\pi)(s-r)\log(s-r)}{s-r} \leqq -x(p)\right] \\ \leqq k_1^2 P[X(1) \leqq -x(p)]. \end{aligned}$$

Proof. The proof is similar to that of Lemma 2.2. Let Γ be the event that there exists some $s \in [t - tp^{-1}, t]$ with $(X(s) - (2/\pi)\, s \log s)/s \leqq -x(p)$. The r.v. S is defined on Γ to be the infimum of these numbers s. Then by the right-continuity

$$(X(S) - (2/\pi)\, S \log S)/S \leqq -x(p).$$

By the strong Markov property we have for $s\in[t-t\,p^{-1}, t)$ after some calculus

$$P\left[\frac{X(t)-(2/\pi)\,t\log t}{t}-\frac{X(s)-(2/\pi)\,s\log s}{s}\leqq 0\,|\,\Gamma\wedge S=s\right]$$
$$=P\left[X(1)\leqq -x(p)+(2/\pi)\big(t\log t-s\log s-(t-s)\log(t-s)\big)/(t-s)\right]$$
$$\geqq P[X(1)\leqq c_1-1]$$

for sufficiently large p. Define the constant k_1 by $k_1^{-1}=P[X(1)\leqq c_1-1]$. Thus

$$P\left[\frac{X(t)-(2/\pi)\,t\log t}{t}-\frac{X(S)-(2/\pi)\,S\log S}{S}\leqq 0\,|\,\Gamma\right]\geqq k_1^{-1}.$$

Part a) follows as in the proof of Lemma 2.2. a).

Again the process $\{\tilde X(r)\colon 0\leqq r\leqq t\}$, with $\tilde X(r)=X(t)-X\big((t-r)-\big)$, is a completely asymmetric stable process with $\alpha=\beta=1$. Define the event Γ_1 and the r.v. S and R by

$$\Gamma_1=\{\omega\colon \exists\,(r,s) \text{ with } r\in[0,t\,p^{-1}] \text{ and } s\in[t-t\,p^{-1},t]$$
$$\text{such that } \big(X(s)-X(r)-(2/\pi)(s-r)\log(s-r)\big)/(s-r)<-x(p)\};$$

on Γ_1

$$S(\omega)=\inf\{s\colon s\in[t-t\,p^{-1},t]\wedge\exists\, r\in[0,t\,p^{-1}]$$
$$\text{such that } \big(X(s)-X(r)-(2/\pi)(s-r)\log(s-r)\big)/(s-r)<-x(p)\}$$

and for $s\in[t-t\,p^{-1},t]$ on $\{\Gamma_1\wedge S=s\}$

$$R(\omega)=\sup\{r\colon r\in[0,t\,p^{-1}]\wedge\big(X(s)-X(r)-(2/\pi)(s-r)\log(s-r)\big)/(s-r)\leqq -x(p)\}.$$

Then we have

$$\frac{X(S)-X(R)-(2/\pi)(S-R)\log(S-R)}{(S-R)}\leqq -x(p)$$

on Γ_1. Remember $X(t,\omega)$ has only positive jumps.

The r.v. S is a stopping time, therefore, by the strong Markov property and similar calculations as in the proof of part a), we have for $s\in[t-t\,p^{-1},t)$ and $r\in[0,t\,p^{-1}]$

$$P\left[\frac{X(t)-X(r)-(2/\pi)(t-r)\log(t-r)}{t-r}-\frac{X(s)-X(r)-(2/\pi)(s-r)\log(s-r)}{s-r}\leqq 0\,|\right.$$
$$\left.\Gamma_1\wedge S=s\wedge R=r\right]$$
$$=P\big[X(1)\leqq -x(p)$$
$$+(2/\pi)\big((t-r)\log(t-r)-(s-r)\log(s-r)-(t-s)\log(t-s)\big)/(t-s)\big]\geqq k_1^{-1}$$

for sufficiently large p. Thus

$$P\left[\frac{X(t)-X(R)-(2/\pi)(t-R)\log(t-R)}{t-R}\right.$$
$$\left.\leqq\frac{X(S)-X(R)-(2/\pi)(S-R)\log(S-R)}{S-R}\,\middle|\,\Gamma_1\right]\geqq k_1^{-1}.$$

As in the proof of Lemma 2.2. b) it follows that

$$P[\Gamma_1] \leqq k_1 P\left[\inf_{0 \leqq r \leqq tp^{-1}} \frac{X(t)-X(r-)-(2/\pi)(t-r)\log(t-r)}{t-r} \leqq -x(p)\right]$$

$$\leqq k_1 P\left[\inf_{t-tp^{-1} \leqq s \leqq t} \frac{\tilde{X}(s)-(2/\pi)s\log s}{s} \leqq -x(p)\right]$$

$$\leqq k_1^2 P[X(1) \leqq -x(p)].$$

The assertion in part b) follows as in Lemma 2.2. b). □

With Lemma 2.2. a) we can prove a generalized law of the iterated logarithm for the completely asymmetric stable laws with $1<\alpha<2$. (This extends Theorem II in [7].)

As in [8] we make use of the following extension of the Borel-Cantelli lemma. (See for the proof [10].)

Lemma 2.4. *Let* $\{A_n\}$ *be a sequence of events with* $\sum P[A_n]=\infty$. *Then* $P[A_n \text{ i.o.}] \geqq c^{-1}$ *if*

$$\liminf \left(\sum_{i=1}^{n} P[A_i]\right)^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} P[A_i \wedge A_j] \leqq c.$$

3. The Case $0<\alpha<1$

In this section we shall prove a generalisation of Theorem 1.2. Let $\{X(t): 0 \leqq t \leqq 1\}$ be the completely asymmetric stable process ($\beta=1$) with characteristic exponent $0<\alpha<1$ and ϕ a non-negative, continuous and monotone non-decreasing function. We define the function ψ by

$$\psi(t^{-1}) = \{2B(\alpha)\}^{\frac{1}{2}} \{\phi(t)\}^{-\frac{\alpha}{2(1-\alpha)}}. \tag{3.1}$$

Before stating the theorem we prove some lemmas. In these lemmas we suppose

$$0<\Delta<t' \leqq t \tag{3.2}$$

and

$$t-\Delta+t' \leqq 1. \tag{3.3}$$

We write

$$P_I = P[X(t) \leqq t^{1/\alpha}\phi(t) \wedge X(t-\Delta+t')-X(t-\Delta) \leqq (t')^{1/\alpha}\phi(t')].$$

X denotes a r.v. with the same distribution as $X(1)$.

Lemma 3.1. *Let* $\phi(s) \to 0$ *for* $s \to 0$. *For all positive* ε *there exist positive constants* t_0 *and* δ *such that for all* $t \leqq t_0$, *all* Δ *satisfying* $\Delta \cdot t^{-1} \cdot \psi^2(t^{-1}) < \delta$ *and all* t' *satisfying* (3.2) *and* (3.3),

$$P_I \leqq (1+\varepsilon) P[X \leqq \phi(t)]\, P[X \leqq \phi(t')].$$

Proof.

$$\begin{aligned} P_I &\leqq P[X(t-\Delta) \leqq t^{1/\alpha}\phi(t) \wedge X(t-\Delta+t')-X(t-\Delta) \leqq (t')^{1/\alpha}\phi(t')] \\ &\qquad \text{the paths are non-decreasing} \\ &\leqq P\left[X \leqq (t/(t-\Delta))^{1/\alpha}\phi(t)\right] P[X \leqq \phi(t')] \\ &\qquad \text{the increments are independent.} \end{aligned} \tag{3.4}$$

From estimate (2.2) it follows that

$$P[X \leqq (t/(t-\Delta))^{1/\alpha} \phi(t)] / P[X \leqq \phi(t)] \leqq 1+\varepsilon$$

for both t and $\Delta \cdot t^{-1} \cdot \psi^2(t^{-1})$ sufficiently small. (This implies Δ/t is small.) □

Lemma 3.2. *Let $\phi(s) \to 0$ for $s \to 0$. For every constant $c \in (0,1)$, all t and Δ with $\Delta/t < c$ and all t' satisfying* (3.2) *and* (3.3) *there exist two positive constants C_1 and C_2 (independent of t, Δ and t') such that*

$$P_I \leqq C_1 e^{-C_2 \psi^2(t^{-1})} P[X \leqq \phi(t')].$$

Proof. Consider the factor

$$P[X \leqq (t/(t-\Delta))^{1/\alpha} \phi(t)]$$

on the right side of (3.4). It follows from estimate (2.2) and $t/(t-\Delta) < (1-c)^{-1}$ that

$$\begin{aligned} P[X \leqq (t/(t-\Delta))^{1/\alpha} \phi(t)] &\leqq P[X \leqq (1-c)^{-1/\alpha} \phi(t)] \\ &\leqq C_1 e^{-C_2 \psi^2(t^{-1})} \quad \text{for all } t \in (0,1). \quad \square \end{aligned}$$

Lemma 3.3. *Let $\phi(s) \to 0$ for $s \to 0$. Let $c \in (0,1)$ and $C > 0$ be two constants. Then, for all t and Δ such that*

$$0 < c \leqq \Delta/t < 1 \tag{3.5}$$

and

$$(t/(t-\Delta))^{(1-\alpha)/\alpha} \phi(t) \leqq C \tag{3.6}$$

and all t' satisfying (3.2) *and* (3.3), *there exist two constants C_3 and C_4 (independent of t, Δ and t') such that*

$$P_I \leqq C_3 e^{-C_4((t-\Delta)/t) \psi^2(t^{-1})} P[X \leqq \phi(t)]. \tag{3.7}$$

Proof.

$$\begin{aligned} P_I = {} & P[X(t) \leqq t^{1/\alpha} \phi(t) \wedge X(t-\Delta+t') - X(t-\Delta) \leqq (t')^{1/\alpha} \phi(t') \\ & \qquad \wedge X(t) - X(t-\Delta) \leqq \Delta^{(1+\alpha)/\alpha} t^{-1} \phi(t)] \\ & + P[X(t) \leqq t^{1/\alpha} \phi(t) \wedge X(t-\Delta+t') - X(t-\Delta) \leqq (t')^{1/\alpha} \phi(t') \\ & \qquad \wedge X(t, -X(t-\Delta) > \Delta^{(1+\alpha)/\alpha} t^{-1} \phi(t)] \\ \leqq {} & P[X(t) - X(t-\Delta) \leqq \Delta^{(1+\alpha)/\alpha} t^{-1} \phi(t)] \\ & + P[X(t-\Delta+t') - X(t-\Delta) \leqq (t')^{1/\alpha} \phi(t') \wedge X(t-\Delta) \leqq t^{1/\alpha} \phi(t) - \Delta^{(1+\alpha)/\alpha} t^{-1} \phi(t)] \\ \leqq {} & P[X \leqq (\Delta/t) \phi(t)] \\ & + P[X \leqq \phi(t')] \, P[X \leqq (t-\Delta)^{-1/\alpha} (t^{1/\alpha} - \Delta^{(1+\alpha)/\alpha} t^{-1}) \phi(t)]. \end{aligned} \tag{3.8}$$

By (2.2) we obtain that

$$P[X \leqq (\Delta/t) \phi(t)] \sim (\Delta/t)^{\overline{2(1-\alpha)}} e^{-\frac{1}{2}((\Delta/t)^{-\overline{1-\alpha}} - 1) \psi^2(t^{-1})} \qquad P[X \leqq \phi(t)] \quad \text{for } t \downarrow 0.$$

Then there exist two constants A_1 and A_2 (independent of t and Δ) such that

$$P[X \leqq (\Delta/t)\,\phi(t)] \leqq A_1\, e^{-A_2((t-\Delta)/t)\,\psi^2(t^{-1})}\, P[X \leqq \phi(t)] \tag{3.9}$$

for all t and Δ satisfying (3.5). We now estimate the last factor on the right side of (3.8). There exists a constant c_1 (independent of t and Δ) such that for all t and Δ satisfying (3.5)

$$(t-\Delta)^{-1/\alpha}(t^{1/\alpha} - \Delta^{(1+\alpha)/\alpha} \cdot t^{-1})\,\phi(t) \leqq c_1\big(t/(t-\Delta)\big)^{(1-\alpha)/\alpha}\,\phi(t).$$

Then by (2.2) and (3.6) it follows that there are two constants B_1 and B_2 (independent of t and Δ) such that

$$P\big[X \leqq c_1\big(t/(t-\Delta)\big)^{(1-\alpha)/\alpha}\,\phi(t)\big] \leqq B_1\, e^{-B_2((t-\Delta)/t)\,\psi^2(t^{-1})}. \tag{3.10}$$

From the estimates (3.9) and (3.10) and the monotonicity of ϕ it easily follows that

$$P_I \leqq C_3\, e^{-C_4((t-\Delta)/t)\,\psi^2(t^{-1})}\, P[X \leqq \phi(t)],$$

where

$$C_3 = 2\max(A_1, B_1) \quad \text{and} \quad C_4 = \min(A_2, B_2). \quad \square$$

We now state our theorem for the case $0<\alpha<1$.

Theorem 3.1. *Let ϕ be a non-negative, continuous and monotone non-decreasing function and $\{X(t)\colon 0 \leqq t \leqq 1\}$ the completely asymmetric stable process ($\beta=1$) with characteristic exponent $0<\alpha<1$. Then*

$$\begin{aligned}P[\{\omega\colon &\text{ there exists some } \Delta_0(\omega)>0 \text{ such that}\\ &X(t+\Delta,\omega) - X(t,\omega) \geqq \Delta^{1/\alpha}\,\phi(\Delta)\\ &\text{for all } 0 \leqq t \leqq 1-\Delta \text{ and } 0<\Delta \leqq \Delta_0(\omega)\}] = 0 \text{ or } 1\end{aligned}$$

according as the integral (1.1) *diverges or converges, where ψ is defined by* (3.1).

Proof. Without loss of generality we may restrict attention to the case where

$$(2\log t - 10\log\log t)^{\frac{1}{2}} \leqq \psi(t) \leqq (2\log t + 10\log\log t)^{\frac{1}{2}}. \tag{3.11}$$

(See Lemma 1 in [3].) This is equivalent with

$$\begin{aligned}\{2B(\alpha)\}^{\frac{1-\alpha}{\alpha}}&(2\log t^{-1} + 10\log\log t^{-1})^{-\frac{1-\alpha}{\alpha}} \leqq \phi(t)\\ &\leqq \{2B(\alpha)\}^{\frac{1-\alpha}{\alpha}}(2\log t^{-1} - 10\log\log t^{-1})^{-\frac{1-\alpha}{\alpha}}\end{aligned} \tag{3.12}$$

and yields

$$\phi(t) \sim \{B(\alpha)\}^{\frac{1-\alpha}{\alpha}}(\log t^{-1})^{-\frac{1-\alpha}{\alpha}} \qquad \text{for } t \downarrow 0. \tag{3.13}$$

Thus the restriction (3.11) implies that $\phi(t) \to 0$ for $t \to 0$.

Suppose the integral (1.1) converges. For $p=1,2,\ldots,k=0,1,\ldots,2^p$ and $l=[p/3],\ldots,p$ we define the event $D^p_{k,l}$ by

$$X\left(\frac{k+l}{2^p}\right) - X\left(\frac{k}{2^p}\right) \leqq \left(\frac{l+2}{2^p}\right)^{1/\alpha} \phi\left(\frac{l+2}{2^p}\right).$$

By (2.2) we have uniformly in k and l

$$P[D^p_{k,l}]=P\left[X(1)\leqq\left(\frac{l+2}{l}\right)^{1/\alpha}\phi\left(\frac{l+2}{2^p}\right)\right]$$

$$\sim(2/\alpha)^{\frac{1}{2}}P\left[U\geqq\left(\frac{l+2}{l}\right)^{-\frac{1}{2(1-\alpha)}}\psi\left(\frac{2^p}{l+2}\right)\right]$$

$$=O(1)\,P\left[U\geqq\psi\left(\frac{2^p}{l+2}\right)\right]\qquad\text{for } p\to\infty,$$

since

$$\left(\frac{l+2}{l}\right)^{-\frac{1}{2(1-\alpha)}}\psi\left(\frac{2^p}{l+2}\right)=\psi\left(\frac{2^p}{l+2}\right)+O\left(1/\psi\left(\frac{2^p}{l+2}\right)\right)\qquad\text{for } p\to\infty.$$

Convergence of the integral implies (see [3])

$$\sum_{p=1}^{\infty}\sum_{k=0}^{2^p}\sum_{l=[p/3]}^{p}P[D^p_{k,l}]<\infty,$$

and hence $P[D^p_{k,l}\text{ i.o.}]=0$.

For arbitrary fixed $t, t+\Delta\in[0,1]$ we define p, k and l by

$$(p+1)\,2^{-p-1}<\Delta\leqq p\,2^{-p} \tag{3.14}$$

and

$$(k-1)\,2^{-p}<t\leqq k\,2^{-p}<(k+l)\,2^{-p}\leqq t+\Delta<(k+l+1)\,2^{-p}. \tag{3.15}$$

This implies $[p/3]\leqq l\leqq p$ for $p\geqq 9$ and

$$X(t+\Delta,\omega)-X(t,\omega)\geqq X\left(\frac{k+l}{2^p},\omega\right)-X\left(\frac{k}{2^p},\omega\right).$$

Hence, for almost all ω, we have for all sufficiently small Δ (i.e. sufficiently large p) and all $t\in[0,1-\Delta]$

$$X(t+\Delta,\omega)-X(t,\omega)>\left(\frac{l+2}{2^p}\right)^{1/\alpha}\phi\left(\frac{l+2}{2^p}\right).$$

Because of the monotonicity of ϕ the right member is larger than $\Delta^{1/\alpha}\phi(\Delta)$. Thus the theorem is proved for the case of convergence.

In the divergent case, we define the event $E^p_{k,l}$ by

$$X\left(\frac{k+l}{2^p}\right)-X\left(\frac{k}{2^p}\right)<\left(\frac{l}{2^p}\right)^{1/\alpha}\phi\left(\frac{l}{2^p}\right)$$

for $p=1,2,\ldots,k=0,1,\ldots,2^p$ and $l=[p/2]+1,\ldots,p$. It is sufficient to prove $P[E^p_{k,l}\text{ i.o.}]=1$. To prove this assertion we apply Lemma 2.4. Rearrange the events $E^p_{k,l}$. If $E_n=E^p_{k,l}$ and $E_{n'}=E^{p'}_{k',l'}$ then $n<n'$ iff one of the following conditions holds:

1. $p<p'$
2. $p=p'$ and $l>l'$
3. $p=p'$, $l=l'$ and $k<k'$.

This rearrangement implies $l\,2^{-p} \geqq l'\,2^{-p'}$ for $n<n'$. Divergence of the integral (1.1) implies $\sum P[E_n]=\infty$. (See [3].)

For estimating the liminf in Lemma 2.4 we have to estimate $P[E_n \wedge E_{n'}]$. Consider two events $E_n = E^p_{k,l}$ and $E_{n'} = E^{p'}_{k',l'}$ with $n<n'$ and let $\Delta_{n,n'}$ denote the length of the intersection of $[k\,2^{-p}, (k+l)\,2^{-p}]$ and $[k'\,2^{-p'}, (k'+l')\,2^{-p'}]$. If

$$k\,2^{-p} < k'\,2^{-p'} < (k+l)\,2^{-p} < (k'+l')\,2^{-p'} \tag{3.16}$$

and $k=0$, we may apply one of the Lemmas 3.1, 3.2 or 3.3. If either (3.16) holds and $k \neq 0$ or if

$$0 \leqq k'\,2^{-p'} < k\,2^{-p} < (k'+l')\,2^{-p'} < (k+l)\,2^{-p} \leqq 1$$

or if

$$0 \leqq k\,2^{-p} \leqq k'\,2^{-p'} < (k'+l')\,2^{-p'} \leqq (k+l)\,2^{-p} \leqq 1$$

similar estimates may be derived.

Proceeding in this manner we arrive at the following three conclusions.

1. For any positive ε, there exist a number p_0 and a positive δ such that

$$P[E_n \wedge E_{n'}] \leqq (1+\varepsilon)\,P[E_n]\,P[E_{n'}] \tag{3.17}$$

for all events E_n and $E_{n'}$ with $n<n'$, $p \geqq p_0$ and

$$\Delta_{n,n'} \cdot l^{-1} \cdot 2^p \cdot \psi^2(2^p/l) < \delta. \tag{3.18}$$

2. Computations similar to those in [3] give that for fixed n'

$$\textstyle\sum^* P[E_n \wedge E_{n'}] \leqq M_1\,P[E_{n'}], \tag{3.19}$$

where $\sum^*$ denotes the summation over all events E_n with $n<n'$ and $\Delta_{n,n'} \cdot l^{-1} \cdot 2^p \leqq c$ and (3.18) does not hold. M_1 is a constant independent of n'.

3. In the case

$$\tfrac{1}{2} \leqq c \leqq \Delta_{n,n'} \cdot l^{-1} \cdot 2^p < 1 \tag{3.20}$$

(for $c \geqq \frac{1}{2}$ (3.20) restricts the values of p' to $p'=p, p+1$ or $p+2$) we apply Lemma 3.3 or its analog. The construction of the events $E^p_{k,l}$ and the assumption (3.12) yield that (3.6) is fulfilled for large p. Following the computations in [3] we get for every fixed n

$$\textstyle\sum^{**} P[E_n \wedge E_{n'}] \leqq M_2\,P[E_n], \tag{3.21}$$

where $\sum^{**}$ restricts the summation to all m where (3.20) holds and $n<n'$. From the estimates (3.17), (3.19) and (3.21) it follows that

$$\begin{aligned}&\operatorname{liminf}\left(\sum_{n=1}^{N} P[E_n]\right)^{-2} \sum_{n=1}^{N} \sum_{n'=1}^{N} P[E_n \wedge E_{n'}]\\ &= \operatorname{liminf}\left(\sum_{n=1}^{N} P[E_n]\right)^{-2} \cdot 2 \cdot \sum_{n<n'}^{N}\sum P[E_n \wedge E_{n'}] \leqq 1+\varepsilon.\end{aligned}$$

Taking $\varepsilon \downarrow 0$ we obtain liminf $\leqq 1$. □

Remark 1. Taking $\phi(\Delta) = \{2B(\alpha)\}^{\frac{1-\alpha}{\alpha}} \{2(1+\delta)\log(\Delta^{-1})\}^{-\frac{1-\alpha}{\alpha}}$ one obtains Theorem 1.2.

Remark 2. Let the function $\phi(\Delta)$ be defined by

$$\{2B(\alpha)\}^{\frac{1}{2}}\,\phi(\Delta)^{-\frac{\alpha}{2(1-\alpha)}}=\{2\log\Delta^{-1}+5\log_{(2)}\Delta^{-1}+2\log_{(3)}\Delta^{-1}+\dots$$
$$+2\log_{(n-1)}\Delta^{-1}+c\log_{(n)}\Delta^{-1}\}^{\frac{1}{2}}$$
$$\left(\log_{(n)}\Delta^{-1}=\log(\log_{(n-1)}\Delta^{-1})\right)$$

then the integral converges if $c>2$ and diverges if $c\leqq 2$.

Remark 3. For every non-negative, continuous and monotone non-decreasing function ϕ we have either

$$\lim_{\varepsilon\downarrow 0}\inf_{\substack{0\leqq t\leqq 1-\Delta\\ 0<\Delta<\varepsilon}}\frac{X(t+\Delta)-X(t)}{\Delta^{1/\alpha}\,\phi(\Delta)}\leqq 1\qquad\text{a.s.}$$

or

$$\lim_{\varepsilon\downarrow 0}\inf_{\substack{0\leqq t\leqq 1-\Delta\\ 0<\Delta<\varepsilon}}\frac{X(t+\Delta)-X(t)}{\Delta^{1/\alpha}\,\phi(\Delta)}\geqq 1\qquad\text{a.s.}$$

4. The Case $1<\alpha<2$

In this section we prove a similar theorem for the case $1<\alpha<2$. The proof does not differ essentially from that of Theorem 3.1. We shall only give the points of difference between the two proofs.

$\{X(t)\colon 0\leqq t\leqq 1\}$ is the completely asymmetric stable process with $1<\alpha<2$. Let ϕ and $\tilde{\phi}$ be non-negative and monotone non-increasing functions with $\phi<\tilde{\phi}$. Define the function ψ by

$$\psi(t^{-1})=\{2B(\alpha)\}^{\frac{1}{2}}\,\phi(t)^{\frac{\alpha}{2(\alpha-1)}}. \tag{4.1}$$

We first give the lemmas corresponding with the Lemmas 3.1, 3.2 and 3.3. Again we suppose (3.2) and (3.3). Now

$$P_I=P[X(t)\leqq -t^{1/\alpha}\,\phi(t)\wedge X(t-\Delta+t')-X(t-\Delta)\leqq -(t')^{1/\alpha}\,\phi(t')].$$

The r. v. X has the same distribution as $X(1)$.

Lemma 4.1. *Let $\phi(s)\to\infty$ for $s\to 0$. For all $\varepsilon>0$ there exist positive constants t_0 and δ such that for all $t\leqq t_0$, all Δ satisfying $\Delta\cdot t^{-1}\cdot\tilde{\phi}^{\alpha}(\Delta)\cdot\psi^2(t^{-1})<\delta$ and all t' satisfying* (3.2) *and* (3.3)

$$P_I\leqq P[X\leqq -\tilde{\phi}(\Delta)]+(1+\varepsilon)\,P[X\leqq -\phi(t)]\,P[X\leqq -\phi(t')].$$

Proof.

$$\begin{aligned}P_I=&P[X(t)\leqq -t^{1/\alpha}\,\phi(t)\wedge X(t-\Delta+t')-X(t-\Delta)\leqq -(t')^{1/\alpha}\,\phi(t')\\&\qquad\wedge X(t)-X(t-\Delta)\leqq -\Delta^{1/\alpha}\,\tilde{\phi}(\Delta)]\\&+P[X(t)\leqq -t^{1/\alpha}\,\phi(t)\wedge X(t-\Delta+t')-X(t-\Delta)\leqq -(t')^{1/\alpha}\,\phi(t')\\&\qquad\wedge X(t)-X(t-\Delta)> -\Delta^{1/\alpha}\,\tilde{\phi}(\Delta)]\\ \leqq& P[X\leqq -\tilde{\phi}(\Delta)]\\&+P[X(t-\Delta+t')-X(t-\Delta)\leqq -(t')^{1/\alpha}\,\phi(t')\wedge X(t-\Delta)\leqq -t^{1/\alpha}\,\phi(t)+\Delta^{1/\alpha}\,\tilde{\phi}(\Delta)]\\ =&P[X\leqq -\tilde{\phi}(\Delta)]+P[X\leqq -\phi(t')]\\&\cdot P\left[X\leqq -\left(t/(t-\Delta)\right)^{1/\alpha}\phi(t)+\left(\Delta/(t-\Delta)\right)^{1/\alpha}\tilde{\phi}(\Delta)\right].\end{aligned} \tag{4.2}$$

By (2.4) it follows that

$$P\left[X \leqq -(t/(t-\Delta))^{1/\alpha}\,\phi(t)+(\Delta/(t-\Delta))^{1/\alpha}\,\tilde{\phi}(\Delta)\right]/P[X \leqq -\phi(t)] \leqq 1+\varepsilon$$

for $t \leqq t_0$ and $\Delta \cdot t^{-1} \cdot \tilde{\phi}^{\alpha}(\Delta) \cdot \psi^2(t^{-1}) < \delta$. □

Lemma 4.2. *Let $\phi(s) \to \infty$ for $s \to 0$. For every constant $c \in (0,1)$, all t and Δ with $(\Delta/t)^{1/\alpha}\,\tilde{\phi}(\Delta)/\phi(t) < c$ and all t' satisfying* (3.2) *and* (3.3) *there exist two positive constants C_1 and C_2 (independent of t, Δ and t') such that*

$$P_I \leqq P[X \leqq -\tilde{\phi}(\Delta)] + C_1\, e^{-C_2 \psi^2(t^{-1})}\, P[X \leqq -\phi(t')].$$

Proof. The last factor in (4.2) is less than

$$\begin{aligned} &P\left[X \leqq -\phi(t)\left(1-(\Delta/t)^{1/\alpha}\,\tilde{\phi}(\Delta)/\phi(t)\right)\right] \\ &\leqq P[X \leqq -(1-c)\,\phi(t)] \leqq C_1\, e^{-C_2 \psi^2(t^{-1})} \qquad \text{by (2.4).} \quad \square \end{aligned}$$

Lemma 4.3. *Let $\phi(s) \to \infty$ for $s \to 0$. Let $c \in (0,1)$ and $C > 0$ be two constants. Then, for all t and Δ such that* (3.5) *and*

$$(t/(t-\Delta))^{(1-\alpha)/\alpha}\,\phi(t) \geqq C$$

hold and for all t' satisfying (3.2) *and* (3.3), *there exist two constants C_3 and C_4 (independent of t, Δ and t') such that*

$$P_I \leqq C_3\, e^{-C_4((t-\Delta)/t)\,\psi^2(t^{-1})}\, P[X \leqq -\phi(t)].$$

Proof. Just as in (4.2) we have

$$\begin{aligned} P_I \leqq\; & P\left[X \leqq -\phi(t)(1+(t-\Delta)A/t)\right] + P[X \leqq -\phi(t')] \\ & \cdot P\left[X \leqq -(t/(t-\Delta))^{1/\alpha}\,\phi(t) + (\Delta/(t-\Delta))^{1/\alpha}\,\phi(t)(1+(t-\Delta)A/t)\right] \end{aligned}$$

where A is some constant with $0 < A < \alpha^{-1}$. From (2.4) we get that there exist two constants A_1 and A_2 such that

$$P\left[X \leqq -\phi(t)(1+(t-\Delta)A/t)\right] \leqq A_1\, e^{-A_2((t-\Delta)/t)\,\psi^2(t^{-1})}\, P[X \leqq -\phi(t)].$$

For any $c \in (0,1)$ there exists a positive constant c_1 such that

$$-(t/(t-\Delta))^{1/\alpha}\,\phi(t) + (\Delta/(t-\Delta))^{1/\alpha}\,\phi(t)\,(1+(t-\Delta)A/t) \leqq -c_1 (t/(t-\Delta))^{(1-\alpha)/\alpha}\,\phi(t).$$

Thus there are two constants B_1 and B_2 such that

$$\begin{aligned} &P\left[X \leqq -(t/(t-\Delta))^{1/\alpha}\,\phi(t) + (\Delta/(t-\Delta))^{1/\alpha}\,\phi(t)\,(1+(t-\Delta)A/t)\right] \\ &\leqq P\left[X \leqq -c_1 (t/(t-\Delta))^{(1-\alpha)/\alpha}\,\phi(t)\right] \leqq B_1\, e^{-B_2((t-\Delta)/t)\,\psi^2(t^{-1})}. \end{aligned}$$

The lemma follows if we take $C_3 = 2\max(A_1, B_1)$, $C_4 = \min(A_2, B_2)$ and making use of the monotonicity of ϕ. □

Theorem 4.1. *Let ϕ be a non-negative, continuous and monotone non-increasing function and $\{X(t)\colon 0 \leqq t \leqq 1\}$ the completely asymmetric stable process with*

$1<\alpha<2$. *Then*

$$\begin{aligned}&P[\{\omega\colon \text{ there exists some } \Delta_0(\omega)>0 \text{ such that}\\ &\quad X(t+\Delta,\omega)-X(t,\omega)\geqq -\Delta^{1/\alpha}\phi(\Delta)\\ &\quad \text{for all } t\in(0,1-\Delta) \text{ and } 0<\Delta<\Delta_0(\omega)\}]=0 \text{ or } 1\end{aligned}$$

according as the integral (1.1) *diverges or converges, where* ψ *is defined by* (4.1).

Proof. Again we may restrict ourselves to functions ψ satisfying (3.11). Hence it follows by (4.1)

$$\phi(t)\sim\{B(\alpha)\}^{-\frac{\alpha-1}{\alpha}}(\log t^{-1})^{\frac{\alpha-1}{\alpha}}\qquad \text{for } t\downarrow 0$$

and this implies $\phi(t)\to\infty$ for $t\to 0$.

Suppose the integral (1.1) converges. For $p=1,2,\ldots,\ k=0,1\ldots 2^p$ and $l=[p/3],\ldots,p$ we define the event $D^p_{k,l}$ by

$$\inf_{0\leqq r,s\leqq 2^{-p}}\left\{X\left(\frac{k+l}{2^p}+s\right)-X\left(\frac{k}{2^p}-r\right)\right\}\leqq-\left(\frac{l}{2^p}\right)^{1/\alpha}\phi\left(\frac{l+2}{2^p}\right).$$

By Lemma 2.2.b) we have

$$\begin{aligned}P[D^p_{k,l}]&\leqq k_\alpha^2 P\left[\left(\frac{l+2}{2^p}\right)^{1/\alpha}X(1)\leqq-\left(\frac{l}{2^p}\right)^{1/\alpha}\phi\left(\frac{l+2}{2^p}\right)\right]\\ &=k_\alpha^2 P\left[X(1)\leqq-\left(\frac{l}{l+2}\right)^{1/\alpha}\phi\left(\frac{l+2}{2^p}\right)\right].\end{aligned}$$

By (2.4) we have uniformly in k and l

$$P[D^p_{k,l}]=O(1)\,P\left[U\geqq\psi\left(\frac{2^p}{l+2}\right)\right]\qquad\text{for } p\to\infty.$$

Hence, as in Section 3, it follows that

$$P[D^p_{k,l}\ \text{i.o.}]=0. \tag{4.3}$$

For any t and Δ we define integers p, k and l by (3.14) and (3.15). For all ω we have

$$X(t+\Delta,\omega)-X(t,\omega)\geqq\inf_{0\leqq r,s\leqq 2^{-p}}\left\{X\left(\frac{k+l}{2^p}+s,\omega\right)-X\left(\frac{k}{2^p}-r,\omega\right)\right\}.$$

By (4.3) we have for almost all ω, sufficiently small Δ and all t

$$X(t+\Delta,\omega)-X(t,\omega)>-\left(\frac{l}{2^p}\right)^{1/\alpha}\phi\left(\frac{l+2}{2^p}\right)>-\Delta^{1/\alpha}\phi(\Delta)$$

by the monotonicity of ϕ.

In the divergent case we define $E^p_{k,l}$ by

$$X\left(\frac{k+l}{2^p}\right)-X\left(\frac{k}{2^p}\right)<-\left(\frac{l}{2^p}\right)^{1/\alpha}\phi\left(\frac{l}{2^p}\right),$$

for $p=1,2,\ldots,$ $k=0,1,\ldots,2^p$ and $l=[p/2]+1,\ldots,p$. We rearrange the events $E^p_{k,l}$ as described in Section 3. It follows that $\sum P[E_n]=\infty$. The remainder of the proof closely resembles the proof of Theorem 3.1. However, the necessary estimation of the liminf occurring in Lemma 2.4 differs on two points. The first difference is the appearance of a term $P[X\leqq -\tilde{\phi}(\Delta)]$ in Lemmas 4.1 cf. 4.2. We choose

$$\tilde{\phi}(t)=(2\log t^{-1}+12\log\log t^{-1})^{\frac{\alpha-1}{\alpha}}(2B(\alpha))^{-\frac{\alpha-1}{\alpha}} \tag{4.4}$$

for t in the neighbourhood of zero. By (2.4) we have for small Δ

$$P[X\leqq -\tilde{\phi}(\Delta)]\leqq P[X\leqq -\tilde{\phi}(t')]\sim(\alpha/(2\pi))^{\frac{1}{2}}\,t'(\log(1/t'))^{-\frac{13}{2}}.$$

For a fixed event $E_{n'}=E^{p'}_{k',l'}$ the number of events E_n with $n<n'$ and $P[E_n\wedge E_{n'}]\neq P[E_n]\cdot P[E_{n'}]$ is less than $3(p')^3$. The number events $E_{n'}$ for a fixed integer p' is less than $2^{p'}\cdot p'$. For every fixed integer p' the sum of all terms $P[X\leqq -\tilde{\phi}(\Delta)]$ occurring in the estimates of $P[E_n\wedge E_{n'}]$ is $O(1)(p')^{-\frac{3}{2}}$ for $p'\to\infty$. Hence the sum of all terms of this kind is finite.

The other difference arises in connection with Lemma 4.2. We want to use this lemma in the case $0<\Delta/t<c_1<1$. In that case $(\Delta/t)^{1/\alpha}\,\tilde{\phi}(\Delta)/\phi(t)$ is not necesseraly less than 1. However, one only has to invoke Lemma 4.2 in the case that $p'-5\log p'<p<p'$. Then by the restriction (3.11) and (4.4) we know that for any pair of constants (c_1,c), with $0<c_1<c<1$, the restriction $\Delta/t<c_1$ implies $(\Delta/t)^{1/\alpha}\,\tilde{\phi}(\Delta)/\phi(t)<c$ for sufficiently large p' (or p). □

Remark. Taking $\phi(\Delta)=\{2B(\alpha)\}^{-\frac{\alpha-1}{\alpha}}(2(1+\delta)\log\Delta^{-1})^{\frac{\alpha-1}{\alpha}}$ we obtain

$$\lim_{\varepsilon\downarrow 0}\inf_{\substack{0\leqq t\leqq 1-\Delta\\ 0<\Delta<\varepsilon}}\frac{X(t+\Delta)-X(t)}{\Delta^{1/\alpha}(2\log\Delta^{-1})^{\frac{\alpha-1}{\alpha}}}=-\{2B(\alpha)\}^{\frac{\alpha-1}{\alpha}}\quad\text{a.s.}$$

5. The Case $\alpha=1$

Let $\{X(t)\colon 0\leqq t\leqq 1\}$ be the completely asymmetric stable process with $\alpha=1$ and let ϕ and $\tilde{\phi}$ be non-negative, monotone non-increasing functions with $\phi<\tilde{\phi}$. Define the function ψ by

$$\psi(t^{-1})=2(\pi e)^{-\frac{1}{2}}\exp(\pi\,\phi(t)/4). \tag{5.1}$$

In the lemmas, corresponding with the Lemmas 3.1, 3.2 and 3.3, we suppose (3.2) and (3.3) and write

$$\begin{aligned}P_I=P[&X(t)-(2/\pi)t\log t\leqq -t\,\phi(t)\\ &\wedge X(t-\Delta+t')-X(t-\Delta)-(2/\pi)t'\log t'\leqq -t'\,\phi(t')].\end{aligned}$$

The r.v. X has the same distribution as $X(1)$.

Lemma 5.1. *Let $\phi(s)\to\infty$ for $s\to 0$. For all $\varepsilon>0$ there exist positive constants t_0 and δ such that for all $t\leqq t_0$, all Δ satisfying $\Delta\cdot t^{-1}\cdot\psi^2(t^{-1})\cdot(\tilde{\phi}(\Delta)+(2/\pi)+(2/\pi)\log\Delta^{-1})<\delta$ and all t' satisfying* (3.2) *and* (3.3)

$$P_I\leqq P[X\leqq -\tilde{\phi}(\Delta)]+(1+\varepsilon)P[X\leqq -\phi(t)]\,P[X\leqq -\phi(t')].$$

Proof.

$$\begin{aligned}P_I = &P[X(t)-(2/\pi)t\log t \leqq -t\,\phi(t)\\ &\wedge X(t-\Delta+t')-X(t-\Delta)-(2/\pi)t'\log t' \leqq -t'\,\phi(t')\\ &\wedge X(t)-X(t-\Delta)-(2/\pi)\Delta\log\Delta \leqq -\Delta\,\tilde{\phi}(\Delta)]\\ &+P[X(t)-(2/\pi)t\log t \leqq -t\,\phi(t)\\ &\wedge X(t-\Delta+t')-X(t-\Delta)-(2/\pi)t'\log t' \leqq -t'\,\phi(t')\\ &\wedge X(t)-X(t-\Delta)-(2/\pi)\Delta\log\Delta > -\Delta\,\tilde{\phi}(\Delta)]\\ \leqq &P[X\leqq -\tilde{\phi}(\Delta)]+P[X\leqq -\phi(t')]\\ &\cdot P[X(t-\Delta)\leqq -t\,\phi(t)+\Delta\,\tilde{\phi}(\Delta)+(2/\pi)t\log t-(2/\pi)\Delta\log\Delta]\,. \end{aligned} \tag{5.2}$$

The last factor is equal to

$$\begin{aligned}P\big[X\leqq &-\big(t/(t-\Delta)\big)\,\phi(t)+\big(\Delta/(t+\Delta)\big)\,\tilde{\phi}(\Delta)\\ &+(2/\pi)\,(t-\Delta)^{-1}\big(t\log t-\Delta\log\Delta-(t-\Delta)\log(t-\Delta)\big)\big]\,.\end{aligned} \tag{5.3}$$

For $\Delta/t\to 0$ we have

$$\begin{aligned}A(t,\Delta)&=\big(\Delta/(t-\Delta)\big)\,\tilde{\phi}(\Delta)+(2/\pi)\,(t-\Delta)^{-1}\{t\log t-\Delta\log\Delta-(t-\Delta)\log(t-\Delta)\}\\ &\sim(\Delta/t)\,\{\tilde{\phi}(\Delta)+(2/\pi)+(2/\pi)\log(t/\Delta)\}\,.\end{aligned}$$

From estimate (2.3) it follows that t_0 and δ exist such that

$$P[X\leqq -\phi(t)+A(t,\Delta)]/P[X\leqq -\phi(t)]\leqq 1+\varepsilon$$

for $t\leqq t_0$ and $\Delta\cdot t^{-1}\cdot\{\tilde{\phi}(\Delta)+(2/\pi)+(2/\pi)\log\Delta^{-1}\}\cdot\psi^2(t^{-1})<\delta$. □

Lemma 5.2. *Let $\phi(s)\to\infty$ for $s\to 0$. For every constant $c\in(0,1)$, all t and Δ with $\Delta(t-\Delta)^{-1}\{\tilde{\phi}(\Delta)-\phi(t)+(2/\pi)\log 2\}\leqq c$ and all t' satisfying* (3.2) *and* (3.3) *there exist two positive constants C_1 and C_2 (independent of t, Δ and t') such that*

$$P_I\leqq P[X\leqq -\tilde{\phi}(\Delta)]+C_1\,e^{-C_2\psi^2(t^{-1})}\,P[X\leqq -\phi(t')]\,.$$

Proof. By convexity of $x\log x$ and (2.3), (5.3) is less than

$$\begin{aligned}&P\big[X\leqq -\big(t/(t-\Delta)\big)\,\phi(t)+\big(\Delta/(t-\Delta)\big)\,\tilde{\phi}(\Delta)+\big(2t/\pi(t-\Delta)\big)\log 2\big]\\ &\leqq P[X\leqq -\phi(t)+c+(2/\pi)\log 2]\\ &\leqq C_1\,e^{-C_2\psi^2(t^{-1})}\qquad\text{for all } t\in(0,1)\,.\ \square\end{aligned}$$

Lemma 5.3. *Let $\phi(s)\to\infty$ for $s\to 0$. Let $c\in(0,1)$ and $C>0$ be two constants. Then, for all t and Δ such that $0<c<\Delta/t<1$ and*

$$\phi(t)-(2/\pi)\log\big(t/(t-\Delta)\big)\geqq C \tag{5.4}$$

and all t' satisfying (3.2) *and* (3.3), *there exist two constants C_3 and C_4 (independent of t, Δ and t') such that*

$$P_I\leqq C_3\,e^{-C_4((t-\Delta)/t)\psi^2(t^{-1})}\,P[X\leqq -\phi(t)]\,.$$

Proof. As in (5.2) we have

$$\begin{aligned} P_I \leqq P[X \leqq -\phi(t)+(2/\pi)\log(\Delta t)] + P[X \leqq -\phi(t')] \\ \cdot P\big[X \leqq -\phi(t) - (2/\pi)\big(\Delta/(t-\Delta)\big)\log(\Delta/t) \\ +(2/\pi)(t-\Delta)^{-1}\big(t\log t - \Delta\log\Delta - (t-\Delta)\log(t-\Delta)\big)\big]. \end{aligned} \tag{5.5}$$

From (2.3) it follows that there exist two constants A_1 and A_2 such that

$$P[X \leqq -\phi(t)+(2/\pi)\log(\Delta/t)] \leqq A_1\, e^{-A_2((t-\Delta)/t)\psi^2(t^{-1})}\, P[X \leqq -\phi(t)]$$

for all $t\in(0,1)$ and $\Delta/t\in(c,1)$. After some algebra one finds that there is a constant c_1 such that

$$\begin{aligned} -(2/\pi)\big(\Delta/(t-\Delta)\big)\log(\Delta/t) + (2/\pi)(t-\Delta)^{-1}\big(t\log t - \Delta\log\Delta - (t-\Delta)\log(t-\Delta)\big) \\ \leqq (2/\pi)\log\big(t/(t-\Delta)\big) + c_1 \end{aligned}$$

for all $\Delta/t\in(c,1)$. Then we can bound the last factor in (5.5) by

$$P\big[X \leqq -\phi(t)+(2/\pi)\log\big(t/(t-\Delta)\big)+c_1\big].$$

Then there exist two constants B_1 and B_2 such that (5.5) can be estimated by

$$B_1\, e^{-B_2((t-\Delta)/t)\psi^2(t^{-1})} \quad \text{for all } t\in(0,1).$$

The lemma follows as in the cases $0<\alpha<1$ and $1<\alpha<2$. □

Theorem 5.1. *Let ϕ be a continuous non-negative, monotone non-increasing function and $\{X(t): 0\leqq t\leqq 1\}$ the completely asymmetric stable process with $\alpha=1$. Then*

$$\begin{aligned} P[\{\omega: \text{there exists some } \Delta_0(\omega)>0 \text{ such that} \\ X(t+\Delta,\omega) - X(t,\omega) - (2/\pi)\Delta\log\Delta \geqq -\Delta\,\phi(\Delta) \\ \text{for all } t\in(0,1-\Delta) \text{ and } 0<\Delta<\Delta_0(\omega)\}] = 0 \text{ or } 1 \end{aligned}$$

according as the integral (1.1) *diverges or converges, where ψ is defined by* (5.1).

Proof. Again we restrict ourselves to functions ψ satisfying (3.11). Hence

$$\phi(t) \sim (2/\pi)\log\log t^{-1} \quad \text{for } t\to\infty$$

and this implies $\phi(t)\to\infty$ for $t\downarrow 0$.

Assume (1.1) is convergent. For $p=1,2,\ldots$, $k=0,1,\ldots,2^p$ and $l=[p/3],\ldots,p$ we define the event $D^p_{k,l}$ by

$$\inf_{0\leqq r,s\leqq 2^{-p}} \left\{ \frac{X((k+l)2^{-p}+s) - X(k2^{-p}-r) - (2/\pi)(l2^{-p}+r+s)\log(l2^{-p}+r+s)}{l2^{-p}+r+s} \right\} \leqq -\phi\left(\frac{l+2}{2^p}\right).$$

The restriction (3.11) implies that the conditions in Lemma 2.3 are fulfilled uniformly in l. Thus

$$P[D^p_{k,l}] \leqq k_1^2\, P\left[X(1) \leqq -\phi\left(\frac{l+2}{2^p}\right)\right].$$

By (2.3) we have uniformly in k and l

$$P[D^p_{k,l}]=O(1)\,P\left[U\geqq\psi\left(\frac{2^p}{l+2}\right)\right]\quad\text{for } p\to\infty.$$

Convergence of (1.1) gives $P[D^p_{k,l}\text{ i.o.}]=0$. For arbitrary t, $t+\Delta\in[0,1]$ we define p, k and l by (3.14) and (3.15). For almost all ω, we have for sufficiently large p

$$\begin{aligned}&\frac{X(t+\Delta)-X(t)-(2/\pi)\Delta\log\Delta}{\Delta}\\ &\geqq\inf_{0\leqq r,s\leqq 2^{-p}}\left\{\frac{X((k+l)2^{-p}+s)-X(k2^{-p}-r)-(2/\pi)(l2^{-p}+r+s)\log(l2^{-p}+r+s)}{l2^{-p}+r+s}\right\}\\ &>-\phi\left(\frac{l+2}{2^p}\right)>-\phi(\Delta).\end{aligned}$$

In the divergent case we define $E^p_{k,l}$ by

$$X((k+l)2^{-p})-X(k2^{-p})-(2/\pi)(l2^{-p})\log(l2^{-p})\leqq -l2^{-p}\phi(l2^{-p})$$

for $p=1,2\ldots,k=0,1,\ldots,2^p$ and $l=[p/2]+1,\ldots,p$ and the function $\tilde\phi(s)$ by

$$2(\pi e)^{-\frac{1}{2}}\exp(\pi\tilde\phi(s)/4)=(2\log(s^{-1})+12\log\log(s^{-1}))^{\frac{1}{2}}.$$

There is no difference with the proof of the divergent part of Theorem 4.1. □

Remark. Taking

$$\phi(t)=(2/\pi)\log(\pi e/2)+(2/\pi)\log\log(t^{-1})+(2/\pi)\log\lambda$$

the integral (1.1) converges if $\lambda>1$ and diverges if $\lambda\leqq 1$. Then

$$\lim_{\varepsilon\downarrow 0}\inf_{\substack{0\leqq t<t+\Delta\leqq 1\\ 0<\Delta<\varepsilon}}\left\{\frac{X(t+\Delta)-X(t)-(2/\pi)\Delta\log\Delta}{\Delta}+(2/\pi)\log\log\Delta^{-1}\right\}=(2/\pi)\log(\pi e/2)\quad\text{a.s.}$$

and

$$\lim_{\varepsilon\downarrow 0}\inf_{\substack{0\leqq t<t+\Delta\leqq 1\\ 0<\Delta<\varepsilon}}\left\{\frac{X(t+\Delta)-X(t)-(2/\pi)\Delta\log\Delta}{(2/\pi)\Delta\log\log(\Delta^{-1})}\right\}=-1\quad\text{a.s.}$$

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